

# **IA168 Algorithmic Game Theory**

Tomáš Brázdil

# Organization of This Course

Sources:

- ▶ Lectures (slides, notes)
  - ▶ based on several sources
  - ▶ slides are prepared for lectures, some stuff on greenboard ( $\Rightarrow$  attend the lectures)

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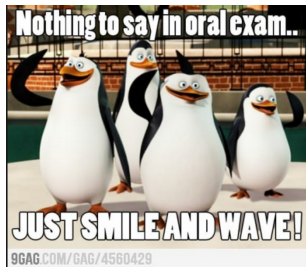
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- ▶ Books:
  - ▶ Nisan/Roughgarden/Tardos/Vazirani, **Algorithmic Game Theory**, Cambridge University, 2007.  
Available online for free:  
[http://www.cambridge.org/journals/nisan/downloads/Nisan\\_Non-printable.pdf](http://www.cambridge.org/journals/nisan/downloads/Nisan_Non-printable.pdf)
  - ▶ Tadelis, **Game Theory: An Introduction**, Princeton University Press, 2013

(I use various resources, so please, attend the lectures)

# Evaluation

- ▶ **Oral exam**
- ▶ **Homework**



- ▶ 3 homework assignments
- ▶ (*possibly* a computer implementation of a strategy)

## Notable features of the course

- ▶ No computer games course!
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An example of an instruction email (from another course with the same system):

It is typically not sufficient to devote a single afternoon to the preparation for the exam.

You have to know `_everything_` (which means every single thing) starting with the slide 42 and ending with the slide 245 with notable exceptions of slides: 121 - 123, 137 - 140, 165, 167.

Proofs presented on the whiteboard are also mandatory.

Most importantly,

The previous slide is not  
a joke!



# What is Algorithmic Game Theory?

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What does the "algorithmic" mean?

- ▶ It means that we are "concerned with the computational questions that arise in game theory, and that enlighten game theory. In particular, questions about finding efficient algorithms to 'solve' games."

Let's have a look at some examples ....

# Prisoner's Dilemma

Prisoners' dilemma




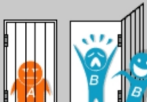
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		confess 	remain silent 		
prisoner A	confess 	 5 years   5 years	 0 year   20 years		
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
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The problem: What would the suspects do?

# Prisoner's Dilemma – Solution(?)

	C	S
C	-5, -5	0, -20
S	-20, 0	-1, -1

Rational "row" suspect (or his adviser) may reason as follows:

# Prisoner's Dilemma – Solution(?)

	$C$	$S$
$C$	$-5, -5$	$0, -20$
$S$	$-20, 0$	$-1, -1$

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- ▶ If my colleague chooses  $C$ , then playing  $C$  gives me  $-5$  and playing  $S$  gives  $-20$ .

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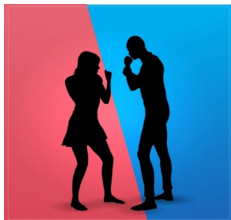
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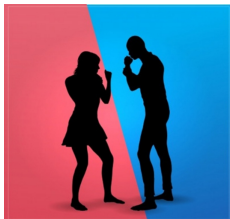
Are there always "dominant" strategies?

# Nash equilibria – Battle of Sexes



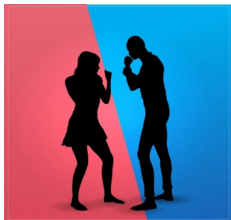
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If they cannot communicate, where should they go?

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Battle of Sexes can be modeled as a game of two players (the couple) with the following payoffs:

	<i>O</i>	<i>F</i>
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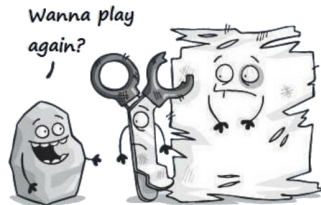
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$(O, O)$  is an example of a *Nash equilibrium* (as is  $(F, F)$ )



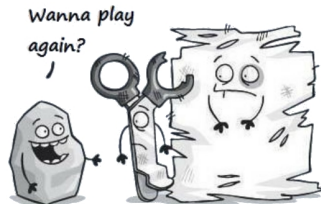
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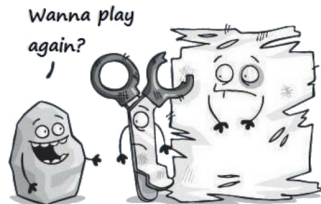
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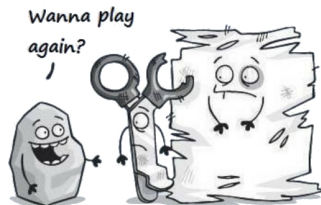
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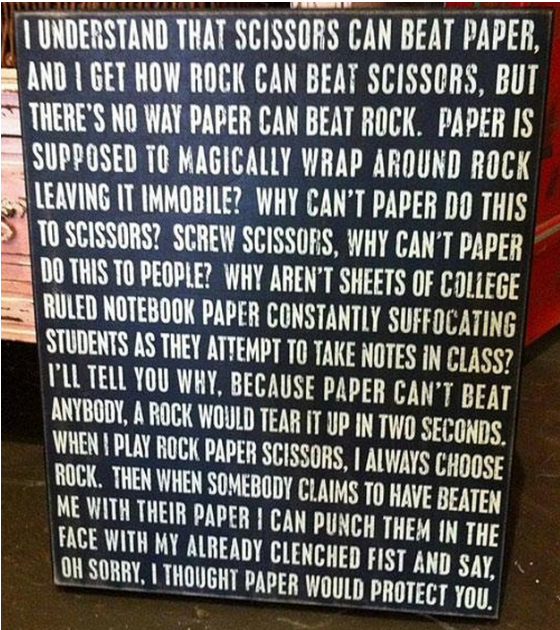
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Use *mixed strategies*: Each player plays each pure strategy with probability  $1/3$ . The expected payoff of each player is 0 (even if one of the players changes his strategy, he still gets 0!).

## Philosophical Issues in Games



I UNDERSTAND THAT SCISSORS CAN BEAT PAPER,  
AND I GET HOW ROCK CAN BEAT SCISSORS, BUT  
THERE'S NO WAY PAPER CAN BEAT ROCK. PAPER IS  
SUPPOSED TO MAGICALLY WRAP AROUND ROCK  
LEAVING IT IMMOBILE? WHY CAN'T PAPER DO THIS  
TO SCISSORS? SCREW SCISSORS, WHY CAN'T PAPER  
DO THIS TO PEOPLE? WHY AREN'T SHEETS OF COLLEGE  
RULED NOTEBOOK PAPER CONSTANTLY SUFFOCATING  
STUDENTS AS THEY ATTEMPT TO TAKE NOTES IN CLASS?  
I'LL TELL YOU WHY, BECAUSE PAPER CAN'T BEAT  
ANYBODY, A ROCK WOULD TEAR IT UP IN TWO SECONDS.  
WHEN I PLAY ROCK PAPER SCISSORS, I ALWAYS CHOOSE  
ROCK. THEN WHEN SOMEBODY CLAIMS TO HAVE BEATEN  
ME WITH THEIR PAPER I CAN PUNCH THEM IN THE  
FACE WITH MY ALREADY CLENCHED FIST AND SAY,  
OH SORRY, I THOUGHT PAPER WOULD PROTECT YOU.

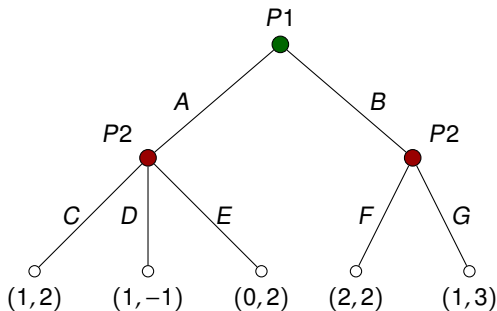
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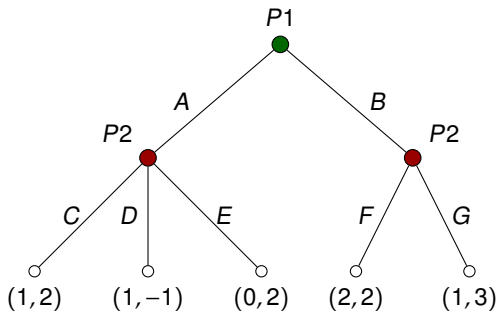
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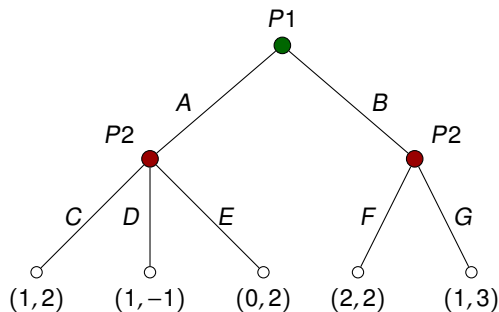
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How to "solve" such games?

What is their relationship to the strategic form games?

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Some decisions in the game tree may be by chance and controlled by neither player (e.g. Poker, Backgammon, etc.)

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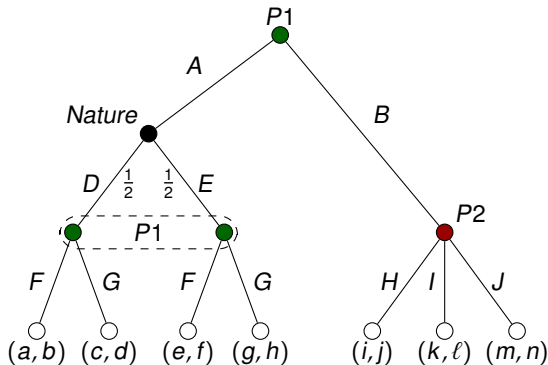
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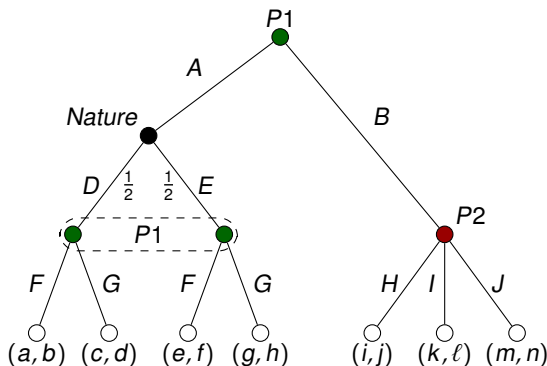
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Again, how to solve such games?

# Games of Incomplete Information

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$$u_1(b_1, b_2) = \begin{cases} v_1 - b_1 & b_1 > b_2 \\ \frac{1}{2}(v_1 - b_1) & b_1 = b_2 \\ 0 & b_1 < b_2 \end{cases}$$

Here  $v_1$  is the private value that player 1 assigns to the item and so the player 2 **does not know**  $u_1$ .

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How to deal with such a game? Assume the “worst” private value?  
What if we have a partial knowledge about the private values?

# Inefficiency of Equilibria

In Prisoner's Dilemma, the selfish behavior of suspects (the Nash equilibrium) results in somewhat worse than ideal situation.

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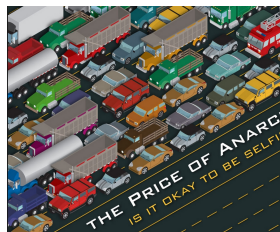
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*Price of Anarchy* is the maximum ratio between values of equilibria and the value of an optimal solution.

# Inefficiency of Equilibria – Selfish Routing

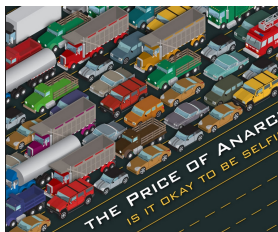
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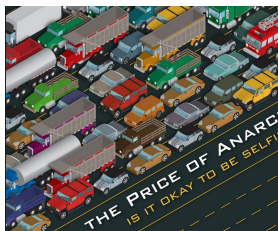




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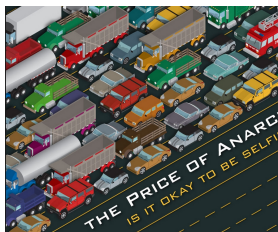


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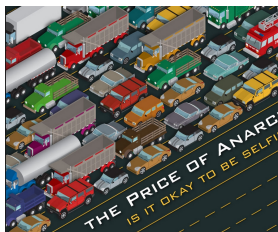
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Problem: Bound the price of anarchy over all routing games?



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- ▶ Games in Logic: modal and temporal logics, Ehrenfeucht-Fraisse games, etc.

Games, the Internet and E-commerce: An extremely active research area at the intersection of CS and Economics

Basic idea: “The internet is a HUGE experiment in interaction between agents (both human and automated)”

How do we set up the rules of this game to harness “socially optimal” results?

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- ▶ Remaining time will be devoted to selected topics from extensive form games, games on graphs etc.

# Static Games of Complete Information

## Strategic-Form Games

### Solution concepts

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Proceed in two steps:

1. Players *simultaneously and independently* choose their *strategies*. This means that players play without observing strategies chosen by other players.

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## Definition 1

A fact  $E$  is a *common knowledge* among players  $\{1, \dots, n\}$  if for every sequence  $i_1, \dots, i_k \in \{1, \dots, n\}$  we have that  $i_1$  knows that  $i_2$  knows that ...  $i_{k-1}$  knows that  $i_k$  knows  $E$ .

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The goal of each player is to maximize his payoff (and this fact is a common knowledge).

# Strategic-Form Games

To formally represent static games of complete information we define *strategic-form games*.

## Definition 2

A game in **strategic-form** (or normal-form) is an ordered triple  $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ , in which:

- ▶  $N = \{1, 2, \dots, n\}$  is a finite set of *players*.
- ▶  $S_i$  is a set of (*pure*) *strategies* of player  $i$ , for every  $i \in N$ .

A *strategy profile* is a vector of strategies of all players  $(s_1, \dots, s_n) \in S_1 \times \dots \times S_n$ .

We denote the set of all strategy profiles by  $S = S_1 \times \dots \times S_n$ .

- ▶  $u_i : S \rightarrow \mathbb{R}$  is a function associating each strategy profile  $s = (s_1, \dots, s_n) \in S$  with the *payoff*  $u_i(s)$  to player  $i$ , for every player  $i \in N$ .



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A *zero-sum* game  $G$  is one in which for all  $s = (s_1, \dots, s_n) \in S$  we have  $u_1(s) + u_2(s) + \dots + u_n(s) = 0$ .

## Example: Prisoner's Dilemma

- ▶  $N = \{1, 2\}$
- ▶  $S_1 = S_2 = \{S, C\}$
- ▶  $u_1, u_2$  are defined as follows:
  - ▶  $u_1(C, C) = -5, u_1(C, S) = 0, u_1(S, C) = -20,$   
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We usually write payoffs in the following form:

	C	S
C	-5, -5	0, -20
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or as two matrices:

	C	S
C	-5	0
S	-20	-1

	C	S
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## Example: Cournot Duopoly

- ▶ Two identical firms, players 1 and 2, produce some good.  
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Strategic-form game model  $(N, (S_i)_{i \in N}, (u_i)_{i \in N})$

- ▶  $N = \{1, 2\}$
- ▶  $S_i = [0, \infty)$
- ▶  $u_1(q_1, q_2) = q_1(\kappa - q_1 - q_2) - q_1 c_1$   
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## Example 4

Nash equilibrium is a solution concept. That is, we “solve” games by finding Nash equilibria and declare them to be reasonable outcomes.

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4. **Self-enforcement**: Any prediction (or equilibrium) of a solution concept must be *self-enforcing*.

Here 4. implies non-cooperative game theory: Each player is in control of his actions, and he will stick to an action only if he finds it to be in his best interest.

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For now, let us concentrate on

**pure strategies only!**

I.e., no mixed strategies are allowed. We will generalize to mixed setting later.



- ▶ Let  $N = \{1, \dots, n\}$  be a finite set and for each  $i \in N$  let  $X_i$  be a set. Let  $X := \prod_{i \in N} X_i = \{(x_1, \dots, x_n) \mid x_j \in X_j, j \in N\}$ .

- ▶ For  $i \in N$  we define  $X_{-i} := \prod_{j \neq i} X_j$ , i.e.,

$$X_{-i} = \{(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \mid x_j \in X_j, \forall j \neq i\}$$

- ▶ An element of  $X_{-i}$  will be denoted by

$$x_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$$

We slightly abuse notation and write  $(x_i, x_{-i})$  to denote  $(x_1, \dots, x_i, \dots, x_n) \in X$ .

