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by

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FI MU Report Series FIMU-RS-2004-04

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An Effective Characterization of Properties Definable by LTL Formulae with a Bounded Nesting Depth of the Next-Time Operator*

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Abstract

It is known that an LTL property is expressible by an LTL formula without any next-time operator if and only if the property is stutter invariant. It is also known that the problem whether a given LTL property is stutter invariant is PSPACE-complete. We extend these results to fragments of LTL obtained by restricting the nesting depth of the next-time operator by a given $n \in \mathbb{N}_0$. Some interesting facts about the logic LTL follow as simple corollaries.

1 Introduction

Lamport [Lam83] observed that LTL formulae without any next-time operator cannot distinguish between *stutter equivalent* ω-words, i.e., ω-words which are the same up to replacing all substrings of the form a^+ with a single a (here a is a letter and a^+ denotes a non-empty finite string of a's). Hence, properties (ω-languages) definable in this fragment of LTL are stutter invariant. Later, Peled and Wilke [PW97] proved that every stutter invariant property definable in LTL is also definable by an LTL formula

*This work has been supported by GAČR, grant No. 201/03/1161.

without any next-time operator. This was achieved by designing a translation algorithm which for a given LTL formula φ computes another formula $\tau(\varphi)$ without any next-time operator such that φ and $\tau(\varphi)$ are equivalent iff the property defined by φ is stutter invariant. Since the equivalence problem for LTL formulae is PSPACE-complete [SC85], one can also decide if a given LTL formula φ defines a stutter invariant property—it suffices to compute $\tau(\varphi)$ and decide if it is equivalent to φ. This algorithm requires exponential space because the size of $\tau(\varphi)$ is exponentially larger than the size of φ in general. Hence, it is surely not optimal—due to [PWW98] we know that the problem whether a given LTL formula φ defines a stutter invariant property is PSPACE-complete. However, the space complexity of the aforementioned algorithm can be improved from exponential to polynomial space by employing an alternative translation algorithm due to Etessami [Ete00]. In this case, the resulting formula $\tau(\varphi)$ can be represented by a circuit of polynomial size (though the size of $\tau(\varphi)$ is still exponential in the nesting depth of the next-time operator in φ). See Section 3 for further comments.

In our paper, we generalize the above discussed results to fragments of LTL where the nesting depth of the next-time operator is bounded by a given $n \in \mathbb{N}_0$. We provide a characterization of LTL properties which are expressible in these fragments, and design a polynomial-space algorithm which decides whether a given LTL formula is expressible in a given fragment (the matching PSPACE-lower bound is due to [PWW98]). Some interesting observations about the logic LTL follow as simple corollaries to our results. For example, it can be easily shown that by increasing the nesting depth of the next-time operator one always yields a strictly more expressive fragment of LTL (this is intuitively clear but a formal proof is not completely trivial), that the '$\mathsf{G}_2 p$' formula is not expressible in LTL, etc.

2 Background

The syntax of linear temporal logic (LTL) [Pnu77] is given by the following abstract syntax equation:

$$\varphi \quad ::= \quad p \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \mathsf{X}\varphi \mid \varphi_1 \mathsf{U} \varphi_2$$

Here p ranges over a countable set $AP = \{p, q, \dots\}$ of *atomic propositions*.

An *alphabet* is a (finite) set $\Sigma = 2^{\mathcal{A}}$, where $\mathcal{A}$ is a finite subset of *AP*. Elements of Σ are called *letters*. An *ω-word over* Σ is an infinite sequence $\alpha = \alpha(0)\alpha(1)\cdots$ of letters from Σ. The set of all ω-words over Σ is denoted by Σ^ω. A *property* (or *ω-language*) over Σ is a set $L \subseteq \Sigma^\omega$. For all $\alpha \in \Sigma^\omega$ and $i \in \mathbb{N}_0$, the symbol α_i denotes the ω-word obtained from α by omitting its first i elements (hence, $\alpha_0 = \alpha$).

