First-order transduction hierarchy among sparse graph classes

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Based on joint work with Jakub Gajarský, Jeremi Gładkowski, Michał Pilipczuk, and Szymon Toruńczyk

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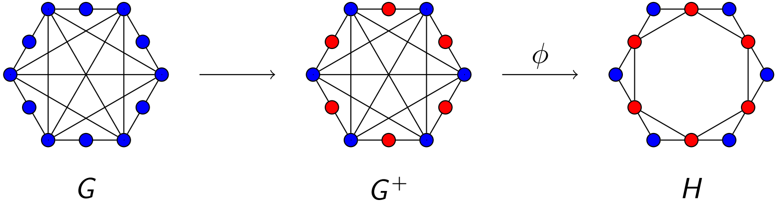
First-order transductions

- Transduction = encoding one class of graphs into another such that we can decode the original class using logic
- **Input:** colored graph G
- **Step 1:** nondeterministically add colors G^+
- **Step 2:** redefine edge relation H

$$uv \in E(H) \iff G^+ \models \phi(u, v)$$

where $\phi(x, y)$ is a first-order (FO) formula

- **Output:** any induced subgraph of H



Transduction hierarchy

- A class $\mathscr{C}$ is *transducibile* from a class $\mathscr{D}$ if there is a transduction τ such that $\mathscr{C} \subseteq \tau(\mathscr{D}) = \{\tau(G) | G \in \mathscr{D}\}$
- Transducibility is quasi-order ($=$ hierarchy) on graph classes
- How does the transduction hierarchy look like?
  - Showing transducibility $=$ finding transduction
  - How to show non-transducibility?
    - Counting
    - Transduction ideals
    - **Combinatorially describing transductions**

Showing non-transducibility: Counting

- $\mathscr{G}_d$:= class of all d-dimensional grids
- $N^r_G(v)$:= set of all vertices at distance at most r in G
- graph $G \in \mathscr{G}_d$, vertex $v \in V(G) \implies |N^r_G(v)| \leqslant \mathcal{O}(r^d)$
- large graph $G \in \mathscr{G}_d$ and vertex $v \in V(G)$ "deep inside G"
  $\implies |N^r_G(v)| \geqslant \Omega(r^d)$
- first-order transduction can be split into
  - local part – only creates edges between nearby vertices; and
  - global part – perturbation – select k sets of vertices and flip adjacency inside each set

$\implies$ $\mathscr{G}_d$ does not transduce $\mathscr{G}_{d+1}$

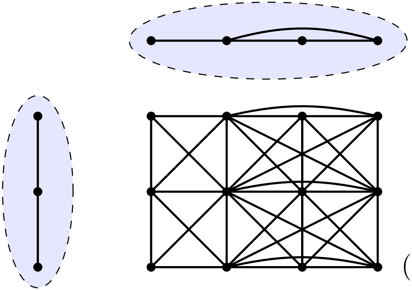
Showing non-transducibility: Transduction ideals

- TW_k := class of graphs of tree-width k
- CW_k := class of graphs of clique-width k
- $\mathcal{G}_d$:= class of all d-dimensional grids
- Example: tree-width and 2D grids
  - For every k and every FO/MSO transduction τ there is a constant ℓ such that $\tau(TW_k) \subseteq CW_\ell$
  - The class $\mathcal{G}_2$ has unbounded clique-width

$\Longrightarrow$ $\mathcal{G}_2$ is not transducible from TW_k for any k

Showing non-transducibility: Transduction ideals

- TW_k := class of graphs of tree-width k
- CW_k := class of graphs of clique-width k
- $\mathcal{G}_d$:= class of all d-dimensional grids
- Example: planarity and 3D grids
  - Planar graphs are subgraphs of the strong product of a path and a graph of tree-width $\leqslant 8$



$$V(G \boxtimes H) = V(G) \times V(H)$$
$$[g_1, h_1][g_2, h_2] \in E(G \boxtimes H)$$
$$\Updownarrow$$
$$(g_1 g_2 \in E(G) \wedge h_1 = h_2) \vee$$
$$(g_1 = g_2 \wedge h_1 h_2 \in E(H)) \vee$$
$$(g_1 g_2 \in E(G) \wedge h_1 h_2 \in E(H))$$

Showing non-transducibility: Transduction ideals

- $TW_k :=$ class of graphs of tree-width k
- $CW_k :=$ class of graphs of clique-width k
- $\mathcal{G}_d :=$ class of all d-dimensional grids
- Example: planarity and 3D grids
  - Planar graphs are subgraphs of the strong product of a path and a graph of tree-width $\leqslant 8$
  - Jakub Gajarský, Michał Pilipczuk, Filip Pokrývka (2025):
    - Transductions of planar graphs admit (up to perturbations) a *slice decomposition* – each graph admits a layering where bounded number of consecutive layers induces a subgraph of bounded clique-width
    - 3D grids do not admit slice decomposition

  $\Longrightarrow$ Planar graphs do not transduce $\mathcal{G}_3$
  - Petr Hliněný, Jan Jedelský (2025):
    - Transductions of planar graphs are (up to perturbations) induced subgraphs of a path and a graph of bounded clique-width
    - 3D grids are not induced subgraphs of such a product