# Strict Dominance in Pure Strategies

## Definition 5

Let  $s_i, s'_i \in S_i$  be strategies of player  $i$ . Then  $s'_i$  is *strictly dominated* by  $s_i$  (write  $s_i \succ s'_i$ ) if for any possible combination of the other players' strategies,  $s_{-i} \in S_{-i}$ , we have

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## Claim 1

*An intelligent and rational player will never play a strictly dominated strategy.*

Clearly, intelligence implies that the player should recognize dominated strategies, rationality implies that the player will avoid playing them.

# Strictly Dominant Strategy Equilibrium in Pure Str.

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$s_i \in S_i$  is *strictly dominant* if every other pure strategy of player  $i$  is strictly dominated by  $s_i$ .

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## Corollary 8

*If the strictly dominant strategy equilibrium exists, it is unique and rational players will play it.*

# Examples

In the Prisoner's dilemma:

	$C$	$S$
$C$	$-5, -5$	$0, -20$
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In the Battle of Sexes:

	<i>O</i>	<i>F</i>
<i>O</i>	2, 1	0, 0
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no strictly dominant strategies exist.

# Examples

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	<i>C</i>	<i>S</i>
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<i>S</i>	-20, 0	-1, -1

$(C, C)$  is the strictly dominant strategy equilibrium.

In the Battle of Sexes:

	<i>O</i>	<i>F</i>
<i>O</i>	2, 1	0, 0
<i>F</i>	0, 0	1, 2

no strictly dominant strategies exist.

# Indiana Jones and the Last Crusade

(Taken from Dixit & Nalebuff's "The Art of Strategy" and a lecture of Robert Marks)

Indiana Jones, his father, and the Nazis have all converged at the site of the Holy Grail. The two Joneses refuse to help the Nazis reach the last step. So the Nazis shoot Indiana's dad. Only the healing power of the Holy Grail can save the senior Dr. Jones from his mortal wound. Suitably motivated, Indiana leads the way to the Holy Grail. But there is one final challenge. He must choose between literally scores of chalices, only one of which is the cup of Christ. While the right cup brings eternal life, the wrong choice is fatal. The Nazi leader impatiently chooses a beautiful gold chalice, drinks the holy water, and dies from the sudden death that follows from the wrong choice. Indiana picks a wooden chalice, the cup of a carpenter. Exclaiming "There's only one way to find out" he dips the chalice into the font and drinks what he hopes is the cup of life. Upon discovering that he has chosen wisely, Indiana brings the cup to his father and the water heals the mortal wound.

# Indiana Jones and the Last Crusade (cont.)

## Indy Goofed

- ▶ Although this scene adds excitement, it is somewhat embarrassing that such a distinguished professor as Dr. Indiana Jones would overlook his dominant strategy.
- ▶ He should have given the water to his father without testing it first.
  - ▶ If Indiana has chosen the right cup, his father is still saved.
  - ▶ If Indiana has chosen the wrong cup, then his father dies but Indiana is spared.
- ▶ Testing the cup before giving it to his father doesn't help, since if Indiana has made the wrong choice, there is no second chance – Indiana dies from the water and his father dies from the wound.



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Thus everyone knows, that nobody will play strictly dominated strategies in the smaller game (and such strategies may indeed exist).

Because it is a common knowledge that all players will perform this kind of reasoning again, the process can continue until no more strictly dominated strategies can be eliminated.

The previous reasoning yields the **Iterated Elimination of Strictly Dominated Strategies (IESDS)**:

Define a sequence  $D_i^0, D_i^1, D_i^2, \dots$  of strategy sets of player  $i$ .  
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**Remark:** If all  $S_i$  are *finite*, then in 2. we may remove only some of the strictly dominated strategies (not necessarily all). The result is *not* affected by the order of elimination since strictly dominated strategies remain strictly dominated even after removing some other strictly dominated strategies.

# IESDS Examples

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all strategies survive all rounds (i.e. IESDS  $\equiv$  anything may happen, sorry)

## A Bit More Interesting Example

	<i>L</i>	<i>C</i>	<i>R</i>
<i>L</i>	4, 3	5, 1	6, 2
<i>C</i>	2, 1	8, 4	3, 6
<i>R</i>	3, 0	9, 6	2, 8

IESDS on greenboard!

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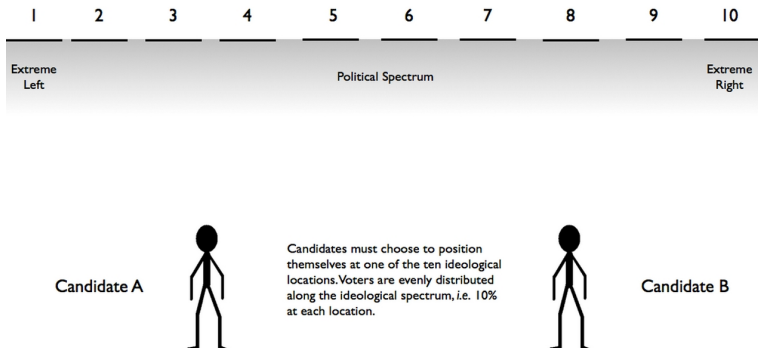
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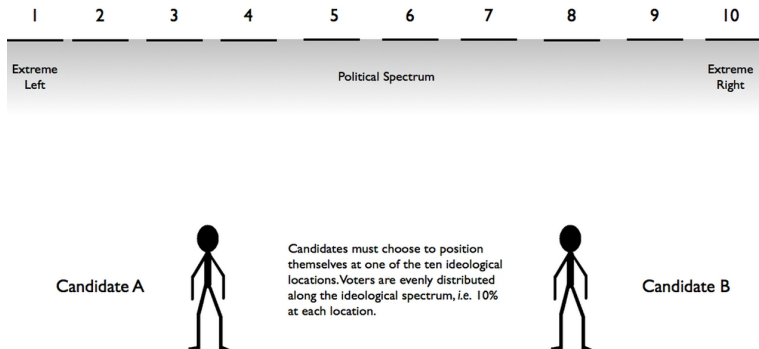
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- ▶ Payoff: The number of voters for the candidate, each candidate (selfishly) strives to maximize this number

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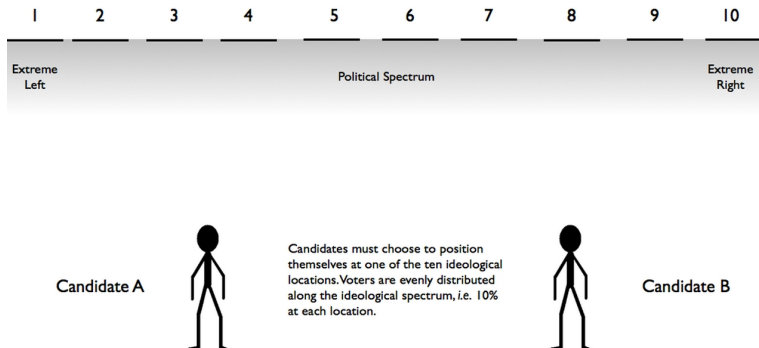


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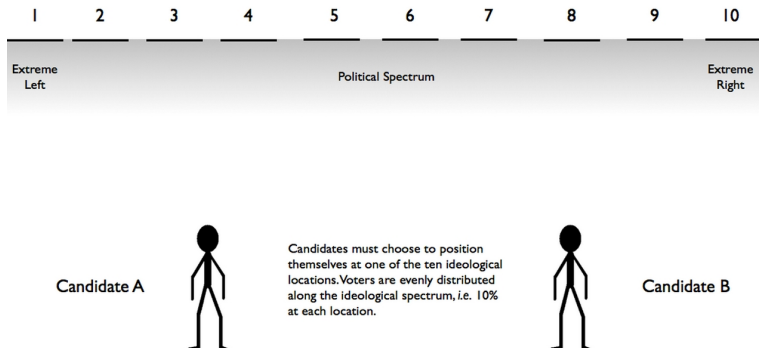
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- ▶ ...
- ▶ only 5, 6 survive IESDS

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Let us formalize this type of reasoning ....

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A rational player never plays any strategy that is never best response.

# Best Response vs Strict Dominance

## Proposition 1

*If  $s_i$  is strictly dominated for player  $i$ , then it is never best response.*



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The opposite does not have to be true in pure strategies:

	X	Y
A	1, 1	1, 1
B	2, 1	0, 1
C	0, 1	2, 1

Here A is never best response but is strictly dominated neither by B, nor by C.

# Elimination of Stupid Strategies = Rationalizability

Using similar iterated reasoning as for IESDS, strategies that are never best response can be iteratively eliminated.

Define a sequence  $R_i^0, R_i^1, R_i^2, \dots$  of strategy sets of player  $i$ .  
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3. Let  $k := k + 1$  and go to 2.

We say that  $s_i \in S_i$  is **rationalizable** if  $s_i \in R_i^k$  for all  $k = 0, 1, 2, \dots$

## Definition 13

A strategy profile  $s = (s_1, \dots, s_n) \in S$  is a **rationalizable equilibrium** if each  $s_i$  is rationalizable.

We say that a game is **solvable by rationalizability** if it has a unique rationalizable equilibrium.

(Warning: For some reasons, rationalizable strategies are almost always defined using mixed strategies!)



# Rationalizability Examples

In the Prisoner's dilemma:

	$C$	$S$
$C$	$-5, -5$	$0, -20$
$S$	$-20, 0$	$-1, -1$

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In the Battle of Sexes:

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all strategies are rationalizable.

# Cournot Duopoly

$$G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$$

- ▶  $N = \{1, 2\}$
- ▶  $S_i = [0, \infty)$
- ▶  $u_1(q_1, q_2) = q_1(\kappa - q_1 - q_2) - q_1 c_1 = (\kappa - c_1)q_1 - q_1^2 - q_1 q_2$   
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Assume for simplicity that  $c_1 = c_2 = c$  and denote  $\theta = \kappa - c$ .

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What is a best response of player 1 to a given  $q_2$  ?

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Since  $q_2 \geq 0$ , we obtain that  $q_1$  is never best response iff  $q_1 > \theta/2$ .  
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Thus  $R_1^1 = R_2^1 = [0, \theta/2]$ .



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Now, in  $G_{Rat}^1$ , we still have that  $q_1 = (\theta - q_2)/2$  is the best response to  $q_2$ , and  $q_2 = (\theta - q_1)/2$  the best resp. to  $q_1$

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Thus  $R_1^2 = R_2^2 = [\theta/4, \theta/2]$ .

....

# Cournot Duopoly (cont.)

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In general, after  $2k$  iterations we have  $R_i^{2k} = R_i^{2k} = [\ell_k, r_k]$  where

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Solving the recurrence we obtain

- ▶  $\ell_k = \theta/3 - \left(\frac{1}{4}\right)^k \theta/3$
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Hence,  $\lim_{k \rightarrow \infty} \ell_k = \lim_{k \rightarrow \infty} r_k = \theta/3$  and thus  $(\theta/3, \theta/3)$  is the only rationalizable equilibrium.

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Are  $q_i = \theta/3$  the best outcomes possible?

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---

Are  $q_i = \theta/3$  the best outcomes possible? NO!

$$u_1(\theta/3, \theta/3) = u_2(\theta/3, \theta/3) = \theta^2/9$$

but

$$u_1(\theta/4, \theta/4) = u_2(\theta/4, \theta/4) = \theta^2/8$$



# IESDS vs Rationalizability in Pure Strategies

## Theorem 14

*Assume that  $S$  is finite. Then for all  $k$  we have that  $R_i^k \subseteq D_i^k$ . That is, in particular, all rationalizable strategies survive IESDS.*

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The opposite inclusion does not have to be true in pure strategies:

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B	2, 1	0, 1
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# Proof of Theorem 14

## Claim

If  $s_i$  is a best response to  $s_{-i}$  in  $G_{Rat}^k$ , then  $s_i$  is a best response to  $s_{-i}$  in  $G$ .

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Then  $s'_i \in G_{Rat}^{k+1}$  since  $s'_i$  is *not* eliminated from  $G_{Rat}^k$ .

However, since  $s_i$  is a best response to  $s_{-i}$  in  $G_{Rat}^{k+1}$ , we get  $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$ .

Thus  $s_i$  is a best response to  $s_{-i}$  in  $G_{Rat}^k$ .

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Assume that the claim is true for some  $k$  and that  $s_i$  is a best response to  $s_{-i}$  in  $G_{Rat}^{k+1}$ . Let  $s'_i$  be a best response to  $s_{-i}$  in  $G_{Rat}^k$ .

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By induction hypothesis,  $s_i$  is a best response to  $s_{-i}$  in  $G$  and the claim has been proved.

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**Keep in mind:** If  $s_i$  is a best response to  $s_{-i}$  in  $G_{Rat}^k$ , then  $s_i$  is a best response to  $s_{-i}$  in  $G$ .

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For  $k = 0$  we have that  $R_i^0 = S_i = D_i^0$  by definition.

Assume that  $R_i^k \subseteq D_i^k$  for some  $k \geq 0$  and prove that  $R_i^{k+1} \subseteq D_i^{k+1}$ .

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Let  $s_i \in R_i^{k+1}$ . Then there must be  $s_{-i} \in R_{-i}^k$  such that

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(This follows from the fact that  $s_i$  has not been eliminated in  $G_{Rat}^k$ .)

By the claim,  $s_i$  is a best response to  $s_{-i}$  in  $G$  as well!

By induction hypothesis,  $s_i \in R_i^{k+1} \subseteq R_i^k \subseteq D_i^k$  and  $s_{-i} \in R_{-i}^k \subseteq D_{-i}^k$ .

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**Keep in mind:** If  $s_i$  is a best response to  $s_{-i}$  in  $G_{Rat}^k$ , then  $s_i$  is a best response to  $s_{-i}$  in  $G$ .

Now we prove  $R_i^k \subseteq D_i^k$  for all players  $i$  by induction on  $k$ .

For  $k = 0$  we have that  $R_i^0 = S_i = D_i^0$  by definition.

Assume that  $R_i^k \subseteq D_i^k$  for some  $k \geq 0$  and prove that  $R_i^{k+1} \subseteq D_i^{k+1}$ .

Let  $s_i \in R_i^{k+1}$ . Then there must be  $s_{-i} \in R_{-i}^k$  such that

$s_i$  is a best response to  $s_{-i}$  in  $G_{Rat}^k$

(This follows from the fact that  $s_i$  has not been eliminated in  $G_{Rat}^k$ .)

By the claim,  $s_i$  is a best response to  $s_{-i}$  in  $G$  as well!

By induction hypothesis,  $s_i \in R_i^{k+1} \subseteq R_i^k \subseteq D_i^k$  and  $s_{-i} \in R_{-i}^k \subseteq D_{-i}^k$ .

However, then  $s_i$  is a best response to  $s_{-i}$  in  $G_{DS}^k$ .

(This follows from the fact that the “best response” relationship of  $s_i$  and  $s_{-i}$  is preserved by removing arbitrarily many other strategies.)



# Proof of Theorem 14

**Keep in mind:** If  $s_i$  is a best response to  $s_{-i}$  in  $G_{Rat}^k$ , then  $s_i$  is a best response to  $s_{-i}$  in  $G$ .

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Thus  $s_i$  is not strictly dominated in  $G_{DS}^k$  and  $s_i \in D_i^{k+1}$ . □

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Criticism of previous approaches:

- ▶ Strictly dominant strategy equilibria often do not exist
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But are all strategy profiles really equally reasonable?

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$(O, O)$  can be obtained as a profile where each player plays the best response to his belief and the **beliefs are correct**.

# Nash Equilibrium

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A usual definition is following:

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A pure-strategy profile  $s^* = (s_1^*, \dots, s_n^*) \in S$  is a (pure) Nash equilibrium if  $s_i^*$  is a best response to  $s_{-i}^*$  for each  $i \in N$ , that is

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Note that this definition is equivalent to the previous one in the sense that  $s_{-i}^*$  may be considered as the (consistent) belief of player  $i$  to which he plays a best response  $s_i^*$

# Nash Equilibria Examples

In the Prisoner's dilemma:

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<i>C</i>	-5, -5	0, -20
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In Cournot Duopoly,  $(\theta/3, \theta/3)$  is the only Nash equilibrium.

(Best response relations:  $q_1 = (\theta - q_2)/2$  and  $q_2 = (\theta - q_1)/2$  are both satisfied only by  $q_1 = q_2 = \theta/3$ )

# Example: Stag Hunt

Story:

- ▶ Two (in some versions more than two) hunters, players 1 and 2, can each choose to hunt
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This is supposed to explain that in real world there are societies that have similar endowments, access to technology and physical environment but have very different achievements, all because of self-fulfilling beliefs (or *norms* of behavior).



