Transductions of Graph Classes Admitting Product Structure

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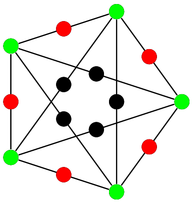
Outline

- First-order transductions
  - Interpretations, locality, and flips
- Global structure of planar graphs
  - Planar Product Structure Theorem
- Generalizing product structure to dense setting
  - $\mathcal{H}$-clique-width
- The main result
- Corollaries

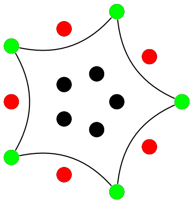
First-order interpretations

Definition

Given a (colored) graph G and a symmetric binary formula $\phi(x, y)$, we denote by $\phi(G)$ the *ϕ-interpretation* of G, that is, a graph obtained from G by redefining edge relation such that $uv \in E(\phi(G)) \iff G \models \phi(u, v)$.



$\phi(x, y) \equiv x$ and y are both green, and they share a red neighbor



First-order transductions

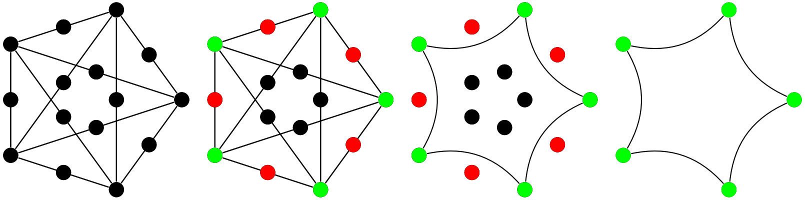
Definition

A (non-copying) *transduction* τ is a composition of the following operations:

- color vertices arbitrarily
- apply fixed interpretation
- take an induced subgraph

Definition

Given a graph G, we denote by $\tau(G)$ the **class** of graphs that can be obtained from G using τ. A class $\mathcal{C}$ is *FO-transducible* from a class $\mathcal{D}$ if there is a transduction τ such that $\mathcal{C} \subseteq \bigcup\{\tau(G) | G \in \mathcal{D}\}$.



Locality

- FO is local – recall the theorems of Gaifman and Hanf

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$\Longrightarrow$ every FO transduction can be split into local and global part

Definition (Local part)

A transduction τ is *strongly r-local* if it does not create edges between vertices x, y at distance greater than r, and the existence of an edge between x and y depends only on the union of r-neighborhoods of x and y.

Definition (Glocal part)

A graph H is a *k-flip* of a graph G if H can be obtained by coloring G using k colors and possibly flipping adjacency (edges became non-edges and vice versa) between some color classes.

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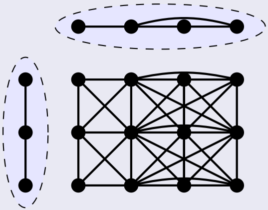
Theorem (Nešetřil, Ossona de Mendez, Siebertz)

If a graph class $\mathcal{C}$ is FO-transducible from a class $\mathcal{D}$ (without copying), then there are number k and r, and a strongly r-local transduction τ such that $\mathcal{C}$ is contained in a k-flip of $\tau(\mathcal{D})$.

Planar Product Structure Theorem

Definition (Strong product $G \boxtimes H$ of graphs G and H)



- $V(G \boxtimes H) = V(G) \times V(H)$
- $[g_1, h_1][g_2, h_2] \in E(G \boxtimes H)$ if one of the following conditions holds:
  - $g_1 = g_2$ and $h_1 h_2 \in E(H)$,
  - $g_1 g_2 \in E(G)$ and $h_1 = h_2$, or
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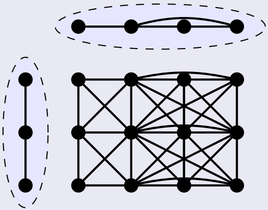

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Definition (Product structure)

A graph class $\mathcal{C}$ *admits product structure* if there is a constant k such that every graph $G \in \mathcal{C}$ is a subgraph of the strong product $P \boxtimes M$ of a path P and a graph M of tree-width at most k.

