$\mathcal{H}$-Clique-Width

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Motivation

- Generalization of clique-width
- Captures more classes of graphs:
  - A graph G has Planar Product Structure $\implies$ G has bounded $\mathcal{P}^\circ$-Clique-Width, where $\mathcal{P}^\circ$ is the class of all reflexive paths
- Allows us to use "clique-width like" arguments in proofs:
  - Bounded $\mathcal{P}^\circ$-Clique-Width $\implies$ does not transduce (using FO logic) the class of all 3D grids (work-in-progress)

Definition

- A graph G has Clique-Width at most k if there is a k-expression valued G.
- k-expression: k colors and the following operations:
  - create_vertex(c): Create a new vertex colored by $c \in [k]$
  - disjoint_union(ψ_1, ψ_2)
  - add_edges($\psi_1, c_1 \neq c_2$): Add edges between every pair of vertices u, v satisfying that:
    - color of u is c_1, and
    - color of v is c_2
  - recolor($\psi_1, c_1 \rightarrow c_2$)

Definition

- A graph G has $\mathcal{H}$-Clique-Width at most k if there is a loop graph $H \in \mathcal{H}$ and a (H,k)-expression valued G. If no such expression exists, then we say that $\mathcal{H}$-Clique-Width is ∞.
- (H,k)-expression: k colors and the following operations:
  - create_vertex(c, p): Create a new vertex colored by $c \in [k]$ with a *parameter vertex* $p \in V(H)$
  - disjoint_union(ψ_1, ψ_2)
  - add_edges$(\psi_1, c_1 \neq c_2)$: Add edges between every pair of vertices u, v satisfying that:
    - color of u is c_1, and
    - color of v is c_2, and
    - the parameter vertices of u and v are adjacent in H
  - recolor$(\psi_1, c_1 \to c_2)$

Example: 2D grid

- 2D grid has $\mathcal{P}^\circ$-clique-width at most 5
- ϕ: create path colored “modulo 2”:

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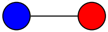

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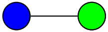

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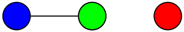

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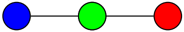

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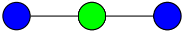

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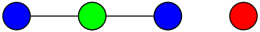

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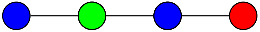

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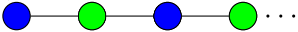

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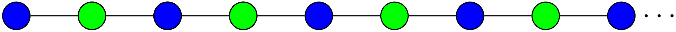

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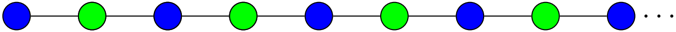

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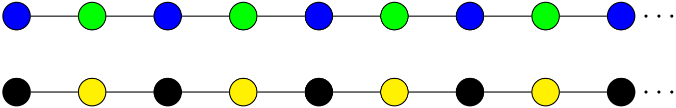

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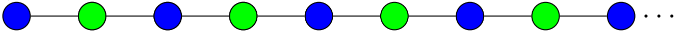

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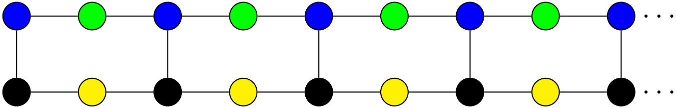

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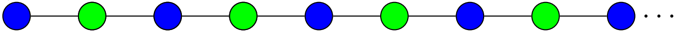

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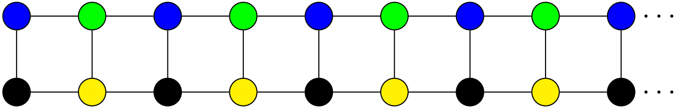

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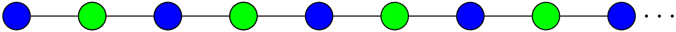

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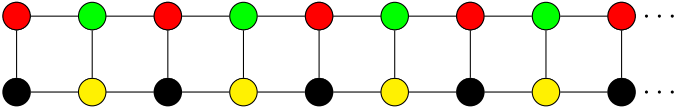

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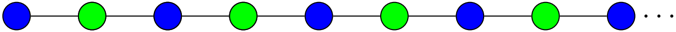

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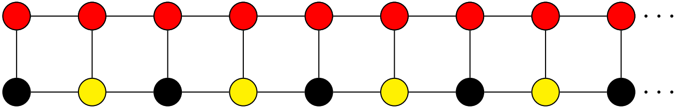

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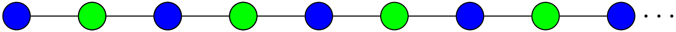

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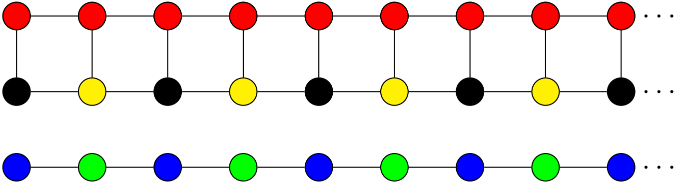