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So it seems to be rational to expect  $(H, H)$  (?)

# Nash Equilibria vs Previous Concepts

## Theorem 16

1. *If  $s^*$  is a strictly dominant strategy equilibrium, then it is the unique Nash equilibrium.*
2. *Each Nash equilibrium is rationalizable and survives IESDS.*
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## Corollary 17

*Assume that  $S$  is finite. If rationalizability or IESDS result in a unique strategy profile, then this profile is a Nash equilibrium.*

# Interpretations of Nash Equilibria

Except the two definitions, usual interpretations are following:

- ▶ When the goal is to give advice to all of the players in a game (i.e., to advise each player what strategy to choose), any advice that was not an equilibrium would have the unsettling property that there would always be some player for whom the advice was bad, in the sense that, if all other players followed the parts of the advice directed to them, it would be better for some player to do differently than he was advised. If the advice is an equilibrium, however, this will not be the case, because the advice to each player is the best response to the advice given to the other players.

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- ▶ When the goal is prediction rather than prescription, a Nash equilibrium can also be interpreted as a potential stable point of a dynamic adjustment process in which individuals adjust their behavior to that of the other players in the game, searching for strategy choices that will give them better results.

# Static Games of Complete Information

## Mixed Strategies



## Let's Mix It

As pointed out before, neither of the solution concepts has to exist in pure strategies

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**Example:** Rock-Paper-sCissors

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How to solve this?

Let the players randomize their choice of pure strategies ....

# Probability Distributions

## Definition 18

Let  $A$  be a finite set. A *probability distribution over  $A$*  is a function  $\sigma : A \rightarrow [0, 1]$  such that  $\sum_{a \in A} \sigma(a) = 1$ .



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## Example 19

Consider  $A = \{a, b, c\}$  and a function  $\sigma : A \rightarrow [0, 1]$  such that  $\sigma(a) = \frac{1}{4}$ ,  $\sigma(b) = \frac{3}{4}$ , and  $\sigma(c) = 0$ . Then  $\sigma \in \Delta(A)$ .

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For example, in rock-paper-scissors, the pure strategy  $R$  corresponds

to  $\sigma_i$  which satisfies  $\sigma_i(X) = \begin{cases} 1 & X = R \\ 0 & \text{otherwise} \end{cases}$



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Thus for  $s = (s_1, s_2) \in S = S_1 \times S_2$  we have that

$$\sigma(s) := \sigma_1(s_1) \cdot \sigma_2(s_2)$$

is the probability that the players randomly select the pure strategy profile  $s$  according to the mixed strategy profile  $\sigma$ .

(We abuse notation a bit here:  $\sigma$  denotes two things, a vector of mixed strategies as well as a probability distribution on  $S$ )

# Mixed Strategies – Example

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$C$	-1,1	1,-1	0,0

## Mixed Strategies – Example

	$R$	$P$	$C$
$R$	0,0	-1,1	1,-1
$P$	1,-1	0,0	-1,1
$C$	-1,1	1,-1	0,0

An example of a mixed strategy  $\sigma_1$ :  $\sigma_1(R) = \frac{1}{2}$ ,  $\sigma_1(P) = \frac{1}{3}$ ,  $\sigma_1(C) = \frac{1}{6}$ .

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Sometimes we write  $\sigma_1$  as  $(\frac{1}{2}(R), \frac{1}{3}(P), \frac{1}{6}(C))$ , or only  $(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$  if the order of pure strategies is fixed.

## Mixed Strategies – Example

	$R$	$P$	$C$
$R$	0, 0	-1, 1	1, -1
$P$	1, -1	0, 0	-1, 1
$C$	-1, 1	1, -1	0, 0

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Consider a mixed strategy profile  $(\sigma_1, \sigma_2)$  where  $\sigma_1 = (\frac{1}{2}(R), \frac{1}{3}(P), \frac{1}{6}(C))$  and  $\sigma_2 = (\frac{1}{3}(R), \frac{2}{3}(P), 0(C))$ .

## Mixed Strategies – Example

	$R$	$P$	$C$
$R$	0,0	-1,1	1,-1
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Consider a mixed strategy profile  $(\sigma_1, \sigma_2)$  where  $\sigma_1 = (\frac{1}{2}(R), \frac{1}{3}(P), \frac{1}{6}(C))$  and  $\sigma_2 = (\frac{1}{3}(R), \frac{2}{3}(P), 0(C))$ .

Then the probability  $\sigma(R, P)$  that the pure strategy profile  $(R, P)$  will be played by players playing the mixed profile  $(\sigma_1, \sigma_2)$  is

$$\sigma_1(R) \cdot \sigma_2(P) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$$



## Expected Payoff

... but now what is the suitable notion of payoff?

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... but now what is the suitable notion of payoff?

## Definition 21

The *expected payoff* of player  $i$  under a mixed strategy profile  $\sigma \in \Sigma$  is

$$u_i(\sigma) := \sum_{s \in S} \sigma(s) u_i(s) \quad \left( = \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} \sigma_1(s_1) \cdot \sigma_2(s_2) \cdot u_i(s_1, s_2) \right)$$

I.e., it is the "weighted average" of what player  $i$  wins under each pure strategy profile  $s$ , weighted by the probability of that profile.

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I.e., it is the "weighted average" of what player  $i$  wins under each pure strategy profile  $s$ , weighted by the probability of that profile.

**Assumption:** Every rational player strives to maximize his own expected payoff.

(This assumption is not always completely convincing ...)

# Expected Payoff – Example

Matching Pennies:

	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

Each player secretly turns a penny to heads or tails, and then they reveal their choices simultaneously. If the pennies match, player 1 (row) wins, if they do not match, player 2 (column) wins.

Consider  $\sigma_1 = (\frac{1}{3}(H), \frac{2}{3}(T))$  and  $\sigma_2 = (\frac{1}{4}(H), \frac{3}{4}(T))$

$$\begin{aligned} u_1(\sigma_1, \sigma_2) &= \sum_{(X,Y) \in \{H,T\}^2} \sigma_1(X) \sigma_2(Y) u_1(X, Y) \\ &= \frac{1}{3} \frac{1}{4} 1 + \frac{1}{3} \frac{3}{4} (-1) + \frac{2}{3} \frac{1}{4} (-1) + \frac{2}{3} \frac{3}{4} 1 = \frac{1}{6} \end{aligned}$$

$$\begin{aligned} u_2(\sigma_1, \sigma_2) &= \sum_{(X,Y) \in \{H,T\}^2} \sigma_1(X) \sigma_2(Y) u_2(X, Y) \\ &= \frac{1}{3} \frac{1}{4} (-1) + \frac{1}{3} \frac{3}{4} 1 + \frac{2}{3} \frac{1}{4} 1 + \frac{2}{3} \frac{3}{4} (-1) = -\frac{1}{6} \end{aligned}$$

# Solution Concepts

We revisit the following solution concepts in mixed strategies:

- ▶ strict dominant strategy equilibrium
- ▶ IESDS equilibrium
- ▶ rationalizable equilibria
- ▶ Nash equilibria

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mixed strategy.

# Solution Concepts

We revisit the following solution concepts in mixed strategies:

- ▶ strict dominant strategy equilibrium
- ▶ IESDS equilibrium
- ▶ rationalizable equilibria
- ▶ Nash equilibria

From now on, when I say a *strategy* I implicitly mean a  
**mixed strategy.**

In order to deal with efficiency issues we assume that the size of the game  $G$  is defined by  $|G| := |N| + \sum_{i \in N} |S_i| + \sum_{i \in N} |u_i|$  where  $|u_i| = \sum_{s \in S} |u_i(s)|$  and  $|u_i(s)|$  is the length of a binary encoding of  $u_i(s)$  (we assume that rational numbers are encoded as quotients of two binary integers)

Note that, in particular,  $|G| > |S|$ .

# Strict Dominance in Mixed Strategies

## Definition 22

Let  $\sigma_1, \sigma'_1 \in \Sigma_1$  be (mixed) strategies of player 1. Then  $\sigma'_1$  is *strictly dominated* by  $\sigma_1$  (write  $\sigma'_1 < \sigma_1$ ) if

$$u_1(\sigma_1, s_2) > u_1(\sigma'_1, s_2) \quad \text{for all } s_2 \in S_2$$

(Symmetrically for player 2.)

**Comment:** The above condition is equivalent to

$$u_1(\sigma_1, \sigma_2) > u_1(\sigma'_1, \sigma_2) \quad \text{for all strategies } \sigma_2 \in \Sigma_2$$



# Strict Dominance in Mixed Strategies

## Example 23

	<i>X</i>	<i>Y</i>
<i>A</i>	3	0
<i>B</i>	0	3
<i>C</i>	1	1

Is there a strictly dominated strategy?

# Strict Dominance in Mixed Strategies

## Example 23

	X	Y
A	3	0
B	0	3
C	1	1

Is there a strictly dominated strategy?

**Question:** Is there a game with at least one strictly dominated strategy but without strictly dominated *pure* strategies?

# Strictly Dominant Strategy Equilibrium

## Definition 24

$\sigma_i \in \Sigma_i$  is *strictly dominant* if every other mixed strategy of player  $i$  is strictly dominated by  $\sigma_i$ .

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A strategy profile  $\sigma \in \Sigma$  is a *strictly dominant strategy equilibrium* if  $\sigma_i \in \Sigma_i$  is strictly dominant for all  $i \in N$ .

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A strategy profile  $\sigma \in \Sigma$  is a *strictly dominant strategy equilibrium* if  $\sigma_i \in \Sigma_i$  is strictly dominant for all  $i \in N$ .

## Proposition 2

*If the strictly dominant strategy equilibrium exists, it is unique, all its strategies are pure, and rational players will play it.*

To compute the strictly dominant strategy equilibrium, it is sufficient to consider only pure strategies (greenboard).

# IESDS in Mixed Strategies

Define a sequence  $D_i^0, D_i^1, D_i^2, \dots$  of strategy sets of player  $i$ .  
(Denote by  $G_{DS}^k$  the game obtained from  $G$  by restricting the pure strategy sets to  $D_i^k, i \in N$ .)

1. Initialize  $k = 0$  and  $D_i^0 = S_i$  for each  $i \in N$ .
2. For all players  $i \in N$ : Let  $D_i^{k+1}$  be the set of all pure strategies of  $D_i^k$  that are *not* strictly dominated in  $G_{DS}^k$  by *mixed strategies*.
3. Let  $k := k + 1$  and go to 2.

We say that  $s_i \in S_i$  *survives IESDS* if  $s_i \in D_i^k$  for all  $k = 0, 1, 2, \dots$

## Definition 26

A strategy profile  $s = (s_1, s_2) \in S$  is an *IESDS equilibrium* if both  $s_1$  and  $s_2$  survive IESDS.

Each  $D_i^{k+1}$  can be computed in polynomial time using *linear programming*.

## IESDS in Mixed Strategie – Example

	<i>X</i>	<i>Y</i>
<i>A</i>	3	0
<i>B</i>	0	3
<i>C</i>	1	1

Let us have a look at the first iteration of IESDS.

## IESDS in Mixed Strategie – Example

	$X$	$Y$
$A$	3	0
$B$	0	3
$C$	1	1

Let us have a look at the first iteration of IESDS.

Observe that  $A, B$  are not strictly dominated by any mixed strategy.



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Let us have a look at the first iteration of IESDS.

Observe that  $A, B$  are not strictly dominated by any mixed strategy.

Let us construct a set of constraints on mixed strategies (possibly) strictly dominating  $C$ :

$$3x_A + 0x_B + x_C > 1$$

Row's payoff against X

$$0x_A + 3x_B + x_C > 1$$

Row's payoff against Y

$$x_A, x_B, x_C \geq 0$$

$$x_A + x_B + x_C = 1$$

$x$ 's must make a distribution

# IESDS in Mixed Strategie – Example

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Row's payoff against  $Y$

$$x_A, x_B, x_C \geq 0$$

$$x_A + x_B + x_C = 1$$

$x$ 's must make a distribution

How to solve this?

# Intermezzo: Linear Programming

Linear programming is a technique for optimization of a linear objective function, subject to linear (non-strict) inequality constraints.

Formally, a linear program in so called *canonical form* looks like this:

$$\text{maximize } \sum_{j=1}^m c_j x_j \quad (\text{objective function})$$

$$\text{subject to } \sum_{j=1}^m a_{ij} x_j \leq b_i \quad 1 \leq i \leq n \quad (\text{constraints})$$

$$x_j \geq 0 \quad 1 \leq j \leq m$$

Here  $a_{ij}$ ,  $b_k$  and  $c_j$  are real numbers and  $x_j$ 's are real variables.

A *feasible solution* is an assignment of real numbers to the variables  $x_j$ ,  $1 \leq j \leq m$ , so that the *constraints* are satisfied.

An *optimal solution* is a feasible solution which maximizes the *objective function*  $\sum_{j=1}^m c_j x_j$ .

## Intermezzo: Complexity of Linear Programming

We assume that coefficients  $a_{ij}$ ,  $b_k$  and  $c_j$  are encoded in binary (more precisely, as fractions of two integers encoded in binary).

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### **Theorem 27 (Khachiyan, Doklady Akademii Nauk SSSR, 1979)**

*There is an algorithm which for any linear program computes an optimal solution in polynomial time.*

The algorithm uses so called ellipsoid method.

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There exist several advanced linear programming solvers (usually parts of larger optimization packages) implementing various heuristics for solving large scale problems, sensitivity analysis, etc.

For more info see

[http://en.wikipedia.org/wiki/Linear\\_programming#Solvers\\_and\\_scripting\\_.28programming.29\\_languages](http://en.wikipedia.org/wiki/Linear_programming#Solvers_and_scripting_.28programming.29_languages)

# IESDS in Mixed Strategie – Example

	X	Y
A	3	0
B	0	3
C	1	1

The linear program for deciding whether C is strictly dominated: The program maximizes  $y$  under the following constraints:

$$3x_A + 0x_B + x_C \geq 1 + y$$

Row's payoff against X

$$0x_A + 3x_B + x_C \geq 1 + y$$

Row's payoff against Y

$$x_A, x_B, x_C \geq 0$$

$$x_A + x_B + x_C = 1$$

x's must make a distribution

$$y \geq 0$$

Here  $y$  just implements the strict inequality using  $\geq$ , we look for a solution with  $y > 0$ .

The maximum  $y = \frac{1}{2}$  is attained at  $x_A = \frac{1}{2}$  and  $x_B = \frac{1}{2}$ .

Note that in step 2 it is not sufficient to consider pure strategies.  
Consider the following zero sum game:

	X	Y
A	3	0
B	0	3
C	1	1

$C$  is strictly dominated by  $(\sigma_1(A), \sigma_1(B), \sigma_1(C)) = (\frac{1}{2}, \frac{1}{2}, 0)$  but no strategy is strictly dominated in pure strategies.

# Best Response in Mixed Strategies

## Definition 28

A *(mixed) belief* of player 1 is a mixed strategy  $\sigma_2$  of player 2 (and vice versa).

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## Definition 29

$\sigma_1 \in \Sigma_1$  is a *best response* to a belief  $\sigma_2 \in \Sigma_2$  if

$$u_1(\sigma_1, \sigma_2) \geq u_1(s_1, \sigma_2) \quad \text{for all } s_1 \in S_1$$

Denote by  $BR_1(\sigma_2)$  the set of all best responses of player 1.  
(Symmetrically for player 2.)

**Comment:** The above condition is equivalent to

$$u_1(\sigma_1, \sigma_2) \geq u_1(\sigma'_1, \sigma_2) \quad \text{for all } \sigma'_1 \in \Sigma_1$$

## Best Response – Example

Consider a game with the following payoffs of player 1:

	$X$	$Y$
$A$	2	0
$B$	0	2
$C$	1	1

- ▶ Player 1 (row) plays  $\sigma_1 = (a(A), b(B), c(C))$ .
- ▶ Player 2 (column) plays  $(q(X), (1 - q)(Y))$  (we write just  $q$ ).

Compute  $BR_1(q)$ .

# Rationalizability in Mixed Strategies (Two Players)

**Assumption:** *A rational player 1 with a belief  $\sigma_2$  always plays a best response to  $\sigma_2$  (the same for player 2).*

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## Definition 30

A pure strategy  $s_1 \in S_1$  of player 1 is *never best response* if it is not a best response to any belief  $\sigma_2$  (similarly for player 2).

No rational player plays a strategy that is never best response.



