Statistical Testing of Randomness

(Yesterday, Today, and Possibly Tomorrow)

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Randomness and Unpredictability

- A random number is
  - an instance of an unbiased random variable
  - the output produced by a uniformly distributed random process (follows a probability distribution)
- Random sequence is
  - a sequence of random variables
  - properties as sample drawn from uniform (rectangular) distribution
- Uniform (rectangular) continuous distribution
  - P(x) = 1/(b–a) for a<=x<=b; 0 otherwise

Basic Idea Behind the Statistical Testing

- Statistical testing of random sequences
- Particular tests are based on test statistic
  - Expected value of some test statistic is known for the reference distribution
  - Generated random stream is subjected to the same test
  - Obtained value is compared against the expected value
- Boundless number of statistical test can be constructed
  - Some of them are accepted as the de facto standard
    - NIST battery consists of 15 such tests (e.g. frequency test)
  - Generators that pass such tests are considered “good”
    - Absolute majority of generated sequences must pass

Statistical Hypothesis Testing (Basics)

- Null hypothesis (H_0) – denotes test hypothesis
  - H_0 = the sequence being tested is random
- Alternative hypothesis (H_A) – negates H_0
  - H_A = the sequence is not random

- Each test is based on some test statistic (TS)
  - TS is quantity calculated from our sample data
  - TS is random variable/vector obtained from transformation of random selection
  - TS has mostly standard normal or chi-square (χ^2) as reference distributions

- After each applied test must be derived conclusion that rejects or not rejects null hypothesis

Statistical Hypothesis Testing (Errors)

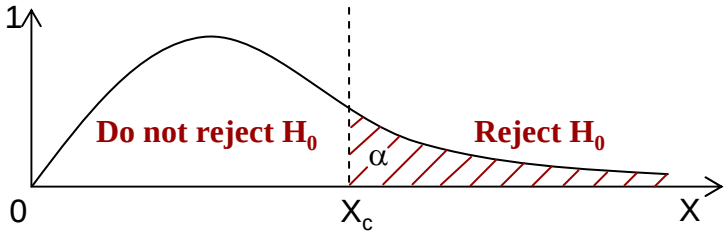
- Conclusion generation procedure and errors

Real situation	Conclusion	
	H_0 is not rejected	**H_0 is rejected**
H_0 is true	good decision	type I error
H_0 is not true	type II error	good decision

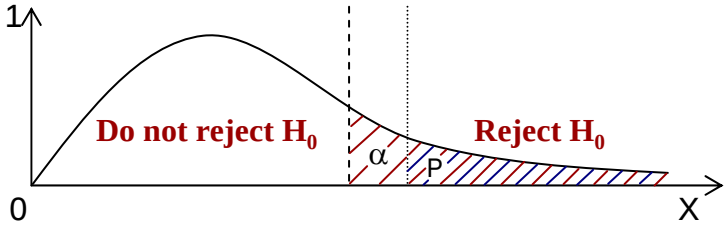
- Probability of type I error (α) = level of significance
  - Set prior the test; typically between 0.0001 and 0.01
  - Nonrandom sequence, produced by “good” generator
- Probability of type II error (β)
  - Random sequence, produced by “bad” generator
- α and β are related to each other and to sample size

Statistical Hypothesis Testing (Evaluation)

- Critical value – threshold between rejection and non-rejection regions
- Two (quite similar) ways of testing
  - α => critical value; test statistic => value; compare



  - Test statistic => P-value; α; compare



- NIST battery uses P-values
  - $P \leq \alpha$ => reject H_0
  - $P > \alpha$ => do not reject H_0

Frequency (Monobit) Test

- Basic idea
  - Number of zeros and ones expected in the truly random sequence should be the same
- Description
  - Length of the bit string: n
  - Sequence of bits: $\varepsilon = \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$
  - Test statistic: $s_{obs} = |S_n|/\sqrt{n}$
    - $S_n = X_1 + X_2 + \ldots + X_n$, where $X_i = 2\varepsilon_i - 1$ (conversion to ± 1)
    - The absolute value => half normal distribution
- Example (for n = 10)
  - $\varepsilon = 1011010101$; $S_n = 1-1+1+1-1+1-1+1-1+1 = 2$
  - $s_{obs} = |2|/\sqrt{10} = 0.632455532$; P-value = 0.5271
  - For $\alpha = 0.01$: $0.5271 > 0.01$ => ε “is random”

NIST Testing Strategy

1. Select (pseudo) random number generator
2. Generate sequences
   a) Generate set of sequences or one long sequence
      i. Divide the long sequence to set of subsequences
3. Execute statistical tests
   a) Select the statistical tests
   b) Select the relevant input parameters (how?)
4. Examine (and analyse) the P-values
   a) For fixed α a certain percentage are expected to failure
5. Assign Pass/Fail

Interpretation of Empirical Results

- Examination of the proportion of sequences that pass the statistical test
  - Determined by confidence interval
    - $p' \pm 3 \cdot \sqrt{(p' \cdot (1-p')/n)}$, where $p' = 1 - \alpha$; n is sample size
  - If proportion falls outside => data are non-random
- Check for uniformity of the distribution of obtained P-values
  - Level of significance $\alpha = 0.0001$ (hardwired)
  - Test statistic: $\chi^2 = \Sigma(o_i - e_i)^2/e_i$
    - o_i is observed number of P-values in i^{th} subinterval
    - e_i is expected number of P-values in i^{th} subinterval
      - Sample size multiplied by probability of occurrence in each subinterval (i.e. for sample size n it is n/10)
  - Total P_T-value is calculated and compared to α
    - P_T-value > 0.0001 => sequence is uniformly distributed

Machine Learning and Random Numbers

- Some problems of classical approach
  - Length of subsequences
  - Parameters setting
  - Independency of tests
- Can machine learning (ML) help?
  - Improved/unified statistical testing
    - Learning of existing ML algorithms should fail
    - Design of new ML algorithm for finding of regularities
  - Visualization techniques
  - Predictions and regression methods
    - Genetic programming
    - Support vector machines