

Twin-width and Transductions of Proper k -Mixed-Thin Graphs

Jakub Balabán, Petr Hliněný, Jan Jedelský

Masaryk University

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Overview

Twin-width Graph parameter describing similarity to cographs

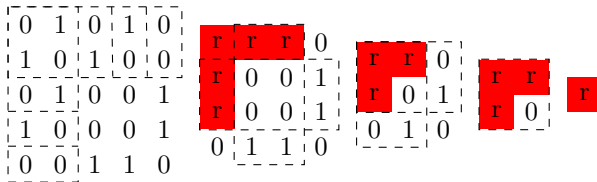
Proper k -Thin Generalization of proper interval graphs

Proper k -Mixed-Thin Our generalization of proper k -mixed graphs

- Twin-width linear in k
- A subclass (inversion-free) transduction equivalent to posets of bounded width

Twin-width

- *Twin-width 1: tractable FO model checking* by Bonnet et al.
- Twin-width using symmetric contraction sequences of adjacency matrices

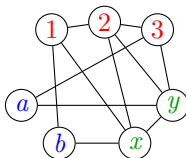


(Proper) k -Thin

- *The stable set problem and the thinness of a graph* by Mannino et al.
- *On the thinness and proper thinness of a graph* by Flavia and Estrada
- Graph $G = (V, E)$, k -partition $(V_1, V_2, \dots, V_k)$ of V , and linear order $\leq$ on V
- For all $u \not\leq v \not\leq w$
 - if $\exists i (u, v \in V_i)$ and $uw \in E$ then $vw \in E$
 - proper if $\exists i (v, w \in V_i)$ and $uw \in E$ then $uv \in E$

Proper k -Mixed-Thin

- Graph $G = (V, E)$, k -partition $(V_1, V_2, \dots, V_k)$ of V ,
a linear order $\leq_{ij}$ on $V_i \cup V_j$, and a choice of $E_{ij} \in \{E, \bar{E}\}$
- Restriction of $\leq_{ij}$ to V_i (resp. V_j) is *aligned* with $\leq_{ii}$ ($\leq_{jj}$)
- For all $1 \leq i, j \leq k$ and all $u \not\leq_{ij} v \not\leq_{ij} w$
if $(u, v \in V_i, w \in V_j$ or $u, v \in V_j, w \in V_i)$ and $uw \in E_{ij}$
then $vw \in E_{ij}$
- proper** if $(u \in V_j, v, w \in V_i$ or $u \in V_i, v, w \in V_j)$ and $uw \in E_{ij}$
then $uv \in E_{ij}$
- Inversion-free if the restriction of $\leq_{ij}$ to V_i (resp. V_j) is equal
to $\leq_{ii}$ (resp. $\leq_{jj}$)


 $\leq_{11}$: 1, 2, 3

 $\leq_{22}$: a, b $(E_{2,2} = \bar{E})$
 $\leq_{33}$: x, y

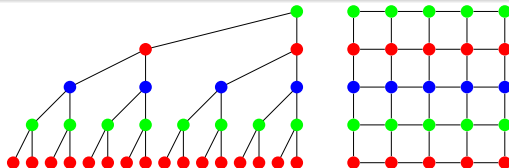
 $\leq_{12}$: a, 1, 2, b, 3 $(E_{1,2} = \bar{E})$
 $\leq_{23}$: x, b, a, y

 $\leq_{13}$: y, 3, 2, 1, x


Proper k -Mixed-Thin Graphs

Proposition 3 and Theorem 4

- Let $d \geq 1$ be an arbitrary integer. Both d -dimensional grids and d -dimensional full grids are inversion-free proper 3^{d-1} -mixed-thin.
- Every tree T is inversion-free proper 3-mixed-thin.



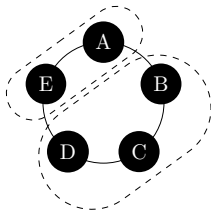
Proposition 2 (Mannino et al., Bonomo and de Estrada)

- The $(r \times r)$ -grid has thinness linear in r .
- The thinness of the complete m -ary tree ($m > 1$) is linear in its height.

Linear Twin-width – Upper Bound

Theorem 6

Let G be a proper k -mixed-thin graph. Then the twin-width of G is at most $9k$. The corresponding contraction sequence for G can be computed in polynomial time from the vertex partition $(V_1, \dots, V_k)$ and the orders $\leq_{ij}$ for G



	B	C	D	A	E
B	0	1	0	1	0
C	1	0	1	0	0
D	0	1	0	0	1
A	1	0	0	0	1
E	0	0	1	1	0

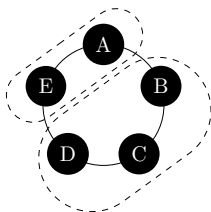
BC	r	r	r	0
D	r	0	0	1
A	r	0	0	1
E	0	1	1	0