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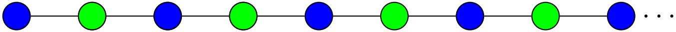

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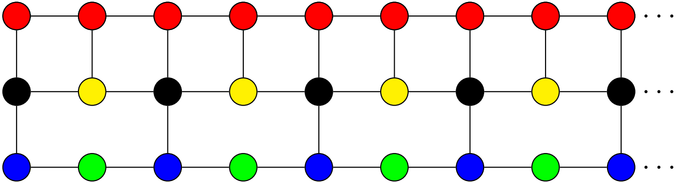

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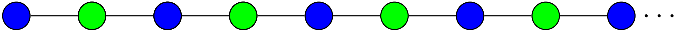

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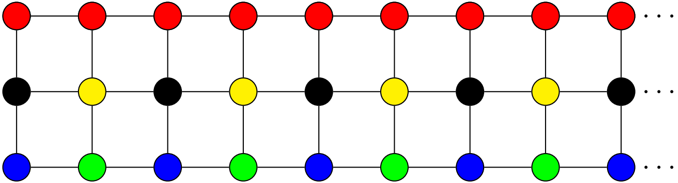

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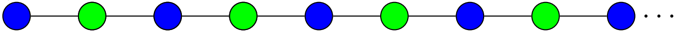

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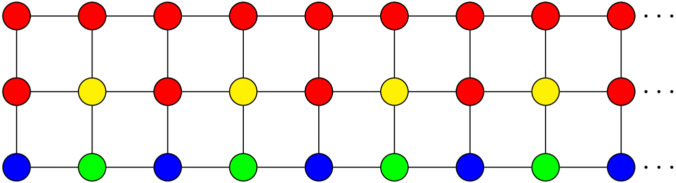

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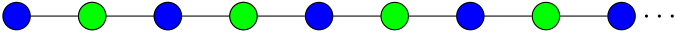

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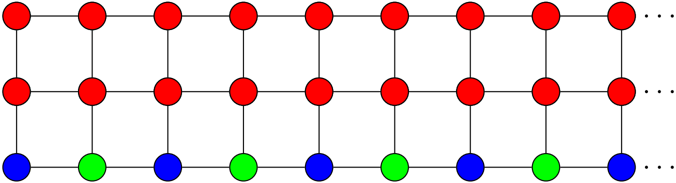

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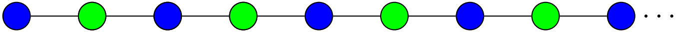

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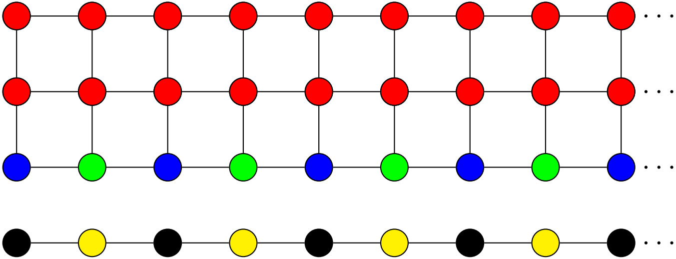

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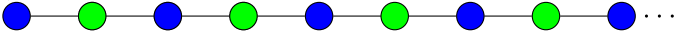

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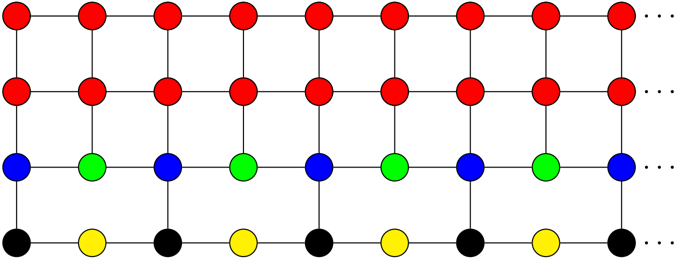

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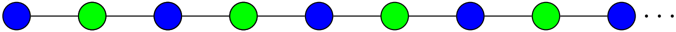

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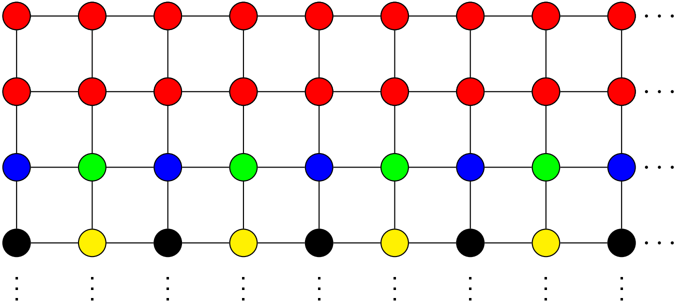