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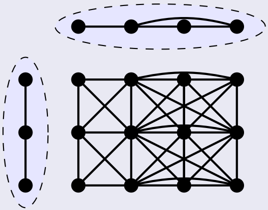

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Theorem (Dujmovic, Joret, Micek, Morin, Ueckerdt, Wood)

Planar graphs as well as graphs embeddable on a fixed surface admit product structure.

Dense analogue of product structure

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Definition (Product structure for dense graphs)

A graph class $\mathcal{C}$ *admits* **hereditary** *product structure* if there is a constant k such that every graph $G \in \mathcal{C}$ is an **induced** subgraph of the strong product $P \boxtimes M$ of a path P and a graph M of clique-width at most k.

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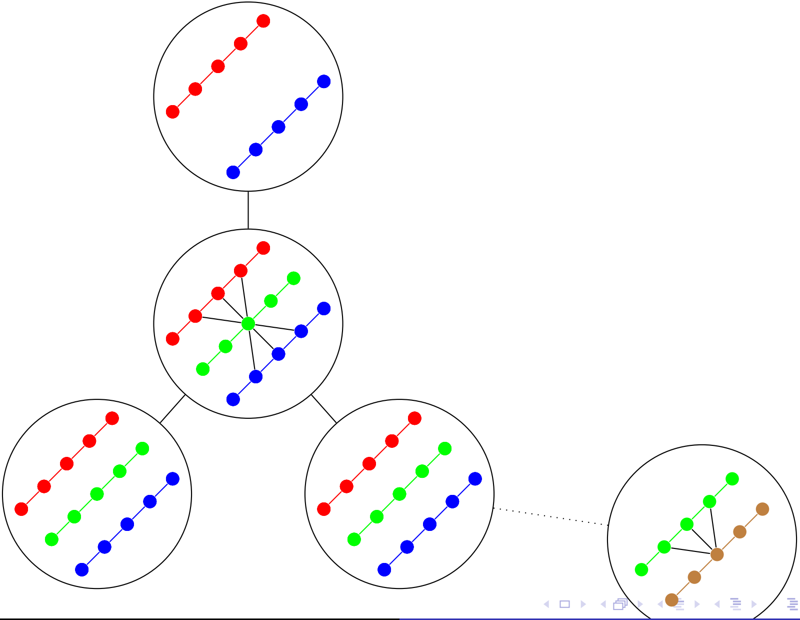
Theorem

Let $\mathcal{C}$ be a graph class admitting product structure, and let $\mathcal{D}$ be a graph class FO-transducible from $\mathcal{C}$. Then, $\mathcal{D}$ is a flip of some graph class $\mathcal{D}'$ which admits hereditary product structure.

Product structure – another view



Hereditary product structure = $\mathcal{H}$-clique-width

Definition

A graph G has *clique-width* at most k if there is a k-expression valued G.

k-expression: k colors and the following operations:

- Given $c \in [k]$, create a graph having single vertex with *color* c
- Take disjoint union
- Given a pair of colors $c_1 \neq c_2$, add edges between every pair of vertices u, v satisfying that:
  - color of u is c_1, and
  - color of v is c_2
- Recolor c_1 to c_2

Hereditary product structure = $\mathcal{H}$-clique-width

Definition

A graph G has *$\mathcal{H}$-clique-width* at most k if there is a loop graph $H \in \mathcal{H}$ and a (H,k)-expression valued G. If no such expression exists, then we say that $\mathcal{H}$-Clique-Width is ∞.

