Uniform Turán densities of 3-uniform hypergraphs

Filip Kučerák

F. Garbe, D. Iľkovič, D. Kráľ, A. Lamaison

Faculty of Informatics, Masaryk University

July 22, 2024

Turán Problems

Question

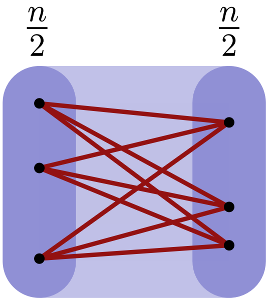
How many edges can a graph have if we forbid a fixed subgraph?

Question

How many edges can a graph have if it is triangle-free?

Theorem (Mantel 1907)

Suppose that for given $n \in \mathbb{N}$, graph G of order n is triangle-free. Then, the number of edges in G is at most $\lfloor n^2/4 \rfloor$.



Theorem (Mantel 1907)

Suppose that for given $n \in \mathbb{N}$, graph G of order n is triangle-free. Then, the number of edges in G is at most $\lfloor n^2/4 \rfloor$.

Theorem (Turán 1941)

Suppose that for given $n, k \in \mathbb{N}$, graph G of order n is K_k-free. Then, the number of edges in G is at most

$$\left(\frac{k-2}{k-1} + o(1)\right)\binom{n}{2}.$$

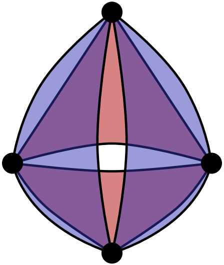
Theorem (Erdős-Stone-Simonovits 1946)

Suppose that graph H has chromatic number $\chi > 2$. Then, the number of edges in any graph G which is H-free is at most

$$\left(\frac{\chi-2}{\chi-1} + o(1)\right)\binom{n}{2}.$$

Question

How dense can a 3-uniform hypergraph be if it does not contain a tetrahedron ($K_4^{(3)}$)?



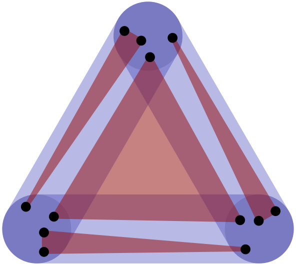
$K_4^{(3)}$

Question (Turán)

How dense can a 3-uniform hypergraph be if it does not contain a tetrahedron ($K_4^{(3)}$)?

Known

$\frac{5}{9} \leq \pi(K_4^{(3)}) \leq 0.5615$ *(Razborov; Baber, Talbot)*



Turán's construction with density $\frac{5}{9}$

Question (Turán)

How dense can a 3-uniform hypergraph be if it does not contain a tetrahedron ($K_4^{(3)}$)?

Question (Erdős and Sós)

How dense can a 3-uniform hypergraph be if it does not contain a tetrahedron ($K_4^{(3)}$) and we require the edges to be uniformly distributed?

Definition

For $d \in [0,1]$ and $\eta > 0$ we say that hypergraph F is (d,η)-dense if for all $U \subseteq V$ the following inequality holds:

$$\left|\binom{U}{3} \cap E\right| \geq d\binom{|U|}{3} - \eta|V|^3.$$

Definition

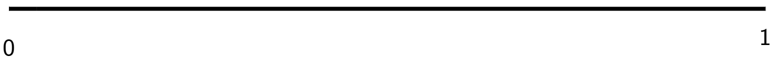
For $d \in [0,1]$ and $\eta > 0$ we say that hypergraph F is (d,η)-dense if for all $U \subseteq V$ the following inequality holds:

$$\left| \binom{U}{3} \cap E \right| \geq d \binom{|U|}{3} - \eta |V|^3.$$

Definition

Let F be a hypergraph; its uniform Turán density is the following:

$$\pi_u(F) = \sup \Big\{ d \in [0,1] : \text{for every } \eta > 0 \text{ and } n \in \mathbb{N}, \text{ there exists an } F\text{-free } (d,\eta)\text{-dense hypergraph } H \text{ of order at least } n \Big\}.$$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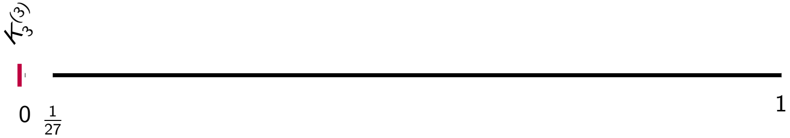
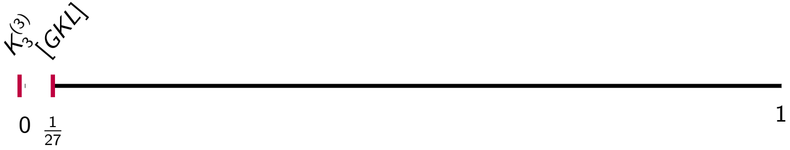