# Rationalizability in Mixed Strategies (Two Players)

Define a sequence  $R_i^0, R_i^1, R_i^2, \dots$  of strategy sets of player  $i$ .  
(Denote by  $G_{Rat}^k$  the game obtained from  $G$  by restricting the pure strategy sets to  $R_i^k, i \in N$ .)

1. Initialize  $k = 0$  and  $R_i^0 = S_i$  for each  $i \in N$ .
2. For all players  $i \in N$ : Let  $R_i^{k+1}$  be the set of all strategies of  $R_i^k$  that are *best responses to some (mixed) beliefs* in  $G_{Rat}^k$ .
3. Let  $k := k + 1$  and go to 2.

We say that  $s_i \in S_i$  is *rationalizable* if  $s_i \in R_i^k$  for all  $k = 0, 1, 2, \dots$

## Definition 31

A strategy profile  $s = (s_1, s_2) \in S$  is a *rationalizable equilibrium* if both  $s_1$  and  $s_2$  are rationalizable.

## Rationalizability vs IESDS (Two Players)

	<i>X</i>	<i>Y</i>
<i>A</i>	3	0
<i>B</i>	0	3
<i>C</i>	1	1

What pure strategies of player 1 are strictly dominated?

What pure strategies of player 1 are never best responses?

## Rationalizability vs IESDS (Two Players)

	X	Y
A	3	0
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What pure strategies of player 1 are strictly dominated?

What pure strategies of player 1 are never best responses?

**Observation:** The set of strictly dominated pure strategies coincides with the set of pure never best responses!

# Rationalizability vs IESDS (Two Players)

	X	Y
A	3	0
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C	1	1

What pure strategies of player 1 are strictly dominated?

What pure strategies of player 1 are never best responses?

**Observation:** The set of strictly dominated pure strategies coincides with the set of pure never best responses!

... and this holds in general for two player games:

## Theorem 32

*A pure strategy  $s_1$  of player 1 is never best response to any belief  $\sigma_2$  iff  $s_1$  is strictly dominated by a strategy  $\sigma_1 \in \Sigma_1$  (similarly for player 2).*

It follows that a strategy of  $S_i$  survives IESDS **iff** it is rationalizable.

# Mixed Nash Equilibrium

## Definition 33

A mixed-strategy profile  $\sigma^* = (\sigma_1^*, \sigma_2^*) \in \Sigma$  is a (mixed) Nash equilibrium if  $\sigma_1^*$  is a best response to  $\sigma_2^*$  and  $\sigma_2^*$  is a best response to  $\sigma_1^*$ . That is

$$u_1(\sigma_1^*, \sigma_2^*) \geq u_1(\mathbf{s}_1, \sigma_2^*) \quad \text{for all } \mathbf{s}_1 \in \mathbf{S}_1$$

$$u_2(\sigma_1^*, \sigma_2^*) \geq u_2(\sigma_1^*, \mathbf{s}_2) \quad \text{for all } \mathbf{s}_2 \in \mathbf{S}_2$$

The above condition is equivalent to

$$u_1(\sigma_1^*, \sigma_2^*) \geq u_1(\sigma_1, \sigma_2^*) \quad \text{for all } \sigma_1 \in \Sigma_1$$

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$$u_1(\sigma_1^*, \sigma_2^*) \geq u_1(s_1, \sigma_2^*) \quad \text{for all } s_1 \in S_1$$

$$u_2(\sigma_1^*, \sigma_2^*) \geq u_2(\sigma_1^*, s_2) \quad \text{for all } s_2 \in S_2$$

The above condition is equivalent to

$$u_1(\sigma_1^*, \sigma_2^*) \geq u_1(\sigma_1, \sigma_2^*) \quad \text{for all } \sigma_1 \in \Sigma_1$$

$$u_2(\sigma_1^*, \sigma_2^*) \geq u_2(\sigma_1^*, \sigma_2) \quad \text{for all } \sigma_2 \in \Sigma_2$$

## Theorem 34 (Nash 1950)

*Every finite game in strategic form has a Nash equilibrium.*

This is THE fundamental theorem of game theory.

## Example: Matching Pennies

	$H$	$T$
$H$	$1, -1$	$-1, 1$
$T$	$-1, 1$	$1, -1$

Player 1 (row) plays  $(p(H), (1 - p)(T))$  (we write just  $p$ ) and player 2 (column) plays  $(q(H), (1 - q)(T))$  (we write  $q$ ).

Compute all Nash equilibria.

---

## Example: Matching Pennies

	$H$	$T$
$H$	$1, -1$	$-1, 1$
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Compute all Nash equilibria.

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What are the expected payoffs of playing pure strategies for player 1?

$$u_1(H, q) = 2q - 1 \text{ and } u_1(T, q) = 1 - 2q$$



## Example: Matching Pennies

	$H$	$T$
$H$	$1, -1$	$-1, 1$
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Player 1 (row) plays  $(p(H), (1 - p)(T))$  (we write just  $p$ ) and player 2 (column) plays  $(q(H), (1 - q)(T))$  (we write  $q$ ).

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What are the expected payoffs of playing pure strategies for player 1?

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Then

$$u_1(p, q) = pu_1(H, q) + (1 - p)u_1(T, q) = p(2q - 1) + (1 - p)(1 - 2q).$$

## Example: Matching Pennies

	$H$	$T$
$H$	$1, -1$	$-1, 1$
$T$	$-1, 1$	$1, -1$

Player 1 (row) plays  $(p(H), (1 - p)(T))$  (we write just  $p$ ) and player 2 (column) plays  $(q(H), (1 - q)(T))$  (we write  $q$ ).

Compute all Nash equilibria.

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What are the expected payoffs of playing pure strategies for player 1?

$$u_1(H, q) = 2q - 1 \text{ and } u_1(T, q) = 1 - 2q$$

Then

$$u_1(p, q) = pu_1(H, q) + (1 - p)u_1(T, q) = p(2q - 1) + (1 - p)(1 - 2q).$$

We obtain the best response correspondence  $BR_1$ :

$$BR_1(q) = \begin{cases} T & \text{if } q < \frac{1}{2} \\ p \in [0, 1] & \text{if } q = \frac{1}{2} \\ H & \text{if } q > \frac{1}{2} \end{cases}$$

## Example: Matching Pennies

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Player 1 (row) plays  $(p(H), (1 - p)(T))$  (we write just  $p$ ) and player 2 (column) plays  $(q(H), (1 - q)(T))$  (we write  $q$ ).

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Similarly for player 2 :

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The only "intersection" of  $BR_1$  and  $BR_2$  is the only Nash equilibrium  $\sigma_1 = \sigma_2 = (\frac{1}{2}, \frac{1}{2})$ .

# Computing Mixed Nash Equilibria

## Lemma 35

Every Nash equilibrium  $\sigma^* = (\sigma_1^*, \sigma_2^*) \in \Sigma$  satisfies

- ▶  $u_1(s_1, \sigma_2^*) = u_1(\sigma^*)$  for  $s_1 \in \text{supp}(\sigma_1^*)$
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The latter assumption implies  $u_1(s_1, \sigma_2^*) \leq u_1(\sigma^*)$  for all  $s_1 \in S_1$ .

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Now, if there exists  $s_1 \in \text{supp}(\sigma_1^*) \subseteq S_1$  satisfying  $u_1(s_1, \sigma_2^*) < u_1(\sigma^*)$ , then because  $\sigma_1^*(s_1) > 0$  we have

$$u_1(\sigma^*) = \sum_{s_1 \in S_1} \sigma_1^*(s_1) u_1(s_1, \sigma_2^*) < \sum_{s_1 \in S_1} \sigma_1^*(s_1) u_1(\sigma^*) = u_1(\sigma^*)$$

A contradiction.

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A contradiction.

Thus  $u_1(s_1, \sigma_2^*) = u_1(\sigma^*)$  for all  $s_1 \in \text{supp}(\sigma_1^*)$ .

## Example: Matching Pennies

	$H$	$T$
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Player 1 (row) plays  $(p(H), (1 - p)(T))$  (we write just  $p$ ) and player 2 (column) plays  $(q(H), (1 - q)(T))$  (we write  $q$ ).

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There are no equilibria where only player 1 randomizes:

Indeed, assume that  $(p, H)$  is such an equilibrium. Then by Lemma 35,

$$1 = u_1(H, H) = u_1(T, H) = -1$$

a contradiction. Also,  $(p, T)$  cannot be an equilibrium.

Similarly, there is no NE where only player 2 randomizes.

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By Lemma 35, such Nash equilibria must satisfy:

$$2q - 1 = 1 - 2q \quad \text{and} \quad 1 - 2p = 2p - 1$$

That is  $p = q = \frac{1}{2}$  is the only Nash equilibrium.

## Example: Battle of Sexes

	$O$	$F$
$O$	2, 1	0, 0
$F$	0, 0	1, 2

Player 1 (row) plays  $(p(O), (1 - p)(F))$  (we write just  $p$ ) and player 2 (column) plays  $(q(O), (1 - q)(F))$  (we write  $q$ ).

Compute all Nash equilibria.

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Now assume that

- ▶ player 1 (row) plays  $(p(O), (1 - p)(F))$  (we write just  $p$ ) and
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where  $p, q \in (0, 1)$ .

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where  $p, q \in (0, 1)$ .

By Lemma 35, such Nash equilibria must satisfy:

$$2q = 1 - q \quad \text{and} \quad p = 2(1 - p)$$

This holds only for  $q = \frac{1}{3}$  and  $p = \frac{2}{3}$ .

# An Algorithm?

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Whenever one of the *supports* was non-singleton, we reduced computation of Nash equilibria to *linear equations*.

# Computing Mixed Nash Equilibria

## Lemma 36

Let  $\sigma^* = (\sigma_1^*, \sigma_2^*) \in \Sigma$  be a mixed profile. Assume that there exist  $w_1, w_2 \in \mathbb{R}$  such that

- ▶  $u_1(s_1, \sigma_2^*) = w_1$  for  $s_1 \in \text{supp}(\sigma_1^*)$
- ▶  $u_1(s_1, \sigma_2^*) \leq w_1$  for  $s_1 \notin \text{supp}(\sigma_1^*)$
- ▶  $u_2(\sigma_1^*, s_2) = w_2$  for  $s_2 \in \text{supp}(\sigma_2^*)$
- ▶  $u_2(\sigma_1^*, s_2) \leq w_2$  for  $s_2 \notin \text{supp}(\sigma_2^*)$

Then  $u_1(\sigma^*) = w_1$  and  $u_2(\sigma^*) = w_2$ , and  $\sigma^*$  is a Nash equilibrium.

**Proof.** Consider just the player 1 (for pl. 2 similarly):

$$\begin{aligned} u_1(\sigma^*) &= \sum_{s_1 \in S_1} \sigma^*(s_1) u_1(s_1, \sigma_2^*) = \sum_{s_1 \in \text{supp}(\sigma_1^*)} \sigma^*(s_1) u_1(s_1, \sigma_2^*) \\ &= \sum_{s_1 \in \text{supp}(\sigma_1^*)} \sigma^*(s_1) w_1 = w_1 \sum_{s_1 \in \text{supp}(\sigma_1^*)} \sigma^*(s_1) = w_1 \end{aligned}$$

Now the fact that  $\sigma^*$  is a Nash equilibrium follows from the definition.

# How to Compute Mixed Nash Equilibria?

Every Nash equilibrium  $\sigma^* = (\sigma_1^*, \sigma_2^*)$  can be computed by finding appropriate  $w_1, w_2$  so that

- ▶  $u_1(s_1, \sigma_2^*) = w_1$  for  $s_1 \in \text{supp}(\sigma_1^*)$
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Indeed,

- ▶ by Lemma 36, all  $\sigma^*$  and  $w_1, w_2$  satisfying the above inequalities give a Nash equilibrium  $\sigma^*$  with  $u_1(\sigma^*) = w_1$  and  $u_2(\sigma^*) = w_2$ ,
- ▶ by Lemma 35, for every Nash equilibrium  $\sigma^*$  choosing  $w_1 = u_1(\sigma^*)$  and  $w_2 = u_2(\sigma^*)$  satisfies the above inequalities.

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Suppose that we somehow know the supports  $\text{supp}(\sigma_1^*), \text{supp}(\sigma_2^*)$  for some Nash equilibrium  $\sigma^* = (\sigma_1^*, \sigma_2^*)$  (which itself is unknown to us).

We may consider all  $\sigma_i^*(s_i)$ 's and both  $w_1, w_2$ 's as variables and use the above conditions to design a system of inequalities capturing Nash equilibria with the given support sets  $\text{supp}(\sigma_1^*), \text{supp}(\sigma_2^*)$ .

# Support Enumeration

To simplify notation, assume that for every  $i$  we have  $S_i = \{1, \dots, m_i\}$ .  
Then  $\sigma_i(j)$  is the probability of the pure strategy  $j$  in the mixed strategy  $\sigma_i$ .

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Fix supports  $\text{supp}_i \subseteq S_i$  for every  $i \in \{1, 2\}$  and consider the following system of constraints with variables

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3. For all  $i \in \{1, 2\}$ :  $\sigma_i(1) + \dots + \sigma_i(m_i) = 1$ .
4. For all  $i \in \{1, 2\}$  and all  $k \in \text{supp}_i$ :  $\sigma_i(k) \geq 0$ .
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**Output:** A Nash equilibrium  $\sigma^*$ .

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- ▶ Check if the corresponding system of linear constraints (from the previous slide) has a feasible solution  $\sigma^*, w_1^*, w_2^*$ .
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**Question:** How many possible subsets  $\text{supp}_1, \text{supp}_2$  are there to try?

# Support Enumeration

The constraints are *linear* for two player games!

How to find  $\text{supp}_1$  and  $\text{supp}_2$ ? ... Just guess!

**Input:** A two-player strategic-form game  $G$  with strategy sets  $S_1 = \{1, \dots, m_1\}$  and  $S_2 = \{1, \dots, m_2\}$  and rational payoffs  $u_1, u_2$ .

**Output:** A Nash equilibrium  $\sigma^*$ .

**Algorithm:** For all possible  $\text{supp}_1 \subseteq S_1$  and  $\text{supp}_2 \subseteq S_2$ :

- ▶ Check if the corresponding system of linear constraints (from the previous slide) has a feasible solution  $\sigma^*, w_1^*, w_2^*$ .
- ▶ If so, STOP: the feasible solution  $\sigma^*$  is a Nash equilibrium satisfying  $u_i(\sigma^*) = w_i^*$ .

**Question:** How many possible subsets  $\text{supp}_1, \text{supp}_2$  are there to try?

**Answer:**  $2^{(m_1+m_2)}$

So, unfortunately, the algorithm requires worst-case exponential time.



# Remarks on Support Enumeration

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(There are algorithms for computing (a finite representation of) a set of all feasible solutions of a given linear constraint system.)
- ▶ The algorithm can be used to compute "good" equilibria.

For example, to find a Nash equilibrium maximizing the sum of all expected payoffs (the "social welfare") it suffices to solve the system of constraints while maximizing  $w_1 + w_2$ . More precisely, the algorithm can be modified as follows:

- ▶ Initialize  $W := -\infty$  ( $W$  stores the current maximum welfare)
- ▶ For all possible  $supp_1 \subseteq S_1$  and  $supp_2 \subseteq S_2$ :
  - ▶ Find the maximum value  $\max(w_1 + w_2)$  of  $w_1 + w_2$  so that the constraints are satisfiable (using linear programming).
  - ▶ Put  $W := \max\{W, \max(w_1 + w_2)\}$ .
- ▶ Return  $W$ .

## Remarks on Support Enumeration (Cont.)

Similar trick works for any notion of "good" NE that can be expressed using a linear objective function and (additional) linear constraints in variables  $\sigma_i(j)$  and  $w_i$ .

(e.g., maximize payoff of player 1, minimize payoff of player 2 and keep probability of playing the strategy 1 below 1/2, etc.)

# Complexity Results – (Two Players)

## Theorem 37

*Given a two-player game in strategic form, a mixed Nash equilibrium can be computed in exponential time.*

## Theorem 38

*All the following problems are NP-complete: Given a two-player game in strategic form, does it have*

- 1. a NE in which player 1 has utility at least a given amount  $v$  ?*
- 2. a NE in which the sum of expected payoffs of the two players is at least a given amount  $v$  ?*
- 3. a NE with a support of size greater than a given number?*
- 4. a NE whose support contains a given strategy  $s$  ?*
- 5. a NE whose support does not contain a given strategy  $s$  ?*
- 6. ....*

NP-hardness can be proved using reduction from SAT.