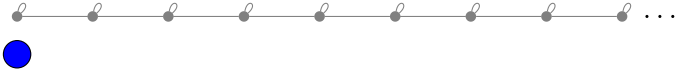
(H,k)-expression: k colors and the following operations:

- Given $c \in [k]$ and $p \in V(H)$, create a graph having single vertex with *color* c and *parameter vertex* $p \in V(H)$
- Take disjoint union
- Given a pair of colors $c_1 \neq c_2$, add edges between every pair of vertices u, v satisfying that:
  - color of u is c_1, and
  - color of v is c_2, and
  - the parameter vertices of u and v are adjacent in H
- Recolor c_1 to c_2 **without** changing parameter vertices

Example: 2D grid

2D grid has $\mathcal{P}^\circ$-clique-width at most 5

Create path colored “modulo 2”:



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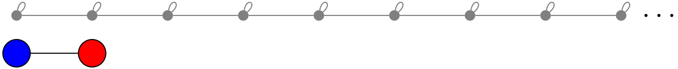

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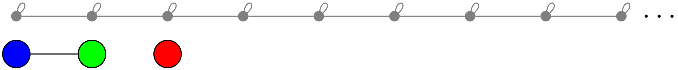

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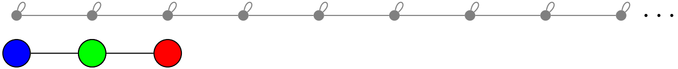

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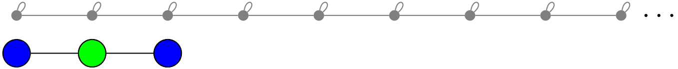

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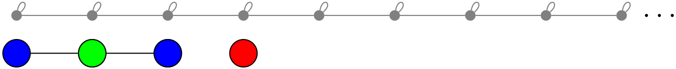

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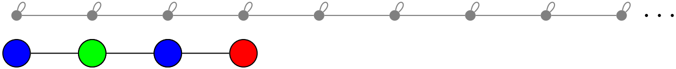

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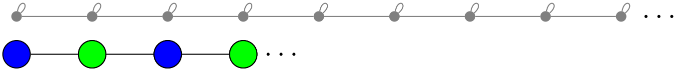

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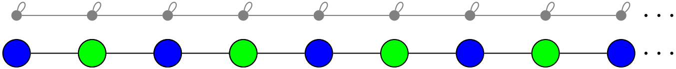

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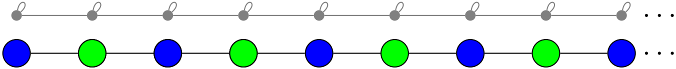

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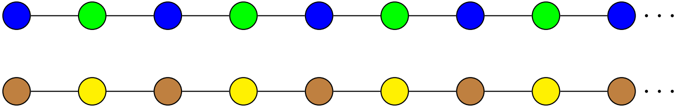

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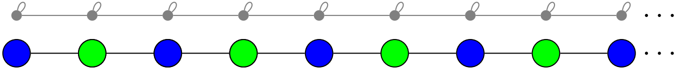

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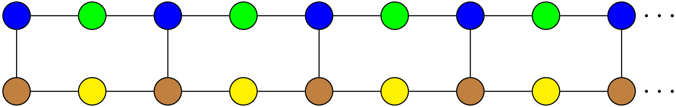

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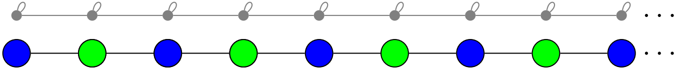

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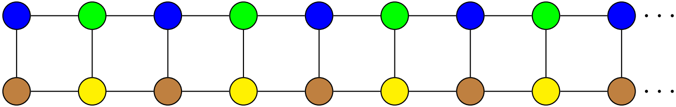

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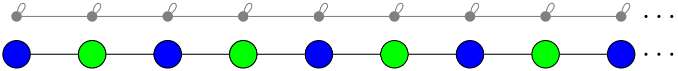

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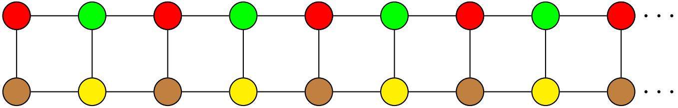