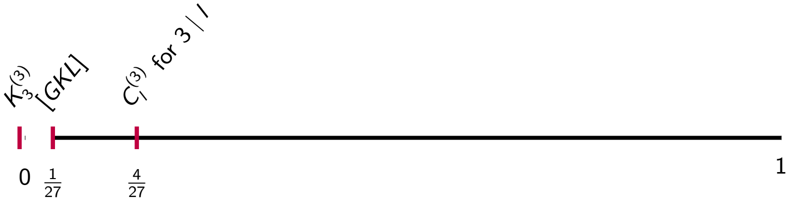
Proposition (Reiher, Rödl, Schacht 2018)

The uniform Turán density of any graph F is zero or at least $1/27$.



Theorem (Garbe, Kráľ, Lamaison 2022)

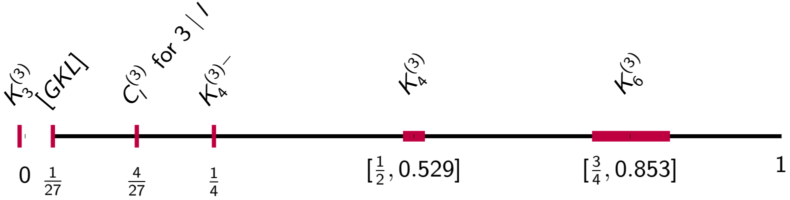
There exists a hypergraph F whose uniform Turán density is $1/27$.



Theorem (Bucić, Cooper, Kráľ, Mohr, Munhá-Correia 2023)

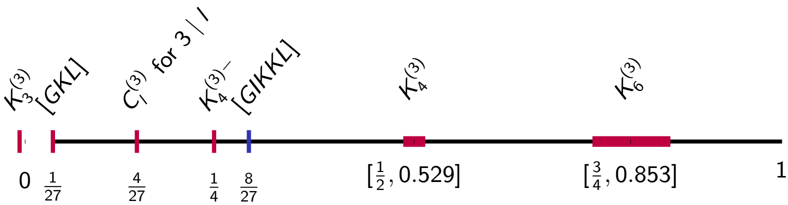
For integer $l \geq 5$, the uniform Turán density of a tight cycle C_l is

1. *zero if l is divisible by three and*
2. $4/27$ *otherwise.*



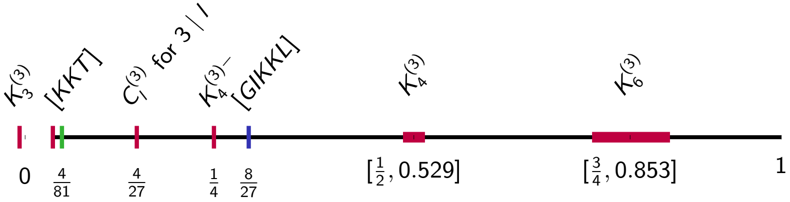
Theorem (Glebov, Kráľ, Volec 2016; Reiher, Rödl, Schacht 2017)

The uniform Turán density of $K_4^{(3)-}$ *(tetrahedron without one edge) is* $1/4$.



Theorem (Garbe, Iľkovič, Kráľ, K., Lamaison 2023+)

There exists a hypergraph F whose uniform Turán density is $8/27$.



Theorem (Garbe, Iľkovič, Kráľ, K., Lamaison 2023+)

There exists a hypergraph F whose uniform Turán density is $8/27$.

Theorem (Kráľ, K., Tardos 2024++)

There exists a hypergraph F whose uniform Turán density is $4/81$.

Definition

Let Φ be a finite set of colors. We call a subset $\mathcal{P} \subseteq \Phi^3$ a coloring palette.