The *validity* of an LTL formula φ for a given $\alpha \in \Sigma^\omega$ is defined inductively as follows:

$$
\begin{array}{lll}
\alpha \models p & \text{iff} & p \in \alpha(0) \\
\alpha \models \neg\varphi & \text{iff} & \alpha \not\models \varphi \\
\alpha \models \varphi_1 \wedge \varphi_2 & \text{iff} & \alpha \models \varphi_1 \wedge \alpha \models \varphi_2 \\
\alpha \models \mathsf{X}\varphi & \text{iff} & \alpha_1 \models \varphi \\
\alpha \models \varphi_1 \mathsf{U} \varphi_2 & \text{iff} & \exists i \in \mathbb{N}_0 : \alpha_i \models \varphi_2 \ \wedge\ \forall 0 \leq j < i : \alpha_j \models \varphi_1
\end{array}
$$

Let Σ be an alphabet. Each LTL formula φ defines a unique property L_φ^Σ over Σ given by $L_\varphi^\Sigma = \{\alpha \in \Sigma^\omega \mid \alpha \models \varphi\}$. Let $AP(\varphi)$ be the set of all atomic propositions which appear in φ. The *canonical alphabet* of φ is the alphabet $\Sigma_\varphi = 2^{AP(\varphi)}$ and the *canonical property* of φ is the property $L_\varphi^{\Sigma_\varphi}$ (denoted just by L_φ for short). A property L is an *LTL property* iff $L = L_\varphi$ for some LTL formula φ. LTL formulae φ, ψ are *equivalent* if $L_\varphi^\Sigma = L_\psi^\Sigma$ for every alphabet Σ.

Remark 2.1. *It can be easily shown that LTL formulae φ, ψ such that $AP(\varphi) = AP(\psi)$ are equivalent iff* $L_\varphi = L_\psi$.

In this paper, we are mainly interested in fragments of LTL obtained by restricting the nesting depth of the X operator to a certain level. Formally, for every LTL formula φ we inductively define its X-*depth* (denoted *depth*(φ)) by

$$
\begin{array}{lcl}
\mathit{depth}(p) & = & 0 \\
\mathit{depth}(\neg\varphi) & = & \mathit{depth}(\varphi) \\
\mathit{depth}(\varphi_1 \wedge \varphi_2) & = & \max\{\mathit{depth}(\varphi_1), \mathit{depth}(\varphi_2)\} \\
\mathit{depth}(\mathsf{X}\varphi) & = & \mathit{depth}(\varphi) + 1 \\
\mathit{depth}(\varphi_1 \mathsf{U} \varphi_2) & = & \max\{\mathit{depth}(\varphi_1), \mathit{depth}(\varphi_2)\}
\end{array}
$$

The set of all LTL formulae whose X-depth is less or equal to a given $n \in \mathbb{N}_0$ is denoted by LTL(X^n). A property L is an LTL(X^n) *property* iff $L = L_\varphi$ for some $\varphi \in$ LTL(X^n).

Let α be an ω-word and $i \in \mathbb{N}_0$. We say that $\alpha(i)$ is *redundant* iff $\alpha(i) = \alpha(i+1)$ and there is $j > i$ such that $\alpha(i) \neq \alpha(j)$. The *canonical form* of α is the ω-word obtained from α by deleting all redundant letters. Two ω-words α, β are *stutter equivalent* iff they have the same canonical form. A property L is *stutter invariant* iff it is closed under stutter equivalence. Stutter invariant LTL properties are classified by the following theorem:

Theorem 2.2. *Let* L *be an LTL property.* L *is stutter invariant iff* L *is an* $\mathrm{LTL}(\mathsf{X}^0)$ *property.*

The '$\Longleftarrow$' direction has been observed by Lamport [Lam83]. The other direction is due to Peled and Wilke [PW97].

Remark 2.3. *Theorem 2.2 cannot be extended to all ω-regular properties*[1]*. For example, the regular and stutter invariant property* $(a^+b^+a^+b^+)^*c^\omega$ *(where* $a, b, c \in \Sigma$*) is not an LTL property. This can be easily shown, e.g., with the help of results presented in [KS02]. See Section 4 for further comments.*

A related result (taken from [PWW98]) is

Theorem 2.4. *Let* φ *be an LTL formula. The problem whether* L_φ *is an* $\mathrm{LTL}(\mathsf{X}^0)$ *property is* PSPACE*-complete.*

3 The Results

In this section we generalize Theorem 2.2 and Theorem 2.4 to $\mathrm{LTL}(\mathsf{X}^n)$ (for arbitrary $n \in \mathbb{N}_0$). Our proofs are obtained by adapting the techniques used for $\mathrm{LTL}(\mathsf{X}^0)$.

