Twin-width and Transductions of Proper k-Mixed-Thin Graphs

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Overview

Twin-width: Graph parameter describing similarity to cographs

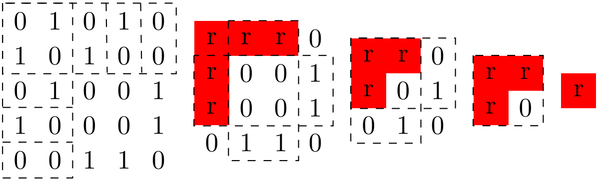
Proper k-Thin: Generalization of proper interval graphs

Proper k-Mixed-Thin: Our generalization of proper k-mixed graphs

- Twin-width linear in k
- A subclass (inversion-free) transduction equivalent to posets of bounded width

Twin-width

- *Twin-width I: tractable FO model checking* by Bonnet et al.
- Twin-width using symmetric contraction sequences of adjacency matrices
- Symmetric k-contraction sequence:
  - mismatched entries replaced by r
  - row and the corresponding column contractions performed simultaneously
  - number of entries r in any row or column $\leq k$



(Proper) k-Thin

- *The stable set problem and the thinness of a graph* by Mannino et al.
- *On the thinness and proper thinness of a graph* by Flavia and Estrada
- Graph $G = (V, E)$, k-partition $(V_1, V_2, \ldots, V_k)$ of V, and linear order $\leq$ on V
- For all $u \lneq v \lneq w$

if $\exists i (u, v \in V_i)$ and $uw \in E$ then $vw \in E$

proper if $\exists i (v, w \in V_i)$ and $uw \in E$ then $uv \in E$

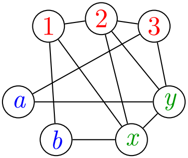
Proper k-Mixed-Thin

- Graph $G = (V, E)$, k-partition $(V_1, V_2, \ldots, V_k)$ of V, a linear order $\leq_{ij}$ on $V_i \cup V_j$, and a choice of $E_{ij} \in \{E, \overline{E}\}$
- Restriction of $\leq_{ij}$ to V_i (resp. V_j) is *aligned* with $\leq_{ii}$ ($\leq_{jj}$)
- For all $1 \leq i, j \leq k$ and all $u \lneq_{ij} v \lneq_{ij} w$

    if ($u, v \in V_i, w \in V_j$ or $u, v \in V_j, w \in V_i$) and $uw \in E_{ij}$ then $vw \in E_{ij}$

    proper if ($u \in V_j, v, w \in V_i$ or $u \in V_i, v, w \in V_j$) and $uw \in E_{ij}$ then $uv \in E_{ij}$

- Inversion-free if the restriction of $\leq_{ij}$ to V_i (resp. V_j) is equal to $\leq_{ii}$ (resp. $\leq_{jj}$)



$\leq_{11}$: 1, 2, 3

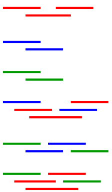
$\leq_{22}$: a, b $(E_{2,2} = \bar{E})$

$\leq_{33}$: x, y

$\leq_{12}$: a, 1, 2, b, 3 $(E_{1,2} = \bar{E})$

$\leq_{23}$: x, b, a, y

$\leq_{13}$: y, 3, 2, 1, x



Proper k-Mixed-Thin Graphs I

Simple cases

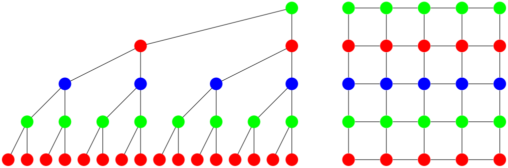
- proper 1-mixed thin = proper 1-thin = proper interval
- proper 2-mixed thin = proper 2-thin $\supsetneq$ proper interval
- proper 3-mixed thin $\supsetneq$ proper 3-thin

$$
\begin{matrix}
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0
\end{matrix}
$$

Proper k-Mixed-Thin Graphs II

Proposition 3 and Theorem 4

- Let $d \geq 1$ be an arbitrary integer. Both d-dimensional grids and d-dimensional full grids are inversion-free proper 3^{d-1}-mixed-thin.
- Every tree T is inversion-free proper 3-mixed-thin.



Proposition 2 (Mannino et al., Bonomo and de Estrada)

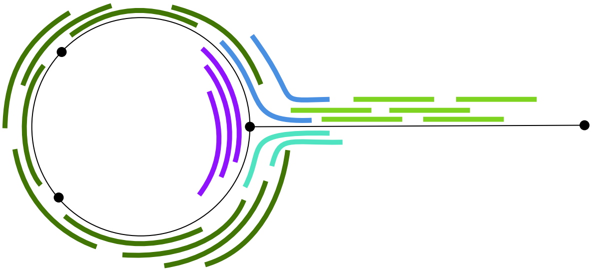
b) The $(r \times r)$-grid has thinness linear in r.

c) The thinness of the complete m-ary tree ($m > 1$) is linear in its height.

Proper k-Mixed-Thin Graphs III

Theorem 5

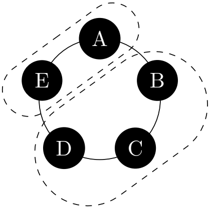
Let $G = (V, E)$ be a proper intersection graph of paths in some subdivision of a fixed connected graph H with m edges, and let k be the number of (all) distinct paths in H. Then G is a proper $(m^2 k)$-mixed-thin graph.



