IA168 Algorithmic Game Theory

Tomáš Brázdil

Organization of This Course

Sources:

- Lectures (slides, notes)
  - based on several sources
  - slides are prepared for lectures, some stuff on greenboard ($\Rightarrow$ attend the lectures)

Organization of This Course

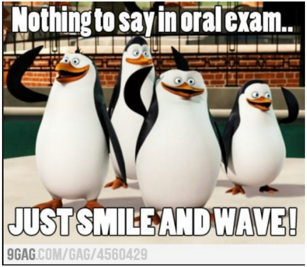
Sources:

- Lectures (slides, notes)
  - based on several sources
  - slides are prepared for lectures, some stuff on greenboard ($\Rightarrow$ attend the lectures)
- Books:
  - Nisan/Roughgarden/Tardos/Vazirani, **Algorithmic Game Theory**, Cambridge University, 2007.
    Available online for free:
    http://www.cambridge.org/journals/nisan/downloads/Nisan_Non-printable.pdf
  - Tadelis, **Game Theory: An Introduction**, Princeton University Press, 2013

(I use various resources, so please, attend the lectures)

Evaluation

- **Oral exam**
- Homework



- 3 homework assignments
- (*possibly* a computer implementation of a strategy)

Notable features of the course

- No computer games course!
- **Very demanding!**
- Mathematical!

Notable features of the course

- No computer games course!
- **Very demanding!**
- Mathematical!

An unusual exam system!

You can repeat the oral exam as many times as needed (only the best grade goes into IS).

Notable features of the course

- No computer games course!
- **Very demanding!**
- Mathematical!

An unusual exam system!

You can repeat the oral exam as many times as needed (only the best grade goes into IS).

An example of an instruction email (from another course with the same system):

```
It is typically not sufficient to devote a single
afternoon to the preparation for the exam.
You have to know _everything_ (which means every
single thing) starting with the slide 42
and ending with the slide 245 with notable exceptions
of slides: 121 - 123, 137 - 140, 165, 167.
Proofs presented on the whiteboard are also mandatory.
```

Most importantly,

The previous slide is not a joke!

What is Algorithmic Game Theory?

First, what is the game theory?

What is Algorithmic Game Theory?

First, what is the game theory?

According to the Oxford dictionary it is "the branch of mathematics concerned with the analysis of strategies for dealing with competitive situations where the outcome of a participant's choice of action depends critically on the actions of other participants"

What is Algorithmic Game Theory?

First, what is the game theory?

According to the Oxford dictionary it is "the branch of mathematics concerned with the analysis of strategies for dealing with competitive situations where the outcome of a participant's choice of action depends critically on the actions of other participants"

According to Myerson it is "the study of mathematical models of conflict and cooperation between intelligent rational decision-makers"

What is Algorithmic Game Theory?

First, what is the game theory?

According to the Oxford dictionary it is "the branch of mathematics concerned with the analysis of strategies for dealing with competitive situations where the outcome of a participant's choice of action depends critically on the actions of other participants"

According to Myerson it is "the study of mathematical models of conflict and cooperation between intelligent rational decision-makers"

What does the "algorithmic" mean?

What is Algorithmic Game Theory?

First, what is the game theory?

According to the Oxford dictionary it is "the branch of mathematics concerned with the analysis of strategies for dealing with competitive situations where the outcome of a participant's choice of action depends critically on the actions of other participants"

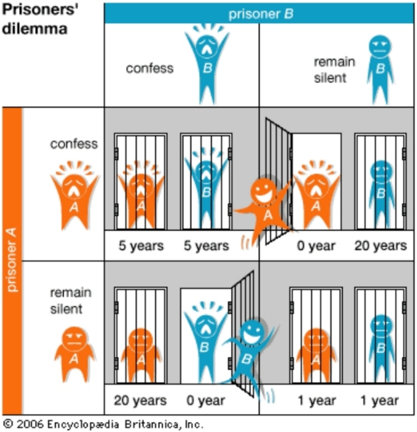
According to Myerson it is "the study of mathematical models of conflict and cooperation between intelligent rational decision-makers"

What does the "algorithmic" mean?