The generalization is based on a simple observation that $\mathrm{LTL}(\mathsf{X}^n)$ formulae cannot distinguish between $n+1$ and more adjacent occurrences of the same letter in a given ω-word. Formally, let Σ be an alphabet, $n \in \mathbb{N}_0$, and $\alpha \in \Sigma^\omega$. A letter $\alpha(i)$ is n-*redundant* if $\alpha(i) = \alpha(i+1) = \cdots = \alpha(i+n+1)$ and there is some $j > i$ such that $\alpha(i) \neq \alpha(j)$. The n-*canonical form* of α, denoted $[n{:}\alpha]$, is obtained from α by deleting all n-redundant letters. Two ω-words α, β are n-*stutter equivalent* iff $[n{:}\alpha] = [n{:}\beta]$. A property L is n-*stutter invariant* iff it is closed under n-stutter equivalence.

[1] ω-regular properties are the properties definable by ω-regular expressions or (equivalently) by Büchi automata [Tho90].

Example 3.1. *Let* $a, b, c \in \Sigma$ *and* $\alpha = aaaa\,b\,ccccc\,aa\,b^\omega$. *Then* $[0{:}\alpha] = a\,b\,c\,a\,b^\omega$, $[1{:}\alpha] = aa\,b\,cc\,aa\,b^\omega$, *and* $[2{:}\alpha] = aaa\,b\,ccc\,aa\,b^\omega$.

Note that for $n = 0$, all of the notions just defined coincide with the ones of Section 2.

Theorem 3.2. *Let* Σ *be an alphabet,* $n \in \mathbb{N}_0$, *and* $\varphi \in \mathrm{LTL}(\mathsf{X}^n)$. *The property* L^Σ_φ *is* n*-stutter invariant.*

Proof. We prove (by induction on the structure of φ) that for every $\alpha \in \Sigma^\omega$ we have that $\alpha \models \varphi$ iff $[n{:}\alpha] \models \varphi$.

- $\varphi \equiv p$. Since $\alpha(0) = [n{:}\alpha](0)$, we are done.
- $\varphi \equiv \neg\psi$ or $\varphi \equiv \psi \wedge \rho$. Immediate.
- $\varphi \equiv \mathsf{X}\psi$. Then $n \geq 1$ and $\psi \in \mathrm{LTL}(\mathsf{X}^{n-1})$. First, observe that the $(n-1)$-canonical form of $[n{:}\alpha]_1$ is exactly $[n-1{:}\alpha_1]$. Now $\alpha \models \mathsf{X}\psi$ iff $\alpha_1 \models \psi$ iff $[n-1{:}\alpha_1] \models \psi$ (we just applied induction hypotheses) iff $[n{:}\alpha]_1 \models \psi$ (here we applied our induction hypotheses to the word $[n{:}\alpha]_1$ using the observation above) iff $[n{:}\alpha] \models \mathsf{X}\psi$.
- $\varphi \equiv \psi\,\mathsf{U}\,\rho$. We define a function $f : \mathbb{N}_0 \to \mathbb{N}_0$ as follows.

$$f(i) = \begin{cases} 0 & \text{if } i = 0 \\ f(i-1) & \text{if } i > 0 \text{ and } \alpha(i-1) \text{ is } n\text{-redundant} \\ f(i-1) + 1 & \text{otherwise} \end{cases}$$

  The function f is nondecreasing, surjective, and for every $i \in \mathbb{N}_0$ it holds that $[n{:}\alpha_i] = [n{:}\alpha]_{f(i)}$. We need to show that $\alpha \models \psi\,\mathsf{U}\,\rho$ iff $[n{:}\alpha] \models \psi\,\mathsf{U}\,\rho$.