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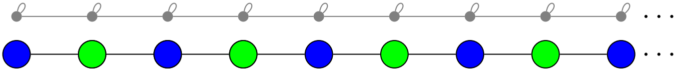

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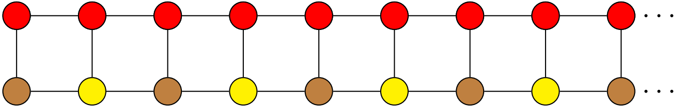

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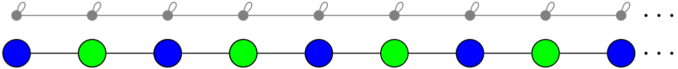

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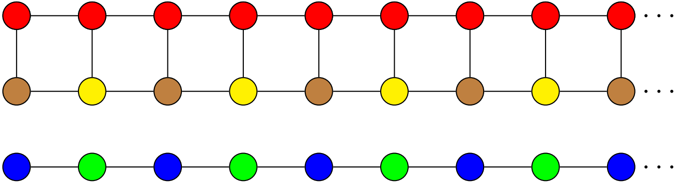

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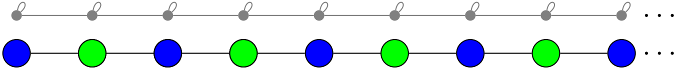

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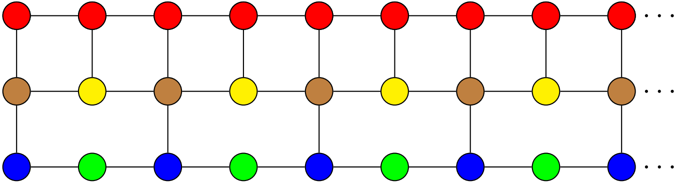

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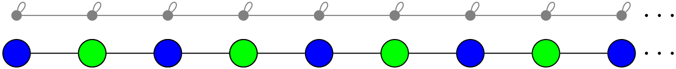

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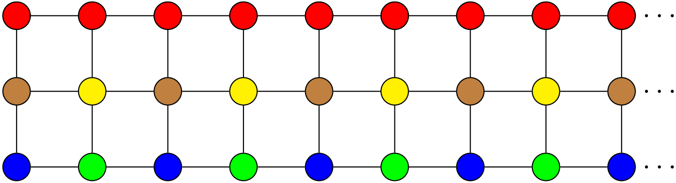

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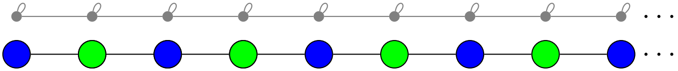

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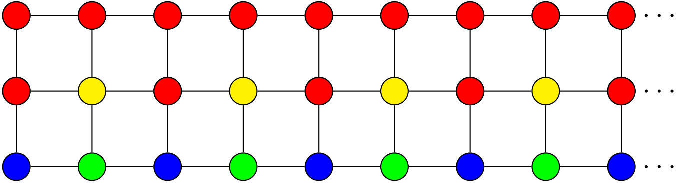

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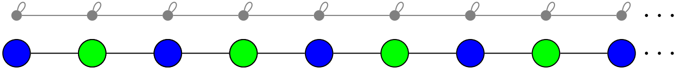

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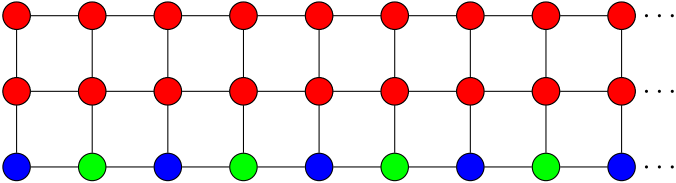

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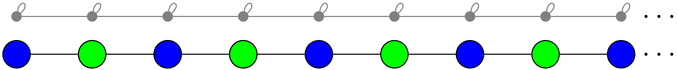