- It means that we are "concerned with the computational questions that arise in game theory, and that enlighten game theory. In particular, questions about finding efficient algorithms to 'solve' games."

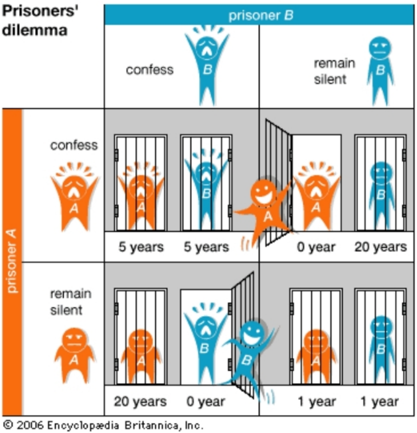
Let's have a look at some examples ....

Prisoner's Dilemma



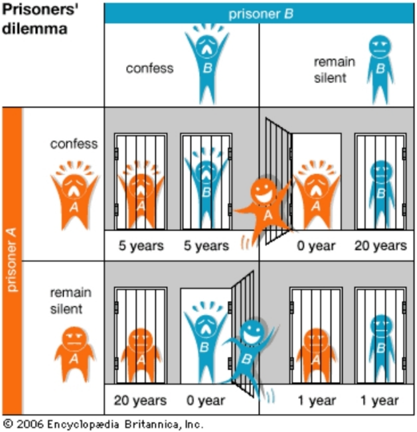
- Two suspects of a serious crime are arrested and imprisoned.

Prisoner's Dilemma



- Two suspects of a serious crime are arrested and imprisoned.
- Police has enough evidence of only petty theft, and to nail the suspects for the serious crime they need testimony from at least one of them.

Prisoner’s Dilemma

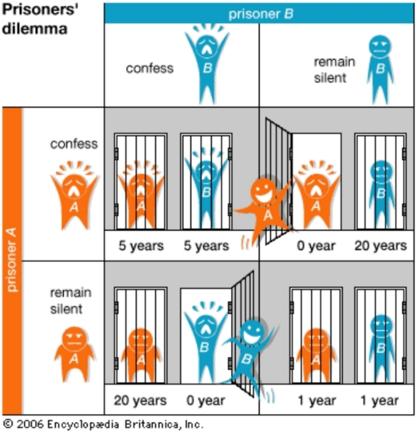


- Two suspects of a serious crime are arrested and imprisoned.
- Police has enough evidence of only petty theft, and to nail the suspects for the serious crime they need testimony from at least one of them.
- The suspects are interrogated separately without any possibility of communication.

Prisoner’s Dilemma

- Two suspects of a serious crime are arrested and imprisoned.
- Police has enough evidence of only petty theft, and to nail the suspects for the serious crime they need testimony from at least one of them.
- The suspects are interrogated separately without any possibility of communication.
- Each of the suspects is offered a deal: If he confesses (C) to the crime, he is free to go. The alternative is not to confess, that is remain silent (S).

Prisoner’s Dilemma



- Two suspects of a serious crime are arrested and imprisoned.
- Police has enough evidence of only petty theft, and to nail the suspects for the serious crime they need testimony from at least one of them.
- The suspects are interrogated separately without any possibility of communication.
- Each of the suspects is offered a deal: If he confesses (C) to the crime, he is free to go. The alternative is not to confess, that is remain silent (S).

Sentence depends on the behavior of both suspects.

Prisoner's Dilemma

© 2006 Encyclopædia Britannica, Inc.

- Two suspects of a serious crime are arrested and imprisoned.
- Police has enough evidence of only petty theft, and to nail the suspects for the serious crime they need testimony from at least one of them.
- The suspects are interrogated separately without any possibility of communication.
- Each of the suspects is offered a deal: If he confesses (C) to the crime, he is free to go. The alternative is not to confess, that is remain silent (S).

Sentence depends on the behavior of both suspects.

The problem: What would the suspects do?

Prisoner's Dilemma – Solution(?)