  "$\Longrightarrow$": If $\alpha \models \psi\,\mathsf{U}\,\rho$ then there is $j \geq 0$ such that $\alpha_j \models \rho$ and for all $i < j$ it holds that $\alpha_i \models \psi$. By induction hypothesis we obtain that $[n{:}\alpha_j] \models \rho$ and $[n{:}\alpha_i] \models \psi$ for every $i < j$. Moreover, $[n{:}\alpha]_{f(j)} \models \rho$ and $[n{:}\alpha]_{i'} \models \psi$ for every $i' < f(j)$ (see the remarks about f above). This means that $[n{:}\alpha] \models \psi\,\mathsf{U}\,\rho$.

  "$\Longleftarrow$": Suppose that $[n{:}\alpha] \models \psi\,\mathsf{U}\,\rho$. Then there is $j \geq 0$ such that $[n{:}\alpha]_j \models \rho$ and for all $i < j$ it holds that $[n{:}\alpha]_i \models \psi$. Let $j' \in \mathbb{N}_0$ be the least number such that $f(j') = j$ (hence, for all $i' < j'$ we have that $f(i') < f(j')$). Then $[n{:}\alpha]_j = [n{:}\alpha_{j'}]$ and by induction hypothesis we get that $\alpha_{j'} \models \rho$. Similarly, for all $i' < j'$ we have that $f(i') < f(j') = j$ and thus $[n{:}\alpha]_{f(i')} \models \psi$. By induction hypothesis, $\alpha_{i'} \models \psi$. To sum up, $\alpha \models \psi \, \mathsf{U} \, \rho$.

$\square$

Theorem 3.2 says that all LTL(X^n) properties are n-stutter invariant. Hence, the theorem can be used to show that a given property is *not* expressible in LTL(X^n) (or even in LTL).

Example 3.3. *The standard example of an ω-regular property which is not definable in LTL is '$\mathsf{G}_2 p$' (see, e.g., [Tho90]). This property consists of all $\alpha \in \{\emptyset, \{p\}\}^\omega$ such that $\alpha(i) = \{p\}$ for every even $i \in \mathbb{N}_0$. With the help of Theorem 3.2 we can easily prove that $\mathsf{G}_2 p$ is not an* LTL(X^n) *property for any $n \in \mathbb{N}_0$ (hence, it is not an LTL property). Suppose the converse, i.e., there are $n \in \mathbb{N}_0$ and $\varphi \in$ LTL(X^n) such that $L_\varphi = \mathsf{G}_2 p$. Now consider the words $\alpha = \{p\}^{2n+2} \emptyset \{p\}^\omega$ and $\beta = \{p\}^{2n+1} \emptyset \{p\}^\omega$. Clearly $\alpha \notin L_\varphi$, $\beta \in L_\varphi$, and $[n{:}\alpha] = [n{:}\beta]$. Hence, L_φ is not n-stutter invariant which contradicts Theorem 3.2.*

Example 3.4. *In a similar way we can also show that the* LTL(X^n) *hierearchy is semantically strict, i.e., for every $n \in \mathbb{N}$ there is $\varphi_n \in$ LTL(X^n) which is not expressible in* LTL(X^{n-1}). *We define*

$$\varphi_n \equiv \overbrace{\mathsf{X} \cdots \mathsf{X}}^{n} p.$$

Let us suppose that L_{φ_n} is an LTL(X^{n-1}) *property. If we put $\alpha = \{p\}^{n+1} \emptyset^\omega$ and $\beta = \{p\}^n \emptyset^\omega$, we see that $\alpha \in L_{\varphi_n}$, $\beta \notin L_{\varphi_n}$, and $[n{-}1{:}\alpha] = [n{-}1{:}\beta]$. It contradicts Theorem 3.2.*

Now we show that every n-stutter invariant LTL property is definable in LTL(X^n). Our proof is similar to the one for 0-stuttering presented by Etessami in [Ete00]. Alternatively, one could also generalize the proof presented earlier in [PW97]. In fact, this would result in a somewhat simpler construction; however, it would not allow to derive the PSPACE-upper bound for the problem whether a given LTL property is an LTL(X^n) property (see Corollary 3.6).