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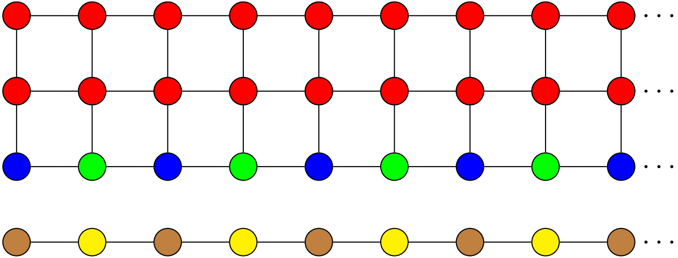

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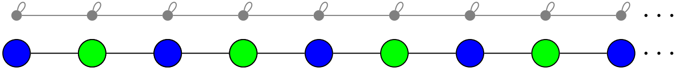

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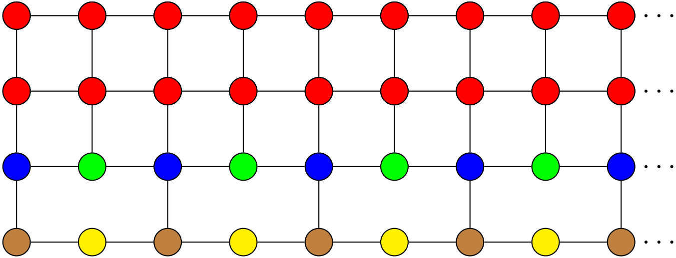

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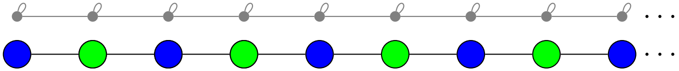

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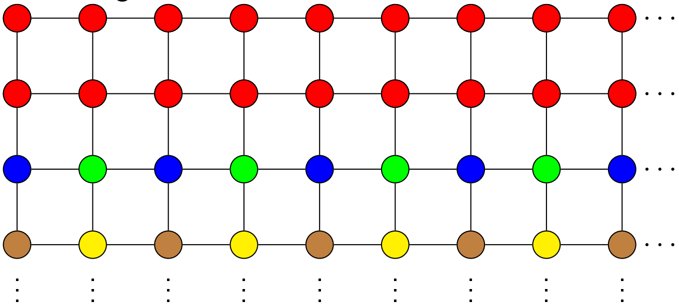


Main result (again and more generally)

Let $\mathcal{Q}$ be a class of bounded degree simple graphs. We denote by $\mathcal{Q}_r^\circ$ the reflexive closure of the r-th power of $\mathcal{Q}$.

Definition

A graph class $\mathcal{C}$ admits $\mathcal{Q}$-product structure if there is a constant k such that every graph $G \in \mathcal{C}$ is a subgraph of the strong product $Q \boxtimes M$ of a graph $Q \in \mathcal{Q}$ and a graph M of tree-width at most k.

Theorem

Let $\mathcal{C}$ be a graph class admitting $\mathcal{Q}$-product structure (such as that class of planar graphs). Let $\mathcal{D}$ be a graph class FO-transducible from $\mathcal{C}$. Then, there are constants k, ℓ, and r such that $\mathcal{D}$ is contained in a k-flip (global path) of a class with $\mathcal{Q}_r^\circ$-clique-width at most ℓ (local part).

Going there and back again

Definition (Product structure for dense graphs)

A graph class $\mathcal{C}$ *admits hereditary product structure* if there is a constant k such that every graph $G \in \mathcal{C}$ is an induced subgraph of the strong product $P \boxtimes M$ of a path P and a graph M of clique-width at most k.

Theorem

Let $\mathcal{D}$ be a class of bounded stable clique-width (stable = does not FO-transduce all half-graphs). Let $\mathcal{C}$ be a class admitting hereditary product structure such that, the graph M from the above definition can be chosen from $\mathcal{D}$. Then, there is a class $\mathcal{G}$ admitting (the classical) product structure such that $\mathcal{C}$ is FO-transducible from $\mathcal{G}$.