BCD	r	r
AE	r	r

BC	r	r	r
D	r	0	r
AE	r	r	r

Linear Twin-width – Upper Bound

- Uses the *red-potential* method developed in *Twin-Width is Linear in the Poset Width* by Balabán and Hliněný
- Matrix ordering: Parts arbitrarily, within part V_i use $\leq_{ii}$
- Submatrices given by parts $V_i \times V_j$ can be split by *diagonal boundaries* into uniform parts (exception – the main diagonal)
- Red entries only next to the boundaries $\rightarrow \mathcal{O}(kn)$ red entries possible $\rightarrow$ there is a contraction with $\mathcal{O}(k)$ red entries



	B	C	D	A	E
B	0	1	0	1	0
C	1	0	1	0	0
D	0	1	0	0	1
A	1	0	0	0	1
E	0	0	1	1	0

BC	$\begin{bmatrix} r & r \\ r & 0 \end{bmatrix}$	$\begin{bmatrix} r & 0 \\ 0 & 1 \end{bmatrix}$
D	$\begin{bmatrix} r & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} r & 0 \\ 0 & 1 \end{bmatrix}$
A	$\begin{bmatrix} r & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} r & 0 \\ 0 & 1 \end{bmatrix}$
E	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

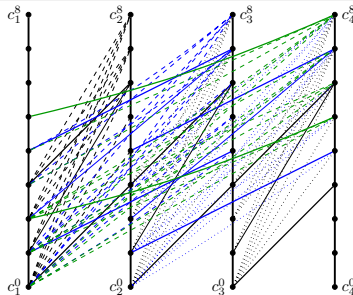
BCD	$\begin{bmatrix} r & r \\ r & r \end{bmatrix}$
AE	$\begin{bmatrix} r & r \\ r & r \end{bmatrix}$

BC	$\begin{bmatrix} r & r \\ r & 0 \end{bmatrix}$	$\begin{bmatrix} r \\ r \end{bmatrix}$
D	$\begin{bmatrix} r & 0 \\ r & 0 \end{bmatrix}$	$\begin{bmatrix} r \\ r \end{bmatrix}$
AE	$\begin{bmatrix} r & r \\ r & r \end{bmatrix}$	$\begin{bmatrix} r \\ r \end{bmatrix}$

Twin-width – Lower Bound – Proof

Proposition 10

For every integer $k \geq 1$, there exists an inversion-free proper $(2k + 1)$ -mixed-thin graph G such that the twin-width of G is at least k .

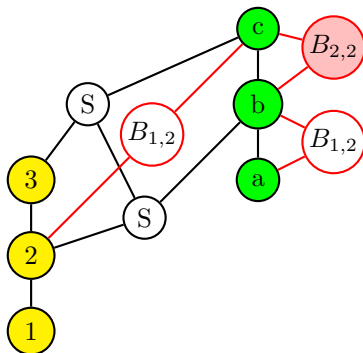


Transductions – Definition

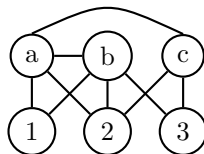
- Start with a relational structure $\sigma = (V, R_1, \dots, R_n)$ on domain V with relations R_i
- Add a fixed number of colors (unary relations) arbitrarily
- Fix FO-formulas $\phi_1(x_1, \dots, x_{ar_1}), \dots, \phi_m(x_m, \dots, x_{ar_m})$ and $\psi(x)$ using the colors and relational symbols R_i
- The result is relational structure $\sigma' = (V', R'_1, \dots, R'_m)$ where $v \in V' \subseteq V$ iff $\sigma \models \psi(x)$ and for all $i = 1, \dots, m$ $(x_i, \dots, x_{ar_i}) \in R'_i \subseteq (V')^{ar_i}$ iff $\sigma \models \phi_i(x_i, \dots, x_{ar_i})$

Transductions: Encoding Inversion-Free Proper Mixed-Thin Graphs in Posets of Bounded Width

- Any inversion-free proper k -mixed-thin graph encoded using poset of width at most $5 \cdot \binom{k}{2} + 5k$
- The poset can be computed in polynomial time given the partition and the orderings

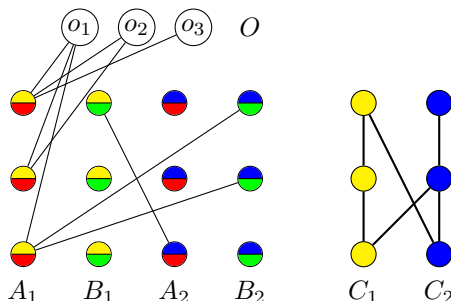


a, 1, 2, b, 3, c



Transductions: Encoding Posets of Bounded Width in Inversion-Free Proper Mixed-Thin Graphs

- Poset of width k encoded using inversion-free proper $2k + 1$ -mixed-thin graph
- The graph can be computed in polynomial time



Conclusions

We have ...

- ... defined the class of proper k -mixed-thin graphs

... and showed that ...

- ... it contains certain natural graph classes (trees, grids) as subclasses

- ... its twin-width is linear in k

- ... its inversion-free subclass is transduction equivalent to posets of bounded-width

Going forward, we ask ...

- ... if the inversion-freeness is necessary in the transduction equivalence

- ... how to recognize proper k -mixed-thin graphs

- ... what are the most general necessary and sufficient conditions to use the red-potential method (elsewhere)