Theorem 3.5. *Every n-stutter invariant LTL property is an* LTL(X^n) *property.*

Proof. Let φ be an LTL formula such that L_φ is n-stutter invariant. We translate φ into an equivalent formula $\tau_n(\varphi)$ whose X-depth is n.

A *literal* is a (possibly negated) proposition of $AP(\varphi)$. For every non-empty sequence $\ell_0 \cdots \ell_k$ of literals we define a formula $\sigma_{\ell_0 \cdots \ell_k}$ as follows:

$$\sigma_{\ell_0 \cdots \ell_k} \equiv \ell_0 \wedge \mathsf{X}(\ell_1 \wedge \mathsf{X}(\ell_2 \wedge \cdots \wedge \mathsf{X}(\ell_{k-1} \wedge \mathsf{X}\ell_k) \cdots))$$

Observe that the X-depth of $\sigma_{\ell_0 \cdots \ell_k}$ is k. A similar notation is used also for sequences of letters; for every $a \in \Sigma_\varphi$ we define

$$\gamma_a \equiv \bigwedge_{p \in a} p \ \wedge \bigwedge_{p \in AP(\varphi) \smallsetminus a} \neg p$$

and for every non-empty sequence $a_0 \cdots a_k$ of letters we put

$$\sigma_{a_0 \cdots a_k} \equiv a_0 \wedge \mathsf{X}(a_1 \wedge \mathsf{X}(a_2 \wedge \cdots \wedge \mathsf{X}(a_{k-1} \wedge \mathsf{X}a_k) \cdots))$$

The sequence consisting of $i \in \mathbb{N}$ copies of an atomic proposition p is denoted p^i, and the same notation is used also for sequences of letters.

The translation $\tau_n(\varphi)$ is defined by induction on the structure of φ.

- $\tau_n(p) = p$
- $\tau_n(\neg\psi) = \neg\tau_n(\psi)$
- $\tau_n(\psi \wedge \rho) = \tau_n(\psi) \wedge \tau_n(\rho)$
- $\tau_n(\psi \,\mathsf{U}\, \rho) = \tau_n(\psi) \,\mathsf{U}\, \tau_n(\rho)$
- $\tau_n(\mathsf{X}\psi) = \Phi(\psi) \vee \Gamma(\psi)$ where

  $$\Phi(\psi) \quad \equiv \quad \bigwedge_{p \in AP(\varphi)} (\mathsf{G}p \vee \mathsf{G}\neg p) \ \wedge \ \tau_n(\psi)$$

  and

  $$\Gamma(\psi) \quad \equiv \quad \bigvee_{p \in AP(\varphi)} (\delta(p) \wedge (\bigvee_{1 < i \leq n+1} \xi(\psi, p, i))).$$

  The subformulae $\delta(p)$ and $\xi(\psi, p, i)$ of $\Gamma(\psi)$ are constructed as follows:

  $$\delta(p) \quad \equiv \bigwedge_{q \in AP(\varphi) \smallsetminus \{p\}} (p \wedge (q \,\mathsf{U} \neg p \vee \neg q \,\mathsf{U} \neg p)) \ \vee \ (\neg p \wedge (q \,\mathsf{U}\, p \vee \neg q \,\mathsf{U}\, p))$$

and

$$\xi(\psi, p, i) \equiv \begin{cases} (\sigma_{p^i \neg p} \wedge p \,\mathsf{U}\, (\sigma_{p^{i-1} \neg p} \wedge \tau_n(\psi))) \vee & \text{if } i \leq n \\ \vee\, (\sigma_{\neg p^i p} \wedge \neg p \,\mathsf{U}\, (\sigma_{\neg p^{i-1} p} \wedge \tau_n(\psi))) & \\ & \\ (\sigma_{p^{n+1}} \wedge p \,\mathsf{U}\, (\sigma_{p^n \neg p} \wedge \tau_n(\psi))) \vee & \text{if } i = n+1 \\ \vee\, (\sigma_{\neg p^{n+1}} \wedge \neg p \,\mathsf{U}\, (\sigma_{\neg p^n p} \wedge \tau_n(\psi))) & \end{cases}$$