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

Rational "row" suspect (or his adviser) may reason as follows:

Prisoner's Dilemma – Solution(?)

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

Rational "row" suspect (or his adviser) may reason as follows:

- If my colleague chooses C, then playing C gives me -5 and playing S gives -20.

Prisoner's Dilemma – Solution(?)

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

Rational "row" suspect (or his adviser) may reason as follows:

- If my colleague chooses C, then playing C gives me -5 and playing S gives -20.
- If my colleague chooses S, then playing C gives me 0 and playing S gives -1.

Prisoner's Dilemma – Solution(?)

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

Rational "row" suspect (or his adviser) may reason as follows:

- If my colleague chooses C, then playing C gives me -5 and playing S gives -20.
- If my colleague chooses S, then playing C gives me 0 and playing S gives -1.

In both cases C is clearly better (it *strictly dominates* the other strategy). If the other suspect's reasoning is the same, both choose C and get 5 years sentence.

Prisoner's Dilemma – Solution(?)

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

Rational "row" suspect (or his adviser) may reason as follows:

- If my colleague chooses C, then playing C gives me -5 and playing S gives -20.
- If my colleague chooses S, then playing C gives me 0 and playing S gives -1.

In both cases C is clearly better (it *strictly dominates* the other strategy). If the other suspect's reasoning is the same, both choose C and get 5 years sentence.

Where is the dilemma? There is a solution (S, S) which is better for both players but needs some "central" authority to control the players.

Prisoner's Dilemma – Solution(?)

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

Rational "row" suspect (or his adviser) may reason as follows:

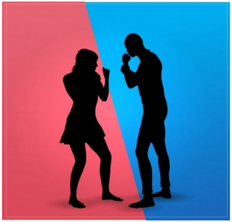
- If my colleague chooses C, then playing C gives me -5 and playing S gives -20.
- If my colleague chooses S, then playing C gives me 0 and playing S gives -1.

In both cases C is clearly better (it *strictly dominates* the other strategy). If the other suspect's reasoning is the same, both choose C and get 5 years sentence.

Where is the dilemma? There is a solution (S, S) which is better for both players but needs some "central" authority to control the players.

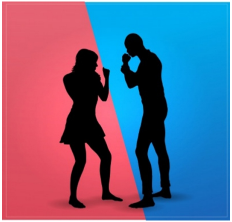
Are there always "dominant" strategies?

Nash equilibria – Battle of Sexes



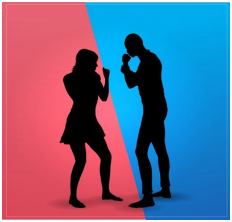
- A couple agreed to meet this evening, but cannot recall if they will be attending the opera or a football match.

Nash equilibria – Battle of Sexes



- A couple agreed to meet this evening, but cannot recall if they will be attending the opera or a football match.
- One of them wants to go to the football game. The other one to the opera. Both would prefer to go to the same place rather than different ones.

Nash equilibria – Battle of Sexes



- A couple agreed to meet this evening, but cannot recall if they will be attending the opera or a football match.
- One of them wants to go to the football game. The other one to the opera. Both would prefer to go to the same place rather than different ones.

If they cannot communicate, where should they go?

Nash equilibria – Battle of Sexes

Battle of Sexes can be modeled as a game of two players (the couple) with the following payoffs:

	O	F
O	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

Nash equilibria – Battle of Sexes

Battle of Sexes can be modeled as a game of two players (the couple) with the following payoffs:

	O	F
O	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

Apparently, no strategy of any player is dominant. A “solution”?

Nash equilibria – Battle of Sexes

Battle of Sexes can be modeled as a game of two players (the couple) with the following payoffs:

	O	F
O	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

Apparently, no strategy of any player is dominant. A “solution”?

Note that whenever *both* players play O, then neither of them wants to *unilaterally* deviate from his strategy!