# The Reduction (It's Short and Sweet)

**Definition 4** Let  $\phi$  be a Boolean formula in conjunctive normal form (representing a SAT instance). Let  $V$  be its set of variables (with  $|V| = n$ ),  $L$  the set of corresponding literals (a positive and a negative one for each variable<sup>6</sup>), and  $C$  its set of clauses. The function  $v : L \rightarrow V$  gives the variable corresponding to a literal, e.g.,  $v(x_1) = v(-x_1) = x_1$ . We define  $G_\epsilon(\phi)$  to be the following finite symmetric 2-player game in normal form. Let  $\Sigma = \Sigma_1 = \Sigma_2 = L \cup V \cup C \cup \{f\}$ . Let the utility functions be

- $u_1(l^1, l^2) = u_2(l^2, l^1) = n - 1$  for all  $l^1, l^2 \in L$  with  $l^1 \neq -l^2$ ;
- $u_1(l, -l) = u_2(-l, l) = n - 4$  for all  $l \in L$ ;
- $u_1(l, x) = u_2(x, l) = n - 4$  for all  $l \in L, x \in \Sigma - L - \{f\}$ ;
- $u_1(v, l) = u_2(l, v) = n$  for all  $v \in V, l \in L$  with  $v(l) \neq v$ ;
- $u_1(v, l) = u_2(l, v) = 0$  for all  $v \in V, l \in L$  with  $v(l) = v$ ;
- $u_1(v, x) = u_2(x, v) = n - 4$  for all  $v \in V, x \in \Sigma - L - \{f\}$ ;
- $u_1(c, l) = u_2(l, c) = n$  for all  $c \in C, l \in L$  with  $l \notin c$ ;
- $u_1(c, l) = u_2(l, c) = 0$  for all  $c \in C, l \in L$  with  $l \in c$ ;
- $u_1(c, x) = u_2(x, c) = n - 4$  for all  $c \in C, x \in \Sigma - L - \{f\}$ ;
- $u_1(x, f) = u_2(f, x) = 0$  for all  $x \in \Sigma - \{f\}$ ;
- $u_1(f, f) = u_2(f, f) = \epsilon$ ;
- $u_1(f, x) = u_2(x, f) = n - 1$  for all  $x \in \Sigma - \{f\}$ .

**Theorem 1** If  $(l_1, l_2, \dots, l_n)$  (where  $v(l_i) = x_i$ ) satisfies  $\phi$ , then there is a Nash equilibrium of  $G_\epsilon(\phi)$  where both players play  $l_i$  with probability  $\frac{1}{n}$ , with expected utility  $n - 1$  for each player. The only other Nash equilibrium is the one where both players play  $f$ , and receive expected utility  $\epsilon$  each.

## ... But What is The Exact Complexity of *Computing* Nash Equilibria in Two Player Games?

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Can we do better than FNP (i.e. exponential time)?

In what follows we show that the sample equilibrium problem can be solved in polynomial time for zero-sum two-player games.

(Using a beautiful characterization of all Nash equilibria)

## Definition 39

$\sigma_1^* \in \Sigma_1$  is a *maxmin* strategy of player 1 if

$$\sigma_1^* \in \underset{\sigma_1 \in \Sigma_1}{\text{argmax}} \min_{s_2 \in S_2} u_1(\sigma_1, s_2) \quad (= \underset{\sigma_1 \in \Sigma_1}{\text{argmax}} \min_{\sigma_2 \in \Sigma_2} u_1(\sigma_1, \sigma_2))$$

(Intuitively, a *maxmin* strategy  $\sigma_1^*$  maximizes player 1's worst-case payoff in the situation where player 2 strives to cause the greatest harm to player 1.)

Similarly,  $\sigma_2^* \in \Sigma_2$  is a *maxmin* strategy of player 2 if

$$\sigma_2^* \in \underset{\sigma_2 \in \Sigma_2}{\text{argmax}} \min_{s_1 \in S_1} u_2(s_1, \sigma_2)$$

Which assuming zero-sum games, i.e.  $u_1 = -u_2$ , becomes

$$\sigma_2^* \in \underset{\sigma_2 \in \Sigma_2}{\text{argmin}} \max_{s_1 \in S_1} u_1(s_1, \sigma_2) \quad (= \underset{\sigma_2 \in \Sigma_2}{\text{argmin}} \max_{\sigma_1 \in \Sigma_1} u_1(\sigma_1, \sigma_2))$$

Note the same payoff function for both players!!

# Zero-Sum Games: von Neumann's Theorem

## Theorem 40 (von Neumann)

Assume a two-player **zero-sum** game. Then

$$\max_{\sigma_1 \in \Sigma_1} \min_{s_2 \in S_2} u_1(\sigma_1, s_2) = \min_{\sigma_2 \in \Sigma_2} \max_{s \in S_1} u_1(s, \sigma_2)$$

Moreover,  $\sigma^* = (\sigma_1^*, \sigma_2^*) \in \Sigma$  is a Nash equilibrium **iff** both  $\sigma_1^*$  and  $\sigma_2^*$  are maxmin.

So to compute a Nash equilibrium it suffices to compute (arbitrary) maxmin strategies for both players.

# Zero-Sum Two-Player Games – Computing NE

Assume  $S_1 = \{1, \dots, m_1\}$  and  $S_2 = \{1, \dots, m_2\}$ .

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Consider a linear program with variables  $\sigma_1(1), \dots, \sigma_1(m_1), v$ :

**maximize:**  $v$

**subject to:** 
$$\sum_{k=1}^{m_1} \sigma_1(k) \cdot u_1(k, \ell) \geq v \quad \ell = 1, \dots, m_2$$

$$\sum_{k=1}^{m_1} \sigma_1(k) = 1$$

$$\sigma_1(k) \geq 0 \quad k = 1, \dots, m_1$$

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## Lemma 41

$\sigma_1^* \in \operatorname{argmax}_{\sigma_1 \in \Sigma_1} \min_{\ell \in S_2} u_1(\sigma_1, \ell)$  **iff** assigning  $\sigma_1(k) := \sigma_1^*(k)$  and  $v := \min_{\ell \in S_2} u_1(\sigma_1^*, \ell)$  gives an optimal solution.

# Zero-Sum Two-Player Games – Computing NE

## Summary:

- ▶ We have reduced computation of NE to computation of maxmin strategies for both players.
- ▶ Maxmin strategies can be computed using linear programming in polynomial time.
- ▶ That is, Nash equilibria in zero-sum two-player games can be computed in polynomial time.

# Strategic-Form Games – Conclusion

We have considered *static games of complete information*, i.e., "one-shot" games where the players know exactly what game they are playing.

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We have considered both pure strategy setting and mixed strategy setting.

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We modeled such games using *strategic-form games*.

We have considered both pure strategy setting and mixed strategy setting.

In both cases, we considered four solution concepts:

- ▶ Strictly dominant strategies
- ▶ Iterative elimination of strictly dominated strategies
- ▶ Rationalizability (i.e., iterative elimination of strategies that are never best responses)
- ▶ Nash equilibria

# Strategic-Form Games – Conclusion

In pure strategy setting:

1. Strictly dominant strategy equilibrium survives IESDS, rationalizability and is the unique Nash equilibrium (if it exists)
2. In finite games, rationalizable equilibria survive IESDS, IESDS preserves the set of Nash equilibria
3. In finite games, rationalizability preserves Nash equilibria

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In mixed setting:

1. In finite two player games, IESDS and rationalizability coincide.
2. Strictly dominant strategy equilibrium survives IESDS (rationalizability) and is the unique Nash equilibrium (if it exists)
3. In finite games, IESDS (rationalizability) preserves Nash equilibria

The proofs for 2. and 3. in the mixed setting are similar to corresponding proofs in the pure setting.



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In the mixed setting, linear programming is needed to implement one step of IESDS (rationalizability).
- ▶ Nash equilibria can be computed for two-player games
  - ▶ in polynomial time for zero-sum games  
(using von Neumann's theorem and linear programming)
  - ▶ in exponential time using support enumeration
  - ▶ in PPAD using Lemke-Howson (omitted)

## Loose Ends – Modes of Dominance

To simplify, let us consider only **pure strategies**.

Let  $s_i, s'_i \in S_i$ . Then  $s'_i$  is *strictly dominated* by  $s_i$  if  $u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$  for all  $s_{-i} \in S_{-i}$ .

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### Claim 4

*Any pure strategy profile  $s \in S$  such that each  $s_i$  is very weakly dominant is a Nash equilibrium.*

The same claim can be proved in the mixed strategy setting.



# Dynamic Games of Complete Information

Extensive-Form Games

Definition

Sub-Game Perfect Equilibria

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Static games (modeled using strategic-form games) cannot capture games that unfold over time.

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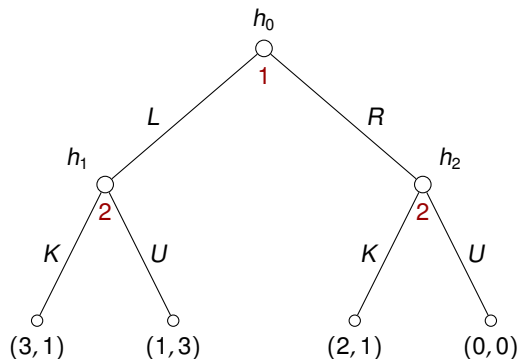
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Then generalize to imperfect information, where players may have only partial knowledge of these results (e.g. most card games).

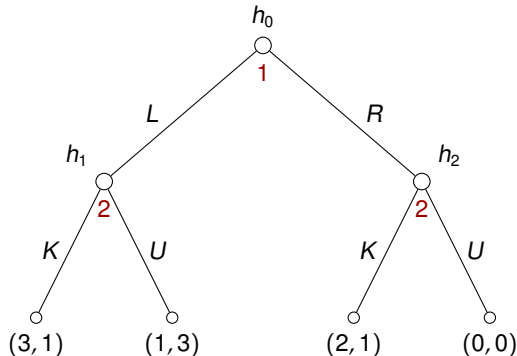


# Perfect-Info. Extensive-Form Games (Example)



Here  $h_0, h_1, h_2$  are non-terminal nodes, leaves are terminal nodes.

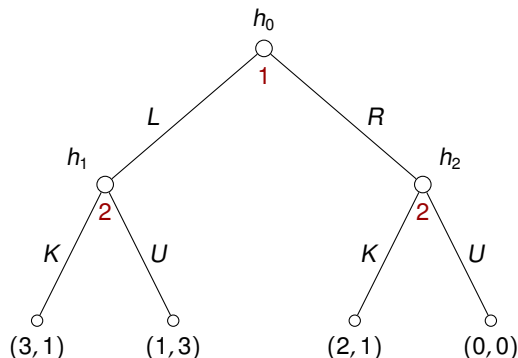
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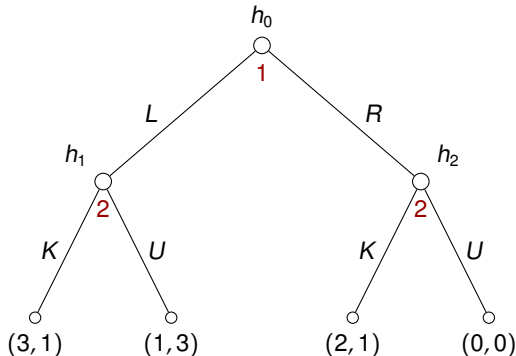
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When a play reaches a terminal node, players collect payoffs.

E.g. the left most terminal node gives 3 to player 1 and 1 to player 2.

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- ▶  $u = (u_1, \dots, u_n)$ , where each  $u_i : Z \rightarrow \mathbb{R}$  is a *payoff function* for player  $i$  in the terminal nodes of  $Z$ .

# Extensive-Form Games as Rooted Trees

$h'$  is a *child* of  $h$ , and  $h$  is a *parent* of  $h'$  if there is  $a \in \chi(h)$  such that  $h' = \pi(h, a)$ .

A *path* from  $h \in \mathcal{H}$  to  $h' \in \mathcal{H}$  is a sequence  $h_1 a_2 h_2 a_3 h_3 \cdots h_{k-1} a_k h_k$  where  $h_1 = h$ ,  $h_k = h'$  and  $\pi(h_{j-1}, a_j) = h_j$  for every  $1 < j \leq k$ .

Note that, in particular,  $h$  is a path from  $h$  to  $h$ .

$h' \in \mathcal{H}$  is *reachable* from  $h \in \mathcal{H}$  if there is a path from  $h$  to  $h'$ .

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**Assumption:** For every  $h \in \mathcal{H}$  there is a unique path from  $h_0$  to  $h$  and there is no infinite path (i.e., a sequence  $h_1 a_2 h_2 a_3 h_3 \cdots$  such that  $\pi(h_{j-1}, a_j) = h_j$  for every  $j > 1$ ).

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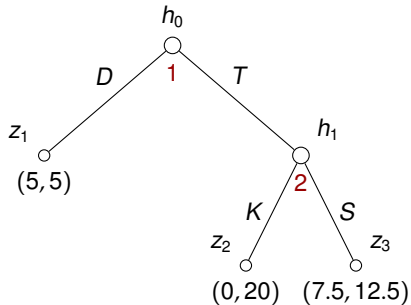
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The assumption implies that every perfect-information extensive-form game can be seen as a game on a *rooted tree*  $(\mathcal{H}, E, h_0)$  where

- ▶  $H \cup Z$  is a set of nodes,
- ▶  $E \subseteq \mathcal{H} \times \mathcal{H}$  is a set of edges defined by  $(h, h') \in E$  iff  $h \in H$  and there is  $a \in \chi(h)$  such that  $\pi(h, a) = h'$ ,
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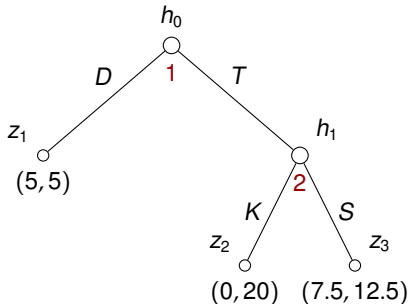


# Example: Trust Game



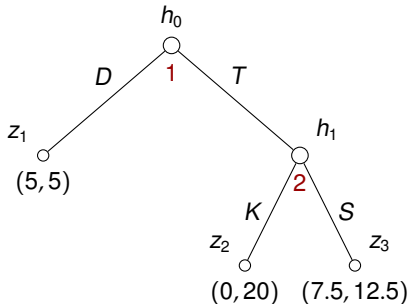
- ▶ Two players, both start with 5\$

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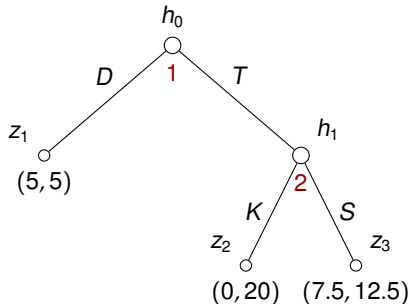
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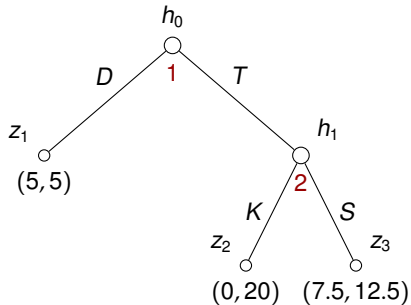
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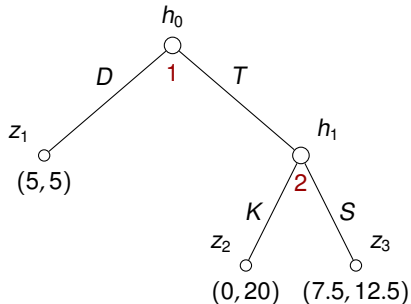
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- ▶ Player 2 may either keep ( $K$ ) the additional 15\$ (resulting in  $(0, 20)$ ), or share ( $S$ ) it with player 1 (resulting in  $(7.5, 12.5)$ )

## Example: Trust Game (Cont.)



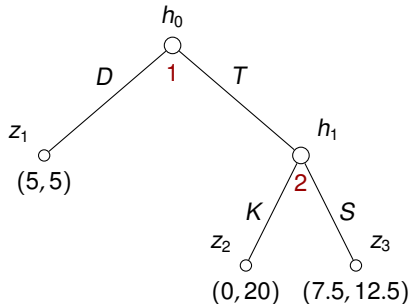
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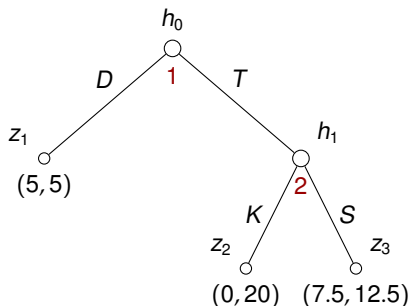
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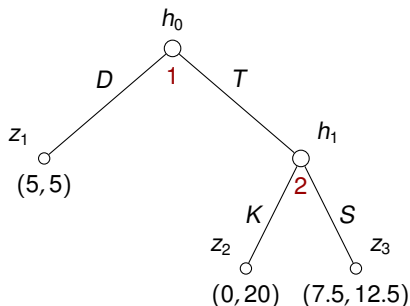
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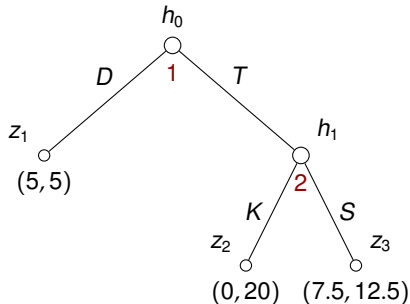


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- ▶  $u_1(z_1) = 5$ ,  $u_1(z_2) = 0$ ,  $u_1(z_3) = 7.5$ ,  $u_2(z_1) = 5$ ,  $u_2(z_2) = 20$ ,  $u_2(z_3) = 12.5$

# Stackelberg Competition

Very similar to Cournot duopoly ...