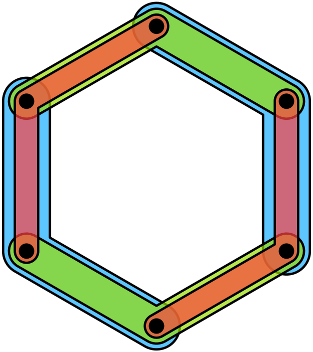
Example

Let $\Phi = \{\ \mathit{red}, \mathit{green}, \mathit{blue}\ \}$ and $\mathcal{P} = \{\ (\mathit{red}, \mathit{green}, \mathit{blue})\ \}$

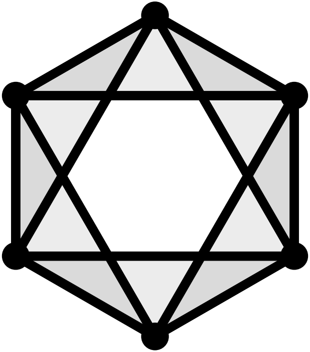
Definition

We say that hypergraph F is $\mathcal{P}$-colorable if there exists an ordering $\prec$ of the vertex set of F and assignment $\varphi : \partial F \to \Phi$ with the property that for all $uvw \in E(F)$ where $u \prec v \prec w$ we have

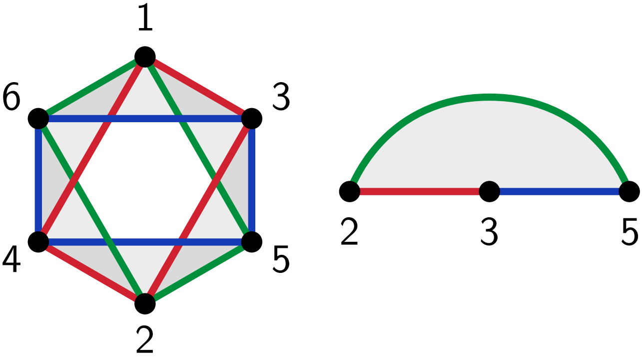
$$(\varphi(uv), \varphi(uw), \varphi(vw)) \in \mathcal{P}.$$



$C_6^{(3)}$



$\Phi = \{\ \textit{red}, \textit{green}, \textit{blue}\ \}$ and $\mathcal{P} = \{\ (\textit{red}, \textit{green}, \textit{blue})\ \}$



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Theorem (Reiher, Rödl, Schacht 2018)

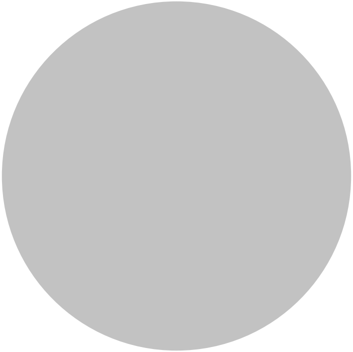
Let the set of colors be $\Phi = \{\ \textit{red}, \textit{green}, \textit{blue}\ \}$ *and let the* Φ*-palette* $\mathcal{P}$ *contain only the pattern* $(\textit{red}, \textit{green}, \textit{blue})$*. Let* F *be a 3-uniform hypergraph, then* $\pi_u(F) = 0$ *if and only if it is* $\mathcal{P}$*-colorable.*

Corollary

Let F *be a hypergraph, then either* $\pi_u(F) = 0$ *or* $\pi_u(F) \geq 1/27$*.*

Proposition

Let the set of colors be $\Phi = \{\ \textit{red}, \textit{green}, \textit{blue}\ \}$ and let the Φ-palette $\mathcal{P}$ contain only the pattern $(\textit{red}, \textit{green}, \textit{blue})$. Let F be a graph such that it is not colorable by $\mathcal{P}$, then its density is at least $1/27$.