Corollaries – 3D grids

Theorem

The class of all 3D grids is not FO-transducible from planar graphs.

Proof.

- Any balanced bipartiton A, B ($|A| \leq 2|B| \leq 4|A|$) of $a \times a \times a$ grid $G_{a\times a\times a}$ induces a matching of size $\Omega(a^2)$
- $G_{a\times a\times a}$ has diameter $\Theta(a)$
- If a is large enough, then any k-flip of $G_{a\times a\times a}$ contains a large induced subgraph H of diameter $d \in \mathcal{O}_k(a)$ such that any balanced bipartiton of H induces a matching or anti-matching of size $m \in \Omega_k(a^2)$
- Suppose that there is (P, ℓ)-expression ϕ valued H
- Some node of ϕ corresponds to balanced bipartition A, B but the maximum size of both matching and anti-matching at every node is at most $\mathcal{O}(\ell \cdot d) = \mathcal{O}_\ell(a)$

□

Consider class $\mathcal{C}$ of graphs G obtained from $a^3 \times a^3$ grid by adding vertices as follows:

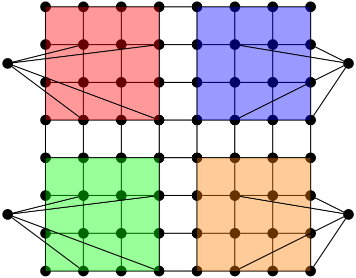
> Observe that G contains a^2 disjoint subgrids of size $a^2 \times a^2$.
> For each such subgrid $H \subseteq G$, we add a single new vertex v_H.
> For each i, j, we add an edge between v_H and the vertex of H which lies in the intersections of ai-th row and aj-th column.

Theorem

The class $\mathcal{C}$ is not FO-transducible from planar graphs.

Observation

The class $\mathcal{C}$ admits slice decomposition.



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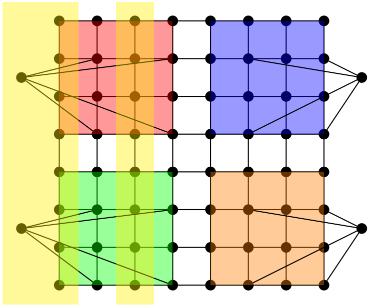

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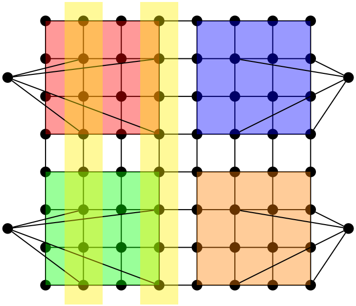

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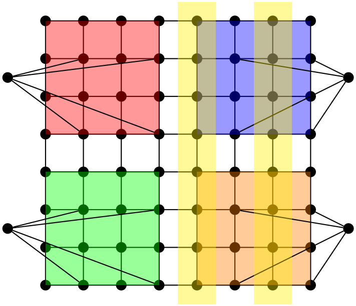

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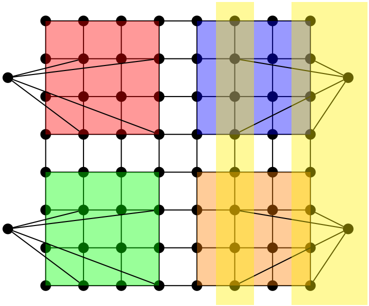

Conclusions

- Transductions of graph classes admitting product structure (subgraphs of Path $\boxtimes$ Small Tree-Width) are k-flips of a class admitting hereditary product structure (induced subgraphs of Path $\boxtimes$ Small Clique-Width)
- $\mathcal{C}$ admits hereditary product structure $\iff$ $\mathcal{C}$ has bounded Path$^{\circ}$-clique-width
- Using (H, k)-expressions, it is easy to prove some non-transducibility results – eq. 3D grids are not transducible from planar graphs