One can readily confirm that the X-depth of $\tau_n(\varphi)$ is n. We prove that if L_φ is n-stutter invariant, then φ is equivalent to $\tau_n(\varphi)$. Since φ and $\tau_n(\varphi)$ use the same set of atomic propositions, it suffices to show that $L_\varphi = L_{\tau_n(\varphi)}$ (see Remark 2.1). Moreover, as both L_φ and $L_{\tau_n(\varphi)}$ are n-stutter closed (in the case of $L_{\tau_n(\varphi)}$ we apply Theorem 3.2), it actually suffices to prove that φ and $\tau_n(\varphi)$ cannot be distinguished by any n-*stutter free* ω-word $\alpha \in \Sigma_\varphi^\omega$ (an ω-word α is n-stutter free if $\alpha = [n{:}\alpha]$).

That is, for every n-stutter free $\alpha \in \Sigma_\varphi^\omega$ we need to show that $\alpha \models \varphi$ iff $\alpha \models \tau_n(\varphi)$. We proceed by induction on the structure of φ. All subcases except for $\varphi = \mathsf{X}\psi$ are trivial. Here we distinguish two possibilities:

- $\alpha = a^\omega$ for some $a \in \Sigma_\varphi$. Then $\alpha_1 = \alpha$ and thus we get $\alpha \models \mathsf{X}\psi$ iff $\alpha_1 \models \psi$ iff $\alpha_1 \models \tau_n(\psi)$ (here we used induction hypotheses) iff $\alpha \models \tau_n(\psi)$. Hence, this subcase is 'covered' by the formula $\Phi(\psi)$ which says that α is of the form a^ω and that $\tau_n(\psi)$ holds.

- $\alpha = a^i b\beta$ where $a, b \in \Sigma_\varphi$, $a \neq b$, $1 \leq i \leq n+1$, and $\beta \in \Sigma_\varphi^\omega$.

  First, let us assume that $i \leq n$. Then $a^i b\beta \models \mathsf{X}\psi$ iff $a^{i-1}b\beta \models \psi$ iff $a^{i-1}b\beta \models \tau_n(\psi)$ (we used induction hypotheses) iff $a^i b\beta \models \sigma_{a^i b} \wedge a \,\mathsf{U}\, (\sigma_{a^{i-1}b} \wedge \tau_n(\psi))$. The structure of the last formula is already similar to the structure of $\xi(\psi, p, i)$. The next step is to realize that since $a \neq b$, there must be some $p \in (a \smallsetminus b) \cup (b \smallsetminus a)$; a characteristic feature of p is that no other $q \in AP(\varphi)$ changes its (in)validity in the word $a^i b\beta$ 'earlier' than p. So, $p \in (a \smallsetminus b) \cup (b \smallsetminus a)$ iff $a^i b\beta \models \delta(p)$. Moreover, if $a^i b\beta \models \delta(p)$, then we also have that $a^i b\beta \models \sigma_{a^i b} \wedge a \,\mathsf{U}\, (\sigma_{a^{i-1}b} \wedge \tau_n(\psi))$ iff $a^i b\beta$ satisfies either the formula

  $$\sigma_{p^i \neg p} \wedge p \,\mathsf{U}\, (\sigma_{p^{i-1} \neg p} \wedge \tau_n(\psi)),$$

  or the formula

  $$\sigma_{\neg p^i p} \wedge \neg p \,\mathsf{U}\, (\sigma_{\neg p^{i-1} p} \wedge \tau_n(\psi)).$$

which is equivalent to $a^i b\beta \models \xi(\psi, p, i)$. Observe that the first formula holds when $p \in a \setminus b$, and the second formula holds when $p \in b \setminus a$.