- ▶ Two identical firms, players 1 and 2, produce some good.  
Denote by  $q_1$  and  $q_2$  quantities produced by firms 1 and 2, resp.
- ▶ The total quantity of products in the market is  $q_1 + q_2$ .
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Except that ...

- ▶ As opposed to Cournot duopoly, the firm 1 moves first, and chooses the quantity  $q_1 \in [0, \infty)$ .
- ▶ Afterwards, the firm 2 chooses  $q_2 \in [0, \infty)$  (knowing  $q_1$ ) and then the firms get their payoffs.

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Let  $G = (N, A, H, Z, \chi, \rho, \pi, h_0, u)$  be a perfect-information extensive-form game.

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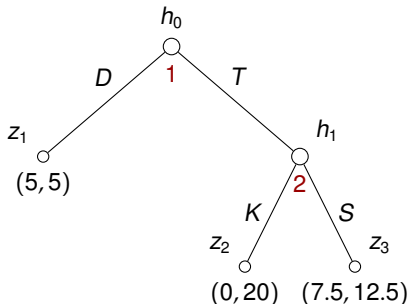
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Abusing notation a bit, we denote by  $u_i(s)$  the value  $u_i(O(s))$  of the payoff for player  $i$  when the terminal node  $O(s)$  is reached using strategies of  $s$ .

## Example: Trust Game



A pure strategy profile  $(s_1, s_2)$  where

$$s_1(h_0) = T \quad \text{and} \quad s_2(h_1) = K$$

is usually written as  $TK$  (BFS & left to right traversal) determines the path  $h_0 T h_1 K z_2$

The resulting payoffs:  $u_1(s_1, s_2) = 0$  and  $u_2(s_1, s_2) = 20$ .

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The extensive-form game  $G$  determines the *corresponding strategic-form game*  $\bar{G} = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$

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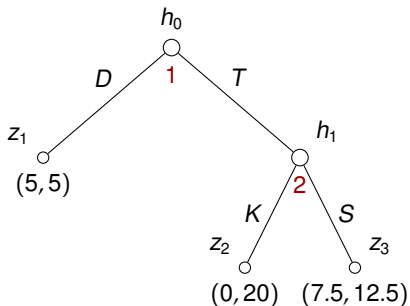
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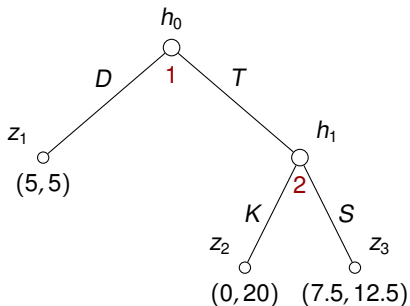
For now, let us consider pure strategies only!

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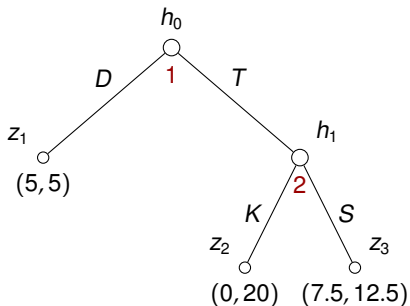


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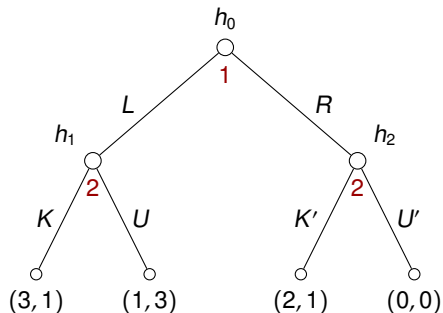


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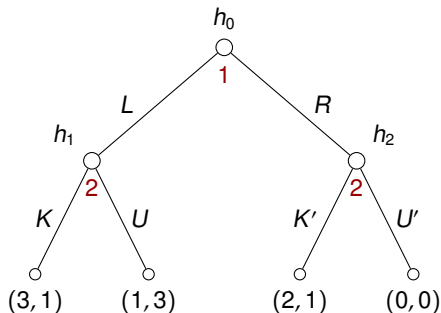
Is there a Nash equilibrium in pure strategies ?

# Example



Find all pure strategies of both players.

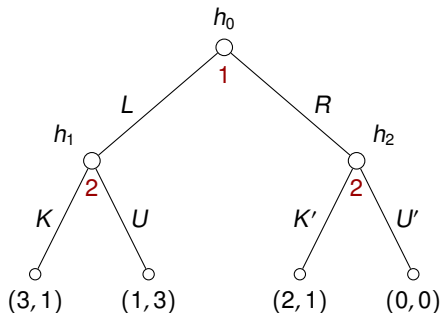
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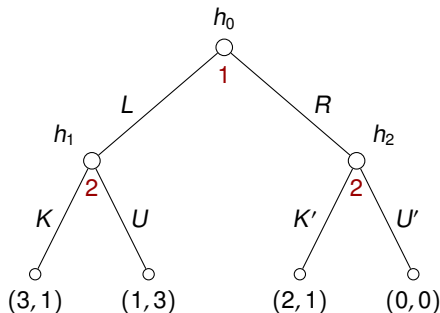


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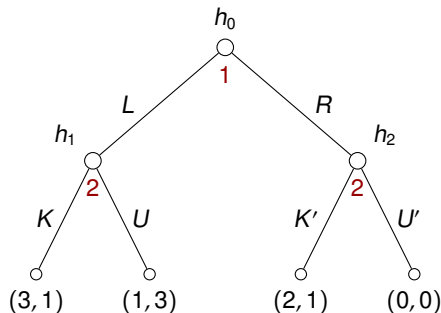
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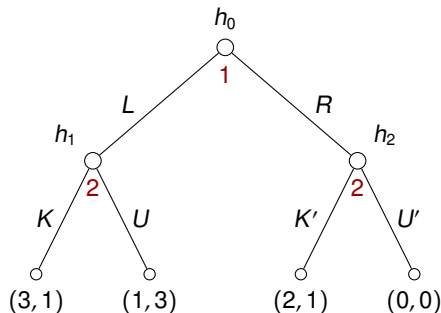
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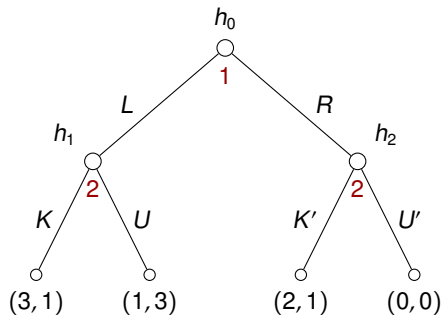
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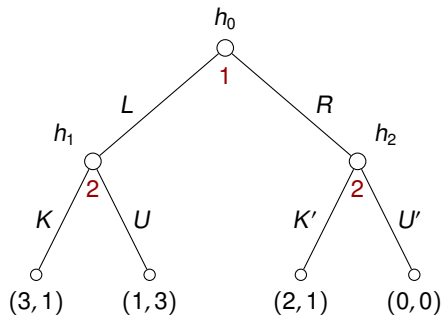


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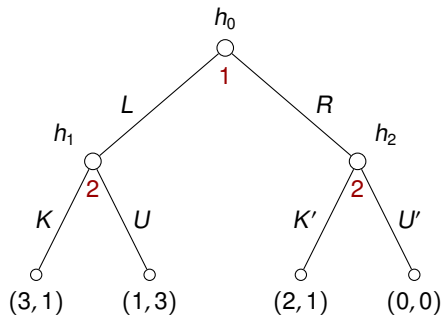


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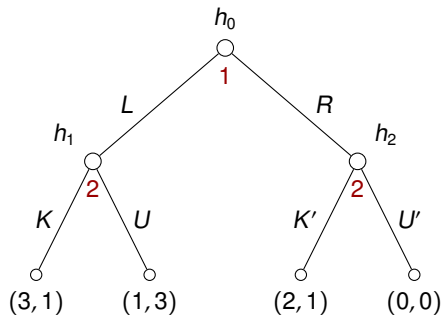
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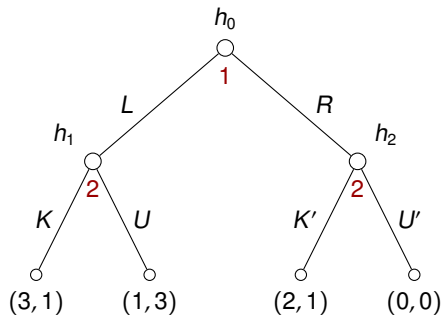
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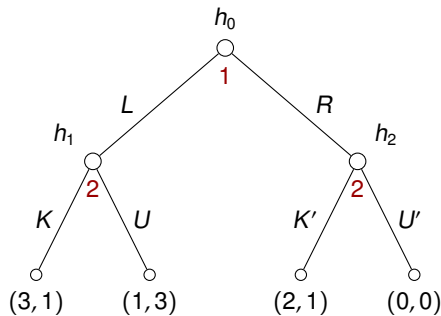
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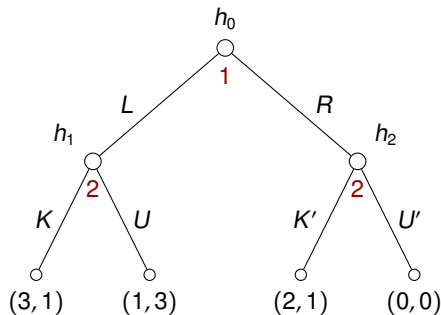
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However, the threat is not *credible*, once a play reaches  $h_2$ , a rational player 2 chooses  $K'$ .

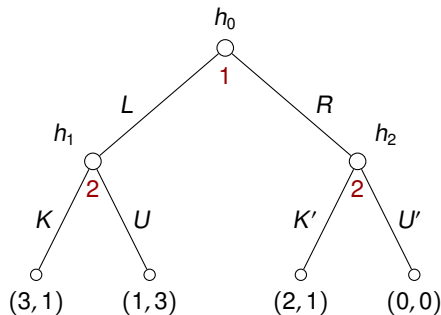
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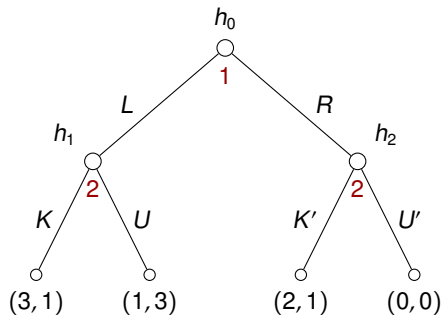


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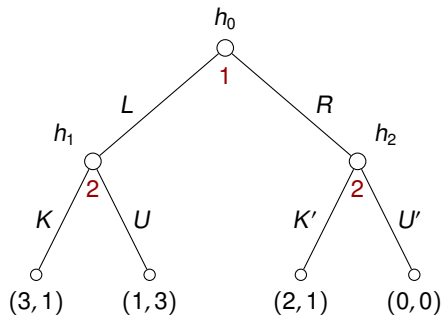
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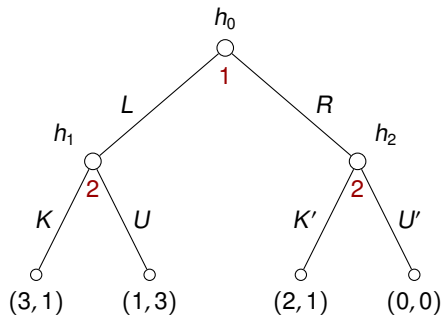
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This equilibrium is called *subgame perfect*.

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- ▶  $\pi^h$  is defined for  $h' \in H^h$  and  $a \in \chi^h(h')$  by  $\pi^h(h', a) = \pi(h', a)$
- ▶ each  $u_i^h$  is a restriction of  $u_i$  to  $Z^h$

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(Given a function  $f : A \rightarrow B$  and  $C \subseteq A$ , a restriction of  $f$  to  $C$  is a function  $g : C \rightarrow B$  such that  $g(x) = f(x)$  for all  $x \in C$ .)

- ▶  $\pi^h$  is defined for  $h' \in H^h$  and  $a \in \chi^h(h')$  by  $\pi^h(h', a) = \pi(h', a)$
- ▶ each  $u_i^h$  is a restriction of  $u_i$  to  $Z^h$

## Definition 44

A *subgame perfect equilibrium (SPE)* in pure strategies is a pure strategy profile  $s \in S$  such that for any subgame  $G^h$  of  $G$ , the restriction of  $s$  to  $H^h$  is a Nash equilibrium in pure strategies in  $G^h$ .

A restriction of  $s = (s_1, \dots, s_n) \in S$  to  $H^h$  is a strategy profile  $s^h = (s_1^h, \dots, s_n^h)$  where  $s_i^h(h') = s_i(h')$  for all  $i \in N$  and all  $h' \in H_i \cap H^h$ .

# Stackelberg Competition – SPE

- ▶  $N = \{1, 2\}$ ,  $A = [0, \infty)$
- ▶  $H = \{h_0, h_1^{q_1} \mid q_1 \in [0, \infty)\}$ ,  $Z = \{z^{q_1, q_2} \mid q_1, q_2 \in [0, \infty)$
- ▶  $\chi(h_0) = [0, \infty)$ ,  $\chi(h_1^{q_1}) = [0, \infty)$ ,  $\rho(h_0) = 1$ ,  $\rho(h_1^{q_1}) = 2$
- ▶  $\pi(h_0, q_1) = h_1^{q_1}$ ,  $\pi(h_1^{q_1}, q_2) = z^{q_1, q_2}$
- ▶ The payoffs are  $u_1(z^{q_1, q_2}) = q_1(\kappa - c - q_1 - q_2)$ ,  
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Note that firm 1 has an advantage as a leader.

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4. Attach to  $h$  the vector of expected payoffs  $u(h) := u(h_{\max})$ .

# Correctness of Backward Induction

## Theorem 45

*For every finite perfect-information extensive-form game and for each node  $h$  the attached  $s^h$  is a SPE and the attached vector  $u(h)$  satisfies  $u(h) = u(s^h) = (u_1(s^h), \dots, u_n(s^h))$ .*

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In both cases the deviation of player  $i$  leads to smaller or equal payoff. Apparently,  $u(s^h) = u(s^{h_{\max}}) = u(h_{\max}) = u(h)$ .

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**Answer:** Nobody knows yet ... the tree is too big!

Even with  $\sim 200$  depth &  $\sim 5$  moves per node:  $5^{200}$  nodes!

# Efficient Algorithms for Pure Nash Equilibria

In the step 2. of the backward induction, the algorithm may choose *an arbitrary*  $h_{\max} \in \operatorname{argmax}_{h' \in K} u_{\rho(h)}(h')$  and always obtain a SPE.

In order to compute all SPE, the algorithm may systematically search through all possible choices of  $h_{\max}$  throughout the induction.

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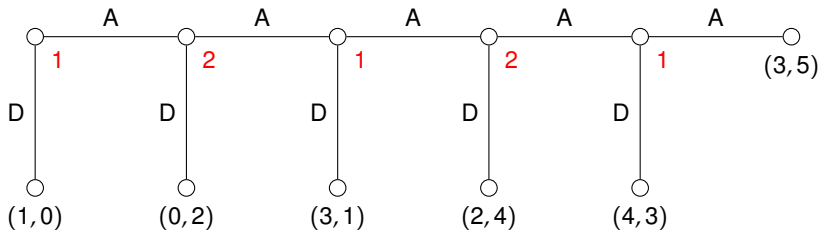
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For details, extensions etc. see e.g.