Linear Twin-width – Upper Bound

Theorem 6

Let G be a proper k-mixed-thin graph. Then the twin-width of G is at most $9k$. The corresponding contraction sequence for G can be computed in polynomial time from the vertex partition $(V_1, \ldots, V_k)$ and the orders $\leq_{ij}$ for G



	B	C	D	A	E
B	0	1	0	1	0
C	1	0	1	0	0
D	0	1	0	0	1
A	1	0	0	0	1
E	0	0	1	1	0

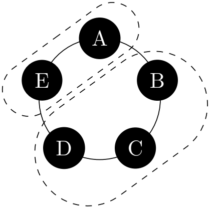
BC	r	r	r	0
D	r	0	0	1
A	r	0	0	1
E	0	1	1	0

BCD	r	r
AE	r	r

BC	r	r	r
D	r	0	r
AE	r	r	r

Linear Twin-width – Upper Bound

- Uses the *red-potential* method developed in *Twin-Width is Linear in the Poset Width* by Balabán and Hliněný
- Matrix ordering: Parts arbitrarily, within part V_i use $\leq_{ii}$
- Submatrices given by parts $V_i \times V_j$ can be split by *diagonal boundaries* into uniform parts (exception – the main diagonal)
- Red entries only next to the boundaries $\rightarrow \mathcal{O}(kn)$ red entries possible $\rightarrow$ there is a contraction with $\mathcal{O}(k)$ red entries
- We can "repair" the boundaries after contractions



	B	C	D	A	E
B	0	1	0	1	0
C	1	0	1	0	0
D	0	1	0	0	1
A	1	0	0	0	1
E	0	0	1	1	0

BC	r	r	r	0
D	r	0	0	1
A	r	0	0	1
E	0	1	1	0

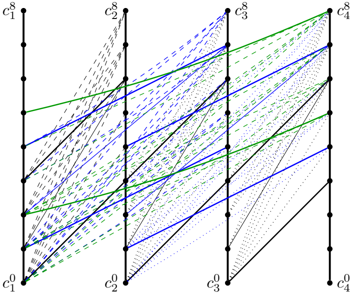
BCD	r	r
AE	r	r

BC	r	r	r
D	r	0	r
AE	r	r	r

Linear Twin-width – Lower Bound – Proof

Proposition 10

For every integer $k \geq 1$, there exists an inversion-free proper $(2k+1)$-mixed-thin graph G such that the twin-width of G is at least k.

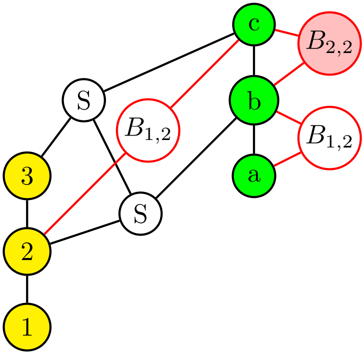


Non-Copying First-Order Transductions – Definition

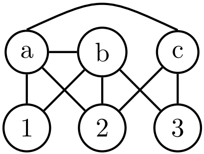
- Start with a relational structure $\sigma = (V, R_1, \ldots, R_n)$ on domain V with relations R_i
- Add a fixed number of colors (unary relations) arbitrarily
- Fix FO-formulas $\phi_1(x_1, \ldots, x_{ar_1}), \ldots, \phi_m(x_m, \ldots, x_{ar_m})$ and $\psi(x)$ using the colors and relational symbols R_i
- The result is relational structure $\sigma' = (V', R'_1, \ldots, R'_m)$ where $v \in V' \subseteq V$ iff $\sigma \models \psi(x)$ and for all $i = 1, \ldots, m$ $(x_i, \ldots, x_{ar_i}) \in R'_i \subseteq (V')^{arity(R'_i)}$ iff $\sigma \models \phi_i(x_i, \ldots, x_{ar_i})$

Transductions: Encoding Inversion-Free Proper Mixed-Thin Graphs in Posets of Bounded Width

- Any inversion-free proper k-mixed-thin graph encoded using poset of width at most $5 \cdot \binom{k}{2} + 5k$
- The poset can be computed in polynomial time given the partition and the orderings

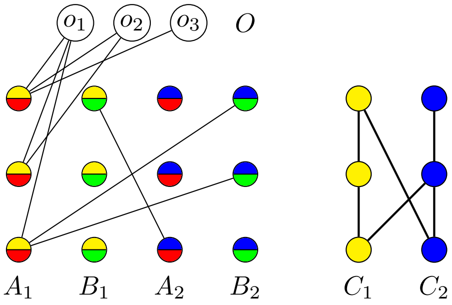


a, 1, 2, b, 3, c



Transductions: Encoding Posets of Bounded Width in Inversion-Free Proper Mixed-Thin Graphs

- Poset of width k encoded using inversion-free proper $2k+1$-mixed-thin graph
- The graph can be computed in polynomial time



Conclusions

We have ...

... defined the class of proper k-mixed-thin graphs

... and showed that ...

... it contains certain natural graph classes (trees, grids) as subclasses

... its twin-width is linear in k

... its inversion-free subclass is transduction equivalent to posets of bounded-width

Going forward, we ask ...

... if the inversion-freeness is necessary in the transduction equivalence

... how to recognize proper k-mixed-thin graphs

... what are the necessary and sufficient conditions to use the red-potential method (elsewhere)