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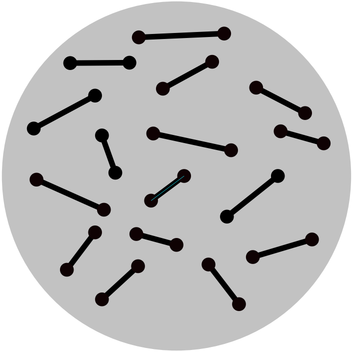

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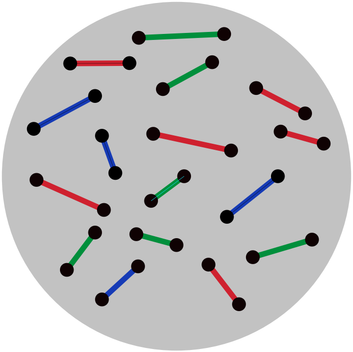
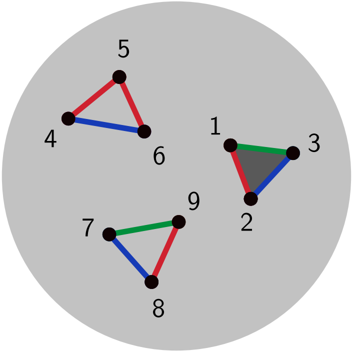


Proposition

Let the set of colors be $\Phi = \{\ red, green, blue\ \}$ *and let the* Φ*-palette* $\mathcal{P}$ *contain only the pattern* $(red, green, blue)$*. Let* F *be a graph such that it is not colorable by* $\mathcal{P}$*, then its density is at least* $1/27$*.*



Theorem (Garbe, Iľkovič, Kráľ, K., Lamaison 2023+)

There exists a hypergraph F with uniform Turán density $\pi_u(F) = 8/27$.

Definition

1. Let $\Psi = \{ \zeta, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_1, \boldsymbol{\beta}_3, \gamma_2, \gamma_3 \}$, and let $\mathcal{R}$ be a Ψ-palette containing patterns $\{ (\boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \zeta), (\boldsymbol{\alpha}_2, \zeta, \gamma_2), (\zeta, \boldsymbol{\beta}_3, \gamma_3) \}$. We will call this palette the *required palette*.
2. Let $\Phi = \{ \boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma \}$ and let $\mathcal{F}$ be a Φ-palette which we define as $\mathcal{F} = \{ \boldsymbol{\alpha}, \boldsymbol{\beta} \} \times \{ \boldsymbol{\beta}, \gamma \} \times \{ \boldsymbol{\alpha}, \gamma \}$. We will call palette $\mathcal{F}$ the *forbidden palette*.

Proposition

Any hypergraph F which is $\mathcal{R}$-colorable but is not $\mathcal{F}$-colorable has uniform Turán density $8/27$.

Proposition

There exists a hypergraph F such that it is $\mathcal{R}$-colorable but is not $\mathcal{F}$-colorable.

Proving upper bound

$(\frac{8}{27}+\delta,\eta)$-dense hypergraph of order N

$\downarrow$ hypergraph regularity lemma

hypergraph regularity partition

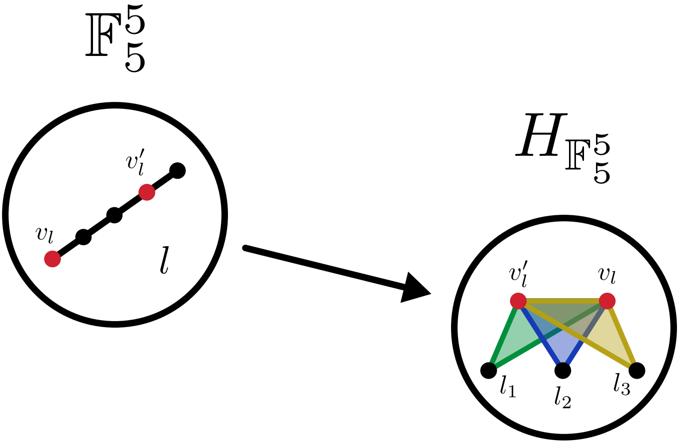
$\downarrow$ cleaning

$(\frac{8}{27}+\delta')$-dense *reduced hypergraph*

$\downarrow$ applications of Ramsey theorems

reduced subhypergraph which "essentially behaves like $\mathcal{R}$"

Construction



Open Problems

Question

Determine for which $d \in [0,1]$ there exists an F such that $\pi_u(F) = d$.

Question

Determine uniform Turán density of graph

$$F = \big([5], \{123, 124, 125, 145, 234\}\big).$$