- ▶ PB016 Artificial Intelligence I
- ▶ Multi-player alpha-beta pruning, R. Korf, *Artificial Intelligence* 48, pages 99-111, 1991
- ▶ Artificial Intelligence: A Modern Approach (3rd edition), S. Russell and P. Norvig, *Prentice Hall*, 2009

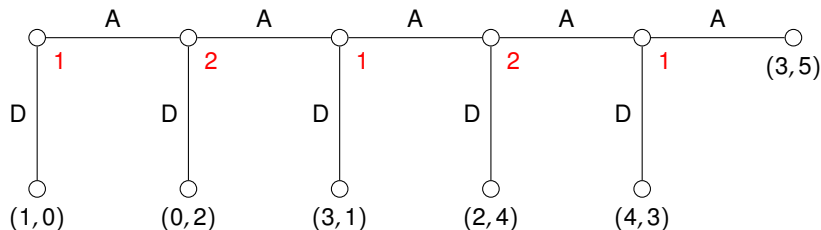
## Example

Centipede game:



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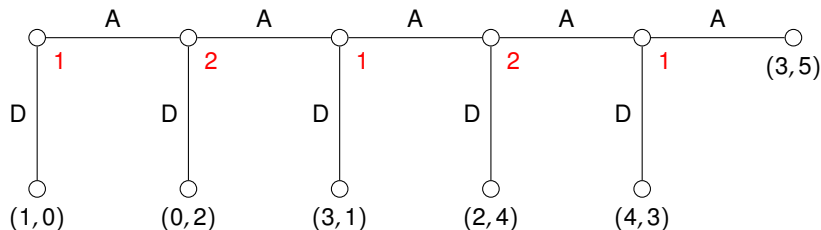
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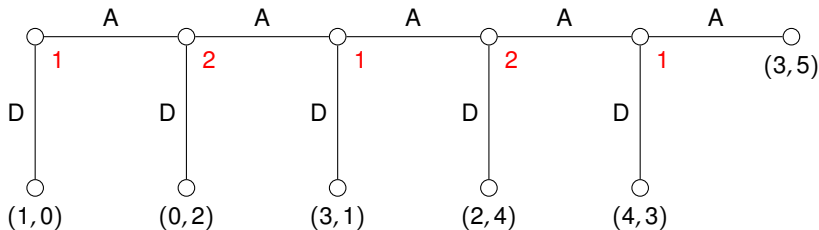
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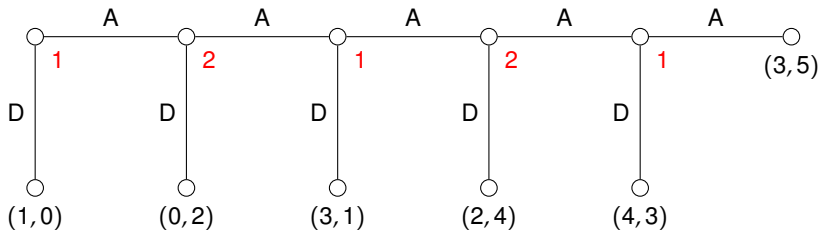


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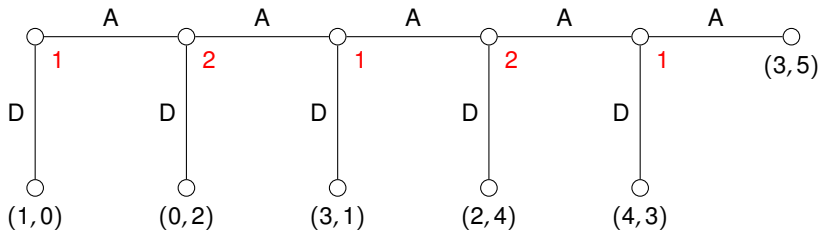
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There are serious issues here ...

- ▶ In laboratory setting, people usually play  $A$  for several steps.
- ▶ There is a theoretical problem: Imagine, that you are player 2. What would you do when player 1 chooses  $A$  in the first step? The SPE analysis says that you should go down, but the same analysis also says that the situation you are in cannot appear :-)



Dynamic Games of Complete Information  
Extensive-Form Games  
**Mixed and Behavioral Strategies**

# Mixed and Behavioral Strategies

Assume two players and a **finite** extensive-form game  $G$ .

## Definition 46

A *mixed strategy*  $\sigma_i$  of player  $i$  in  $G$  is a mixed strategy of player  $i$  in the corresponding strategic-form game.

I.e., a mixed strategy  $\sigma_i$  of player  $i$  in  $G$  is a probability distribution on  $S_i$  (recall that  $S_i$  is the set of all pure strategies, i.e., functions of the form  $s_i : H_i \rightarrow A$ ).

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Given a profile  $\beta = (\beta_1, \beta_2)$  of behavioral strategies, we denote by  $P_\beta(z)$  the probability of reaching  $z \in Z$  when  $\beta$  is used, i.e.,

$$P_\beta(z) = \prod_{\ell=1}^k \beta_{\rho(h_{\ell-1})}(h_\ell)(a_\ell)$$

where  $h_0 a_1 h_1 a_2 h_2 \cdots a_k h_k$  is the unique path from  $h_0$  to  $h_k = z$ .

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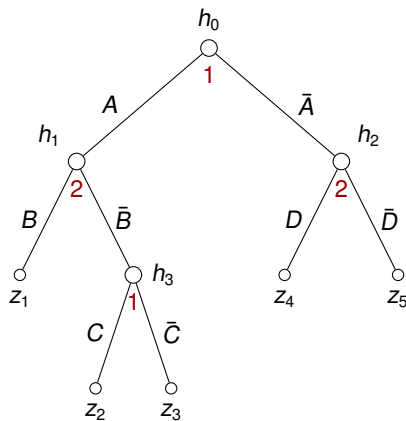
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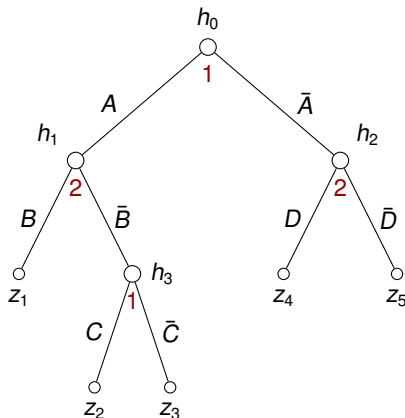
We define  $u_i(\beta) := \sum_{z \in Z} P_\beta(z) \cdot u_i(z)$ .

# Behavioral Strategies: Example



Pure strategies of player 1:

# Behavioral Strategies: Example



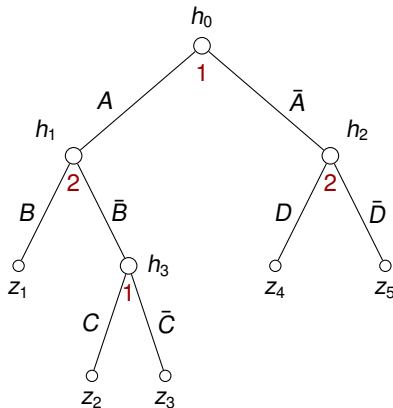
Pure strategies of player 1:  $AC, A\bar{C}, \bar{A}C, \bar{A}\bar{C}$

An example of a mixed strategy  $\sigma_1$  of player 1:

$$\sigma_1(AC) = \frac{1}{3}, \sigma_1(A\bar{C}) = \frac{1}{9}, \sigma_1(\bar{A}C) = \frac{1}{6} \text{ and } \sigma_1(\bar{A}\bar{C}) = \frac{11}{18}$$



# Behavioral Strategies: Example

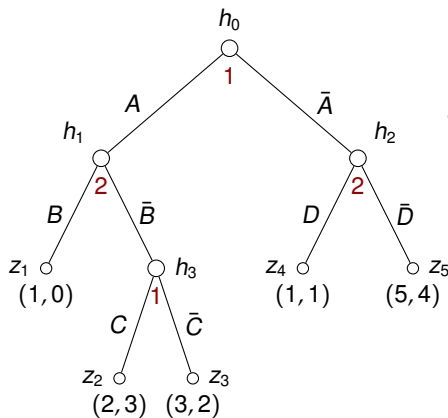


An example of behavioral strategies of both players:

- ▶ player 1:  $\beta_1(h_0)(A) = \frac{1}{3}$  and  $\beta_1(h_3)(C) = \frac{1}{2}$
- ▶ player 2:  $\beta_2(h_1)(B) = \frac{1}{4}$  and  $\beta_2(h_2)(D) = \frac{1}{5}$

$$P_{(\beta_1, \beta_2)}(z_2) = \frac{1}{3} \left(1 - \frac{1}{4}\right) \frac{1}{2} = \frac{1}{8}$$

# Behavioral Strategies: Example



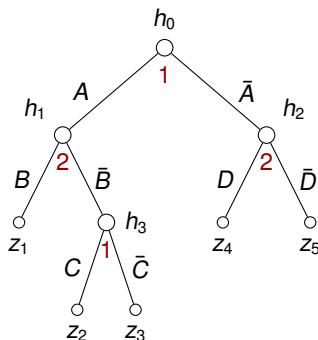
$$\beta = (\beta_1, \beta_2)$$

► player 1:  $\beta_1(h_0)(A) = \frac{1}{3}$   
and  $\beta_1(h_3)(C) = \frac{1}{2}$

► player 2:  $\beta_2(h_1)(B) = \frac{1}{4}$   
and  $\beta_2(h_2)(D) = \frac{1}{5}$

$$\begin{aligned} u_1(\beta) &= P_\beta(z_1) \cdot 1 + P_\beta(z_2) \cdot 2 + P_\beta(z_3) \cdot 3 + P_\beta(z_4) \cdot 1 + P_\beta(z_5) \cdot 5 \\ &= \frac{1}{3} \frac{1}{4} 1 + \frac{1}{3} \frac{3}{4} \frac{1}{2} 2 + \frac{1}{3} \frac{3}{4} \frac{1}{2} 3 + \frac{2}{3} \frac{1}{5} 1 + \frac{2}{3} \frac{4}{5} 5 \approx 3.508 \end{aligned}$$

# Pure Strategies as Behavioral



Each pure strategy can be seen as a behavioral strategy.

Consider e.g.  $s_1 : H_1 \rightarrow A$  defined by  $s_1(h_0) = A$  and  $s_1(h_3) = C$ .

The corresponding behavioral strategy  $\beta_1$  would satisfy  $\beta_1(h_0)(A) = \beta_1(h_3)(C) = 1$  (i.e. select actions chosen by  $s_1$  with prob. 1).

Now given a behavioral strategy  $\beta_2$  of player 2 defined by  $\beta_2(h_1)(B) = \frac{1}{4}$  and  $\beta_2(h_2)(D) = \frac{1}{5}$  we obtain

$$P_{(s_1, \beta_2)}(z_2) = P_{(\beta_1, \beta_2)}(z_2) = 1 \left(1 - \frac{1}{4}\right) 1 = \frac{3}{4}$$

# Mixed/Behavioral Profiles

Let  $\alpha = (\alpha_1, \alpha_2)$  be a strategy profile where each  $\alpha_i$  is either mixed or behavioral.

The game is played as follows:

- ▶ If  $\alpha_1$  mixed, select randomly a pure strategy  $\beta_1$  according to  $\alpha_1$ , else  $\beta_1 := \alpha_1$ .
- ▶ If  $\alpha_2$  mixed, select randomly a pure strategy  $\beta_2$  according to  $\alpha_2$ , else  $\beta_2 := \alpha_2$ .
- ▶ Play  $(\beta_1, \beta_2)$  and collect payoffs.

Denote the resulting payoffs by  $u_1(\alpha)$  and  $u_2(\alpha)$ .

## Lemma 48

*For every mixed/behavioral strategy  $\alpha_1$  of player 1 there is a behavioral/mixed strategy  $\alpha'_1$  such that for every mixed/behavioral strategy  $\alpha_2$  we have that  $u_i(\alpha_1, \alpha_2) = u_i(\alpha'_1, \alpha_2)$  for  $i \in \{1, 2\}$ .*

Dynamic Games of Complete Information

Extensive-Form Games

**Imperfect-Information Games**

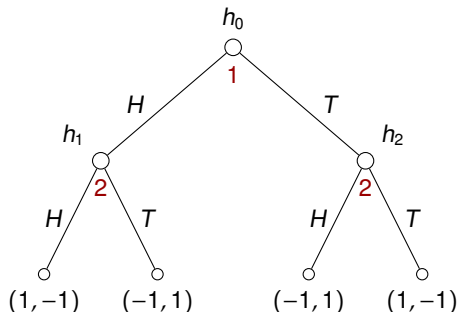
## Extensive-form of Matching Pennies

Is it possible to model Matching pennies using extensive-form games?

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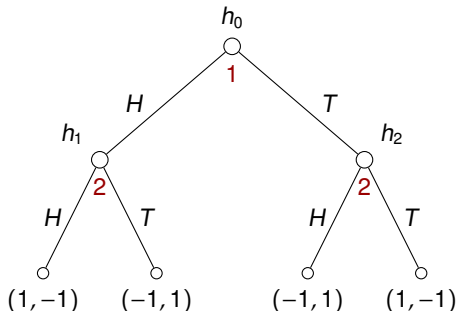
	$H$	$T$
$H$	$1,-1$	$-1,1$
$T$	$-1,1$	$1,-1$



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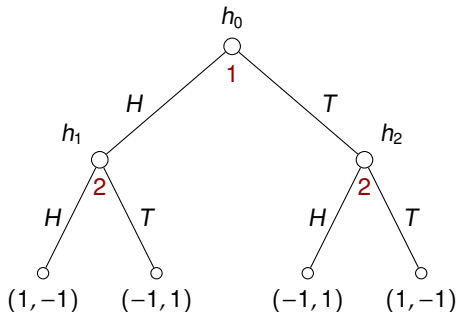
The problem is that player 2 is "perfectly" informed about the choice of player 1. In particular, there are pure Nash equilibria  $(H, TH)$  and  $(T, TH)$  in the extensive-form game as opposed to the strategic-form.



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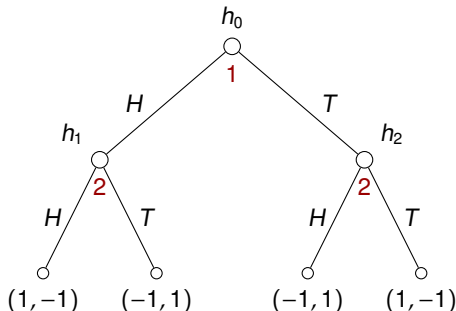
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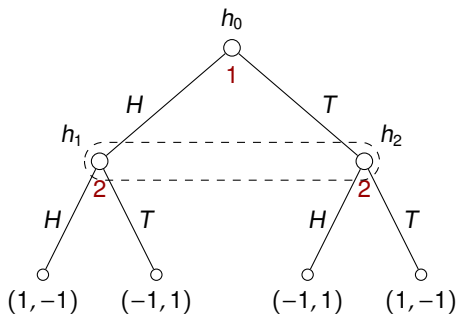
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Reversing the order of players does not help.

We need to extend the formalism to be able to hide some information about previous moves.

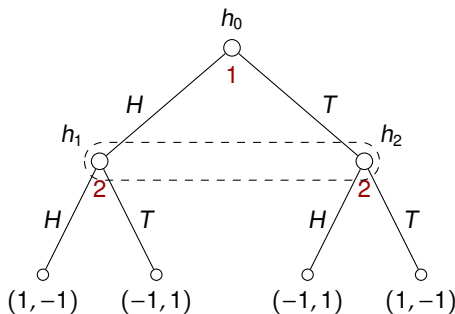
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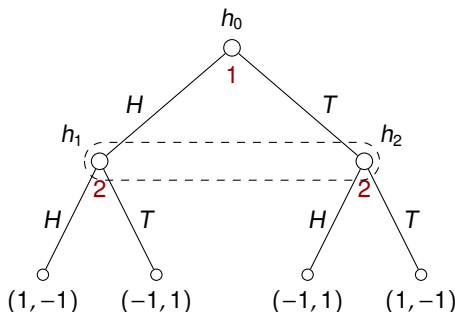
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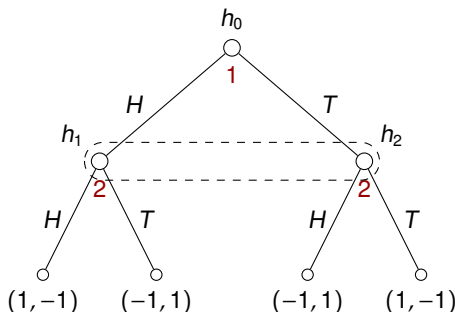


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As a result, player 2 is not able to distinguish between  $h_1$  and  $h_2$ .