The case when $i = n{+}1$ is handled similarly; we have that $a^{n+1}b\beta \models \mathsf{X}\psi$ iff $a^n b\beta \models \psi$ iff $a^n b\beta \models \tau_n(\psi)$ (we used induction hypotheses) iff $a^{n+1}b\beta \models \sigma_{a^{n+1}} \wedge a\,\mathsf{U}\,(\sigma_{a^n b} \wedge \tau_n(\psi))$. Using the same argument as above, we argue that if $a^{n+1}b\beta \models \delta(p)$, then $a^{n+1}b\beta \models \sigma_{a^{n+1}} \wedge a\,\mathsf{U}\,(\sigma_{a^n b} \wedge \tau_n(\psi))$ iff $a^{n+1}b\beta \models \xi(\psi, p, i)$.

To sum up, the case when $\alpha = a^i b\beta$ is 'covered' by the formula $\Gamma(\psi)$.

□

In general, the size of $\tau_n(\varphi)$ is exponential in *depth*(φ). However, the size of the *circuit*[2] representing $\tau_n(\varphi)$ is only $\mathcal{O}(n \cdot |\varphi|^2)$. To see this, realize the following:

(1) The total size of all circuits representing the formulae $\delta(p)$, $\sigma_{p^i \neg p}$, $\sigma_{\neg p^i p}$, $\sigma_{p^{n+1}}$, $\sigma_{\neg p^{n+1}}$ (for all $p \in AP(\varphi)$ and $0 \le i \le n$), is $\mathcal{O}(n^2 \cdot |\varphi|^2)$.

(2) Assuming that the circuits of (1) and the circuit representing $\tau_n(\psi)$ are at our disposal, we need to add only a constant number of new nodes to represent the formula $\xi(\psi, p, i)$ for given $p \in AP(\varphi)$ and $1 \le i \le n{+}1$. It means that we need to add $\mathcal{O}(n \cdot |\varphi|)$ new nodes when constructing the circuit for $\tau_n(\mathsf{X}\psi)$.

(3) Since φ contains $\mathcal{O}(|\varphi|)$ subformulae of the form $\mathsf{X}\psi$, the circuit representing φ has $\mathcal{O}(n^2 \cdot |\varphi|^2)$ nodes in total.

Corollary 3.6. *Let φ be an LTL formula and $n \in \mathbb{N}_0$. The problem if L_φ is an* LTL(X^n) *property is* PSPACE*-complete (assuming unary encoding of n).*

Proof. The PSPACE-lower bound holds even in the special case when $n = 0$ [PWW98]. The matching PSPACE-upper bound is obtained by applying the same argument as in [Ete00]—due to Theorem 3.2 and Theorem 3.5 we have that L_φ is an LTL(X^n) property iff φ is equivalent to $\tau_n(\varphi)$. First, we construct the circuit representing $\tau_n(\varphi)$ (its size is $\mathcal{O}(n^2 \cdot |\varphi|^2)$ as shown above). Then we check the equivalence between the circuit and φ, which can be also done in polynomial space [SC85]. □

[2] The circuit representing a given LTL formula φ is obtained from the syntax tree of φ by identifying all nodes which correspond to the same subformula.

4 Concluding remarks

Theorem 3.5 is closely related to a result presented in [KS02]. Roughly speaking, in this paper it is shown that each property expressible by an LTL formula φ where the X-depth is bounded by n and the U-depth is bounded by m is closed under deleting/pumping of every subword which is 'sufficiently periodic' (the condition depends on n, m, and the length of the subword). For example, if we take the property $(a^+b^+a^+b^+)^*c^\omega$ where $a, b, c \in \Sigma$, and arbitrary $n, m \in \mathbb{N}_0$, then there is (sufficiently large) $k \in \mathbb{N}_0$ such that the leading ab subword becomes 'sufficiently periodic' in the the word $(abab)^k c^\omega$. Hence, the considered (ω-regular and 0-stutter invariant) property is not expressible in LTL, because it does not contain the word $ab(abab)^{k-1}c^\omega$.

Our proof of Theorem 3.2 is based on the proof of the above discussed result presented in [KS02]. Since it is quite simple, we believe it might be of some use in introductory courses on LTL. It is not much longer than the proof for 0-stuttering (which is often included) and it brings interesting consequences 'for free'. Theorem 3.5 and Corollary 3.6 do not follow from the work presented in [KS02] (in fact, if we reformulate Theorem 3.5 for the aforementioned generalized form of stutter invariance, it does *not* hold [KS02]).

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