  $\Longrightarrow$ Planar graphs do not transduce $\mathcal{G}_3$

Beyond transduction ideals – combinatorially describing transductions (for sparse graphs) – definitions

- $\alpha : V(H) \to 2^{V(G)}$ is a *minor model* of H in G if
  - for each vertex $v \in V(H)$, the *branch set* $\alpha(v)$ of v induces a connected subgraph of G, and
  - for each edge $uv \in E(H)$, the minimum distance in G between the branch sets $\alpha(u)$ and $\alpha(v)$ of u and v is at most 1
- Minor model α has congestion at most k if it holds for all vertices v of G that the number of branch sets containing v is at most k
- Minor model α is r-shallow if each branch set has radius at most r in G

Beyond transduction ideals – combinatorially describing transductions (for sparse graphs) – definitions

- Consider a graph G and a linear order $\preceq$ on its vertex set. A vertex v is *distance-r weakly reachable* from a vertex u if there is a path P between u and v in G of length at most r such that $\forall p \in V(P). v \preceq p$.
- *Distance-r weak coloring number* $wcol_r(G)$ of a graph G is the minimum over all vertex orders $\preceq$ of the maximum over all vertices $u \in V(G)$ of the number of distance-r weakly reachable vertices from u w.r. to $\preceq$.

Beyond transduction ideals – combinatorially describing transductions (for sparse graphs)

- Jakub Gajarský, Jeremi Gładkowski, Jan Jedelský, Michał Pilipczuk, and Szymon Toruńczyk (2025, arXiv:2505.15655):
  - Let $\mathscr{D}$ be a weakly sparse graph class first-order transducible from a class $\mathscr{C}$ of bounded expansion. Then, there is $k \in \mathbb{N}$ such that, every graph $G \in \mathscr{D}$ is k-congested k-shallow minor of a graph H° obtained from some $H \in \mathscr{C}$ by adding a universal vertex.
  - Both operations of adding universal vertex and taking k-congested k-shallow minors preserve the degree of the distance-r weak coloring numbers as a polynomial in r.

Using the description

- $TW_k :=$ class of graphs of tree-width k
- $wcol_r(TW_k) = \binom{r+k}{k} = \Theta(r^k)$

$\implies$ TW_k does not transduce TW_{k+1}

- $\mathscr{P} :=$ class of planar graphs
- $wcol_r(\mathscr{P}) \leqslant \mathcal{O}(r^3)$

$\implies$ $\mathscr{P}$ does not transduce TW_4

  - Conjecture by Gwenaël Joret, Piotr Micek (2022): $wcol_r(\mathscr{P}) \leqslant \mathcal{O}(r^2 \log r)$
  - If true, then $\mathscr{P}$ transduces TW_k if and only if $k \leqslant 2$

- $\mathscr{O} :=$ class of outerplanar graphs
- $wcol_r(\mathscr{O}) = \Theta(r \log r)$

$\implies$ $\mathscr{O}$ does not transduce TW_2

Using the description

- Question: Does the class of planar graphs transduce the class of toroidal graphs?
- Observation: Suppose that the class $\mathscr{C}$ of maximum degree $\leqslant d$ toroidal graphs is transducible from planar graphs. Then, there are constants ℓ and k such that every graph $G \in \mathscr{C}$ has subset $X \subseteq V(G)$ of vertices of size $|X| \leqslant \ell$ such that $G - X$ is k-planar.
- Question: Are all bounded degree toroidal graphs k-planar after removal of bounded number of vertices?

Overall picture

$$\text{Paths} \overset{FO}{\subsetneq} \text{PW}_1 \overset{FO}{\subsetneq} \text{PW}_2 \overset{FO}{\subsetneq} \ldots \overset{FO}{\subsetneq} \text{PW}_k \overset{FO}{\subsetneq} \ldots \overset{FO}{\subsetneq} \text{OPlanar} \overset{FO}{\subsetneq} \text{TW}_2 \overset{FO}{\subsetneq} \text{TW}_3 \overset{FO}{\subsetneq} \ldots \overset{FO}{\subsetneq} \text{TW}_k$$

$$\text{FO}\not\forall\not A \qquad\qquad \text{FO}\not\forall\not A$$

$$\text{PW}_1 \overset{FO}{\subsetneq} \text{Trees} \overset{FO}{\equiv} \text{TW}_1 \overset{FO}{\subsetneq} \qquad \text{OPlanar} \overset{FO}{\subsetneq} \text{TW}_2 \overset{FO}{\subsetneq} \qquad \text{Planar}$$

Overall picture

$$\text{Paths} \overset{\text{FO}}{\subsetneq} \text{PW}_1 \overset{\text{FO}}{\subsetneq} \text{PW}_2 \overset{\text{FO}}{\subsetneq} \dots \overset{\text{FO}}{\subsetneq} \text{PW}_k \overset{\text{FO}}{\subsetneq} \dots \overset{\text{FO}}{\subsetneq} \text{OPlanar} \overset{\text{FO}}{\subsetneq} \text{TW}_2 \overset{\text{FO}}{\subsetneq} \text{TW}_3 \overset{\text{FO}}{\subsetneq} \dots \overset{\text{FO}}{\subsetneq} \text{TW}_k$$

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Thank you!