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So even though players do not move simultaneously, the information player 2 has about the current situation is the same as in the simultaneous case.

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Given  $h \in H$ , we denote by  $I(h)$  the information set  $I_{i,j}$  containing  $h$ .

Given an information set  $I_{i,j}$ , we denote by  $\chi(I_{i,j})$  the set of all actions enabled in some (and hence all) nodes of  $I_{i,j}$ .

# Imperfect Information Games – Strategies

Now we define the set of pure, mixed, and behavioral strategies in  $G_{imp}$  as subsets of pure, mixed, and behavioral strategies, resp., in  $G_{perf}$  that respect the information sets.

Let  $G_{imp} = (G_{perf}, I)$  be an imperfect-information extensive-form game where  $G_{perf} = (N, A, H, Z, \chi, \rho, \pi, h_0, u)$ .

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## Definition 49

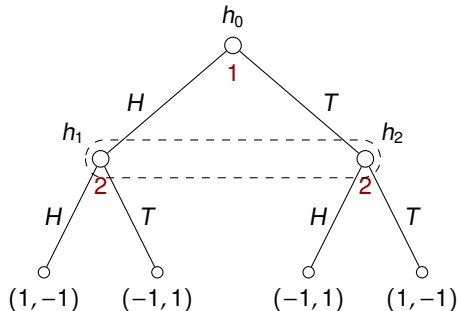
A **pure strategy** of player  $i$  in  $G_{imp}$  is a pure strategy  $s_i$  in  $G_{perf}$  such that for all  $j = 1, \dots, k_i$  and all  $h, h' \in I_{i,j}$  holds  $s_i(h) = s_i(h')$ .

Note that each  $s_i$  can also be seen as a function  $s_i : I_i \rightarrow A$  such that for every  $I_{i,j} \in I_i$  we have that  $s_i(I_{i,j}) \in \chi(I_{i,j})$ .

As before, we denote by  $S_i$  the set of all pure strategies of player  $i$  in  $G_{imp}$ , and by  $S = S_1 \times \dots \times S_n$  the set of all pure strategy profiles.

As in the perfect-information case we have a corresponding strategic-form game  $\bar{G}_{imp} = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ .

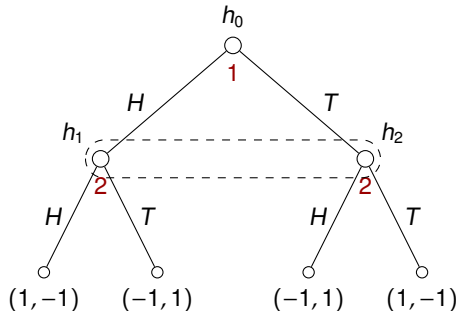
# Matching Pennies



$I_1 = \{I_{1,1}\}$  where  $I_{1,1} = \{h_0\}$

$I_2 = \{I_{2,1}\}$  where  $I_{2,1} = \{h_1, h_2\}$

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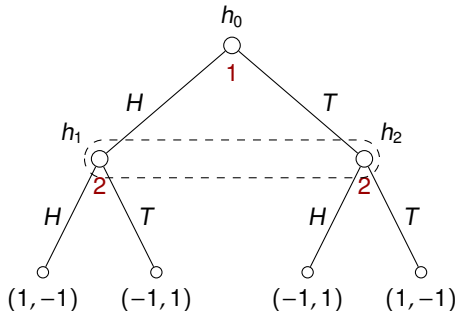
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Example of pure strategies:

- ▶  $s_1(I_{1,1}) = H$  which describes the strategy  $s_1(h_0) = H$
- ▶  $s_2(I_{2,1}) = T$  which describes the strategy  $s_2(h_1) = s_2(h_2) = T$   
(it is also sufficient to specify  $s_2(h_1) = T$  since then  $s_2(h_2) = T$ )



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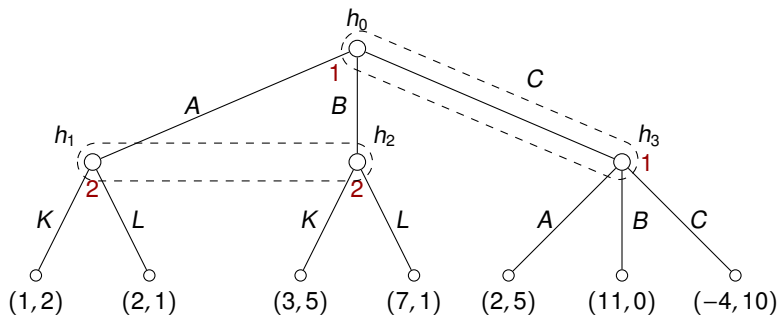
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So we really have strategies  $H, T$  for player 1 and  $H, T$  for player 2.

# Weird Example

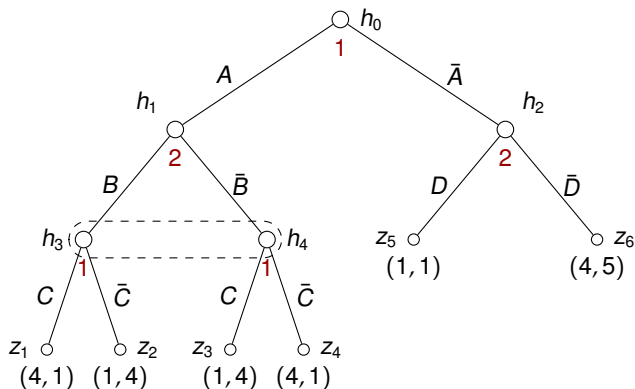


Note that  $I_1 = \{I_{1,1}\}$  where  $I_{1,1} = \{h_0, h_3\}$

and that  $I_2 = \{I_{2,1}\}$  where  $I_{2,1} = \{h_1, h_2\}$

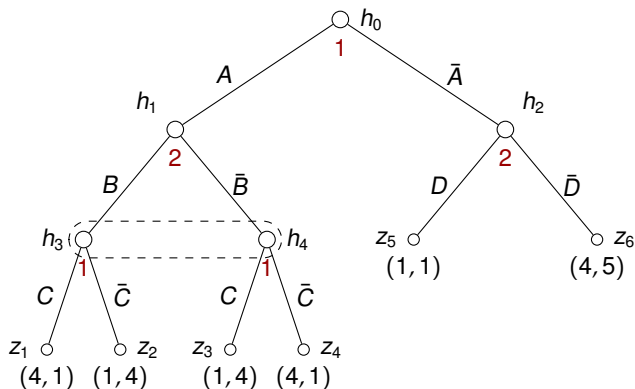
What pure strategies are in this example?

# SPE with Imperfect Information



What we designate as subgames to allow the backward induction?

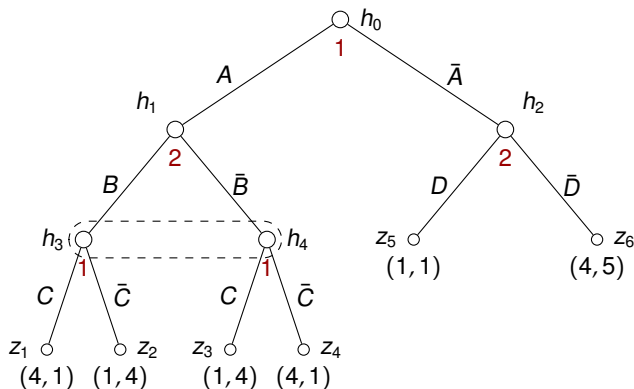
# SPE with Imperfect Information



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Only subtrees rooted in  $h_1$ ,  $h_2$ , and  $h_0$  (together with all subtrees rooted in terminal nodes)

# SPE with Imperfect Information



What we designate as subgames to allow the backward induction?

Only subtrees rooted in  $h_1$ ,  $h_2$ , and  $h_0$  (together with all subtrees rooted in terminal nodes)

Note that subtrees rooted in  $h_3$  and  $h_4$  cannot be considered as "independent" subgames because their individual solutions cannot be combined to a single best response in the information set  $\{h_3, h_4\}$ .

# SPE with Imperfect Information

Let  $G_{imp} = (G_{perf}, I)$  be an imperfect-information extensive-form game where  $G_{perf} = (N, A, H, Z, \chi, \rho, \pi, h_0, u)$  is the underlying perfect-information extensive-form game.

# SPE with Imperfect Information

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## Definition 50

For every  $h \in H_{proper}$  we define a subgame  $G_{imp}^h$  to be the imperfect information game  $(G_{perf}^h, I^h)$  where  $I^h$  is the restriction of  $I$  to  $H^h$ .

Note that as subgames of  $G_{imp}$  we consider only subgames of  $G_{perf}$  that respect the information sets, i.e., are rooted in nodes of  $H_{proper}$ .



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## Definition 51

A strategy profile  $s \in S$  is a subgame perfect equilibrium (SPE) if  $s^h$  is a Nash equilibrium in every subgame  $G_{imp}^h$  of  $G_{imp}$  (here  $h \in H_{proper}$ ).

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3. Consider  $h \in H_{proper}$ , let  $K$  be the set of all  $h' \in (H_{proper} \cup Z) \setminus \{h\}$  that are  $h$ 's **closest descendants out of  $H_{proper} \cup Z$** .  
I.e.,  $h' \in K$  iff  $h' \neq h$  is reachable from  $h$  and the unique path from  $h$  to  $h'$  visits only nodes of  $\mathcal{H} \setminus H_{proper}$  (except the first and the last node).

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4. Now consider all nodes of  $K$  as terminal nodes where each  $h' \in K$  has payoffs  $u(h')$ . This gives a new game in which we compute an equilibrium  $\bar{s}^h$  together with the vector  $u(h)$ .  
The equilibrium  $s^h$  is then obtained by "concatenating"  $\bar{s}^h$  with all  $s^{h'}$ , here  $h' \in K$ , in the subgames  $G_{imp}^{h'}$  of  $G_{imp}^h$ .

# Mutually Assured Destruction

Analysis of Cuban missile crisis of 1962  
(as described in *Games for Business and Economics* by R. Gardner)

- ▶ The crisis started with United States' discovery of Soviet nuclear missiles in Cuba.
- ▶ The USSR then backed down, agreeing to remove the missiles from Cuba, which suggests that US had a credible threat "if you don't back off we both pay dearly".

**Question:** Could this indeed be a credible threat?



## Mutually Assured Destruction (Cont.)

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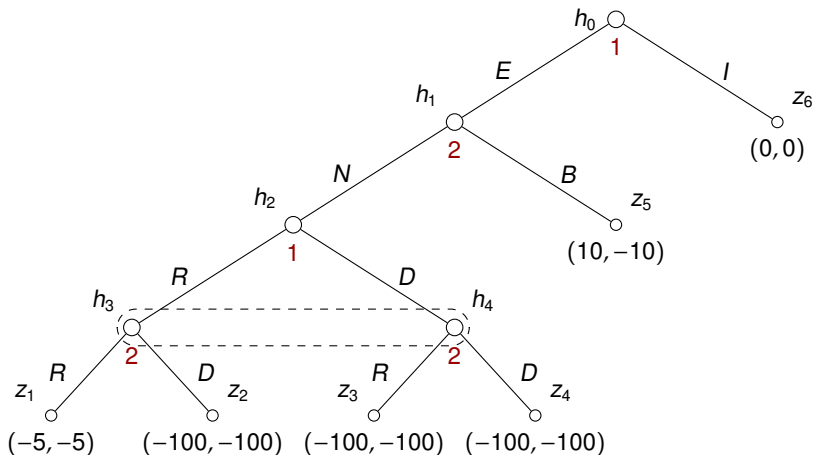
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  - ▶ If both retreat, the payoffs are  $(-5, -5)$ , a small loss due to a mobilization process.
  - ▶ If either of them chooses doomsday, then the world destructs and payoffs are  $(-100, -100)$ .

Find SPE in pure strategies.

# Mutually Assured Destruction (Cont.)



Solve  $G_{imp}^{h_2}$  (a strategic-form game). Then  $G_{imp}^{h_1}$  by solving a game rooted in  $h_1$  with terminal nodes  $h_2, z_5$  (payoffs in  $h_2$  correspond to an equilibrium in  $G_{imp}^{h_2}$ ). Finally solve  $G_{imp}$  by solving a game rooted in  $h_0$  with terminal nodes  $h_1, z_6$  (payoffs in  $h_1$  have been computed in the previous step).

# Mixed and Behavioral Strategies

## Definition 52

A *mixed strategy*  $\sigma_i$  of player  $i$  in  $G_{imp}$  is a mixed strategy of player  $i$  in the corresponding strategic-form game  $\bar{G}_{imp} = (N, (S_i)_{i \in N}, u_i)$ .

Do not forget that now  $s_i \in S_i$  iff  $s_i$  is a pure strategy that assigns the same action to all nodes of every information set. Hence each  $s_i \in S_i$  can be seen as a function  $s_i : I_i \rightarrow A$ .

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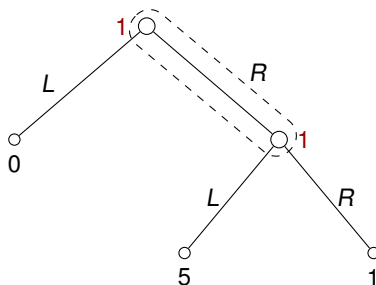
## Definition 53

A *behavioral strategy* of player  $i$  in  $G_{imp}$  is a behavioral strategy  $\beta_i$  in  $G_{perf}$  such that for all  $j = 1, \dots, k_i$  and all  $h, h' \in I_{i,j} : \beta_i(h) = \beta_i(h')$ .

Each  $\beta_i$  can be seen as a function  $\beta_i : I_i \rightarrow \Delta(A)$  such that for all  $I_{i,j} \in I_i$  we have  $\text{supp}(\beta_i(I_{i,j})) \subseteq \chi(I_{i,j})$ .

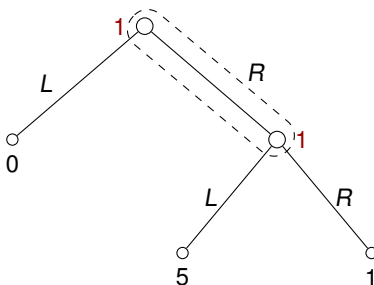
Are they equivalent as in the perfect-information case?

## Example: Absent Minded Driver



Only one player: A driver who has to take a turn at a particular junction. There are two identical junctions, the first one leads to a wrong neighborhood where the driver gets completely lost (payoff 0), the second one leads home (payoff 5). If the driver misses both, there is a longer way home (payoff 1). The problem is that after missing the first turn, the driver forgets that he missed the turn.

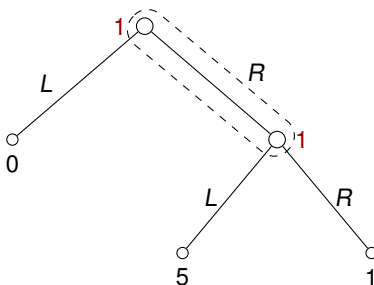
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Behavioral strategy:  $\beta_1(I_{1,1})(L) = \frac{1}{2}$  has the expected payoff  $\frac{3}{2}$ .

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No mixed strategy gives a larger payoff than 1 since no pure strategy ever reaches the terminal node with payoff 5.

# Kuhn's Theorem

Player  $i$  has *perfect recall* in  $G_{imp}$  if the following holds:

- ▶ Every information set of player  $i$  (i.e. *his own*) intersects every path from the root  $h_0$  to a terminal node at most once.
- ▶ Every two paths from the root that end in the same information set of player  $i$ 
  - ▶ pass through the same information sets of player  $i$ ,
  - ▶ and in the same order,
  - ▶ and in every such information set the two paths choose the same action.

May, however, pass through *different* information sets of other players and other players may choose different actions along each of the paths!

I.e. each information set  $J$  of player  $i$  determines the sequence of information sets of player  $i$  and actions taken by player  $i$  along any path reaching  $J$ .

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## Theorem 54 (Kuhn, 1953)

*Assuming perfect recall, every mixed strategy can be translated to a behavioral strategy (and vice versa) so that the payoff for the resulting strategy is the same in any mixed profile.*