Some Properties

- cw$(G) = \{K_1^\circ\}$-cw(G)
- $\{K_1\}$-cw$(G) < \infty \iff G$ has no edges
- $\{H\}$-cw$(G) \leq 2$, where G is H without loops
- Deciding whenever $\{K_3\}$-cw$(G) < \infty$ is NP-hard.
- $\{H\}$-cw$(G) < \infty \iff \exists$ homomorphism from G to H.
- $(\exists f. \forall G.\ \text{cw}(G) \leq f(\mathcal{H}\text{-cw}(G))) \iff \mathcal{H}$ has component-bounded total neighbourhood diversity
  - total neighbourhood type of x: set of neighbourhood of x including x if it has self-loop
  - total neighbourhood diversity: number of different total neighbourhood types
  - component-bounded total neighbourhood diversity: maximum total neighbourhood diversity over all connected components
- $\mathcal{H}$ has maximum degree at most $\Delta \implies$ local clique-width of any graph G is bounded by a function of $\mathcal{H}-cw(G)$ and Δ

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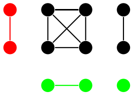

- Strong product of two graphs $G \boxtimes H$:
  - $V(G \boxtimes H) = V(G) \times V(H)$
  - $(u, x)(v, y) \in E(G \boxtimes H) \iff$
    $(u = v \vee uv \in E(G)) \wedge (x = y \vee xy \in E(H))$
- Dujmovic, Esperet, Joret, Walczak, and Wood: Every planar graph is isomorphic to a subgraph of $P \boxtimes H$, where P is a path and H has tree-width at most 8 (later improved to 6).
- Let H° be a reflexive loop graph. Let G be a simple graph. Then, G has $\{H^\circ\}$-clique-width at most ℓ iff G is isomorphic to an induced subgraph of $H \boxtimes M$, where H is obtained from H° by removing all loops and M has clique-width at most ℓ.
- Let G be a subgraph of $P \boxtimes H$, where P is a path and H has tree-width at most k. Then, G has $\mathcal{P}^\circ$-clique-width at most $6(k+1) \cdot 8^{k+1}$. Moreover, G is an induced subgraph of $P \boxtimes H$, where P is a path and H has tree-width at most $6(k+1) \cdot 8^{k+1}$.

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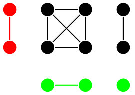

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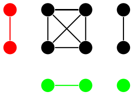
Relation to Planar Product Structure



- Strong product of two graphs $G \boxtimes H$:
  - $V(G \boxtimes H) = V(G) \times V(H)$
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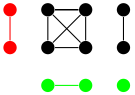

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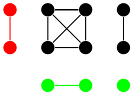

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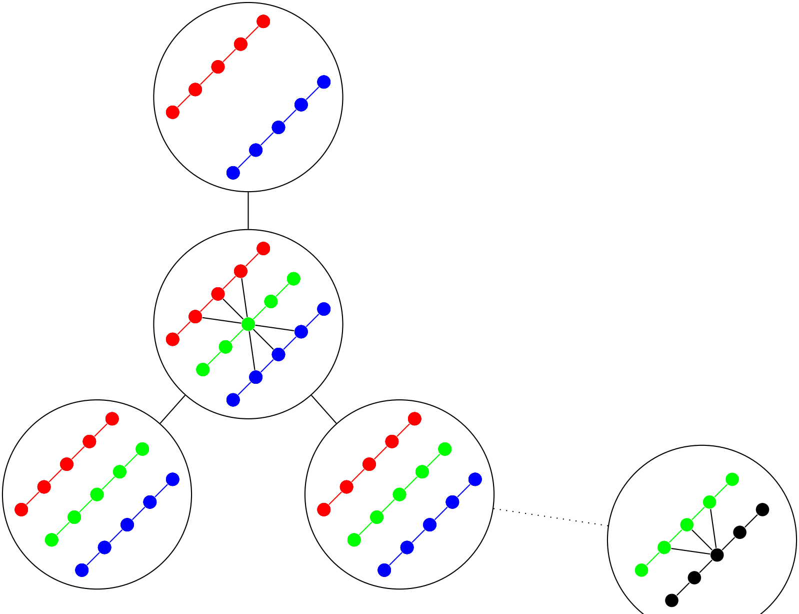
Proof Idea



Open Questions

- Can one approximate $\mathcal{P}^{\circ}$-clique-width in FPT time parameterized by the solution value?
- Is it the case that, for every graph H there is a graph H' such that, for every graph G, $\{H'\}$-clique-width of G is bounded by a fixed function of $\{H\}$-clique-width of the complement of G?
- For which classes $\mathcal{H}$ does the following hold: For every transduction τ there is a function f and a transduction σ such that, for every integer k and every graph G of $\mathcal{H}$-clique-width at most k, it holds that $\sigma(\mathcal{H})$-clique-width of $\tau(G)$ is at most $f(k)$?
- Is there a function f such that the following holds? Let G be a $K_{t,t}$-free graph of $\mathcal{P}^{\circ}$-clique-width at most ℓ. Then, G is isomorphic to a subgraph of $P \boxtimes H$, where P is a path and H has tree-width at most $f(\ell)$.