Nash equilibria – Battle of Sexes

Battle of Sexes can be modeled as a game of two players (the couple) with the following payoffs:

	O	F
O	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

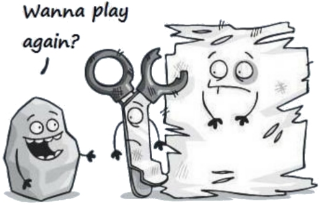
Apparently, no strategy of any player is dominant. A "solution"?

Note that whenever *both* players play O, then neither of them wants to *unilaterally* deviate from his strategy!

(O, O) is an example of a *Nash equilibrium* (as is (F, F))

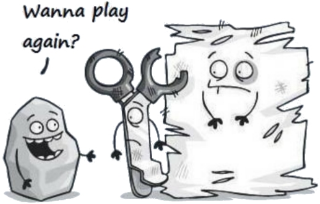
Mixed Equilibria – Rock-Paper-Scissors

	R	P	S
R	$0, 0$	$-1, 1$	$1, -1$
P	$1, -1$	$0, 0$	$-1, 1$
S	$-1, 1$	$1, -1$	$0, 0$



Mixed Equilibria – Rock-Paper-Scissors

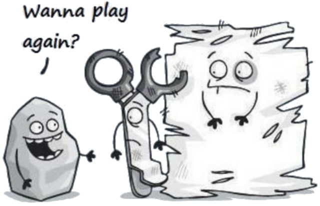
	R	P	S
R	$0, 0$	$-1, 1$	$1, -1$
P	$1, -1$	$0, 0$	$-1, 1$
S	$-1, 1$	$1, -1$	$0, 0$



- This is an example of *zero-sum* games: whatever one of the players wins, the other one looses.

Mixed Equilibria – Rock-Paper-Scissors

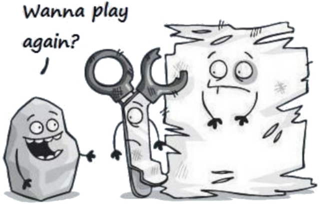
	R	P	S
R	$0, 0$	$-1, 1$	$1, -1$
P	$1, -1$	$0, 0$	$-1, 1$
S	$-1, 1$	$1, -1$	$0, 0$



- This is an example of *zero-sum* games: whatever one of the players wins, the other one looses.
- What is an optimal behavior here? Is there a Nash equilibrium?

Mixed Equilibria – Rock-Paper-Scissors

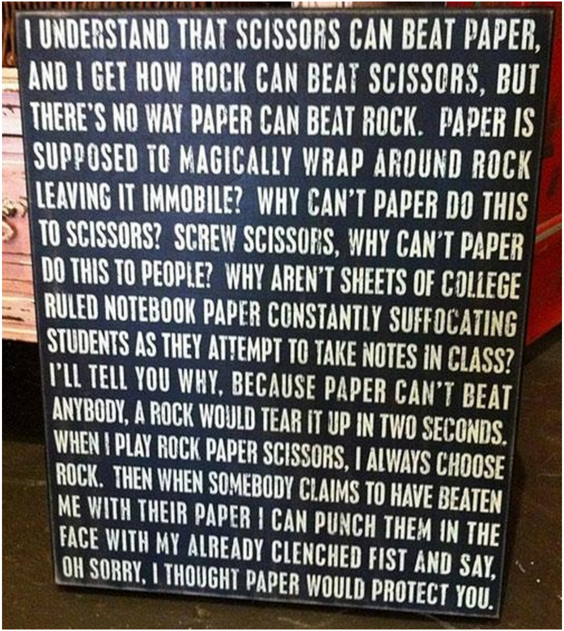
	R	P	S
R	$0, 0$	$-1, 1$	$1, -1$
P	$1, -1$	$0, 0$	$-1, 1$
S	$-1, 1$	$1, -1$	$0, 0$



- This is an example of *zero-sum* games: whatever one of the players wins, the other one looses.
- What is an optimal behavior here? Is there a Nash equilibrium?

  Use *mixed strategies*: Each player plays each pure strategy with probability 1/3. The expected payoff of each player is 0 (even if one of the players changes his strategy, he still gets 0!).

Philosophical Issues in Games



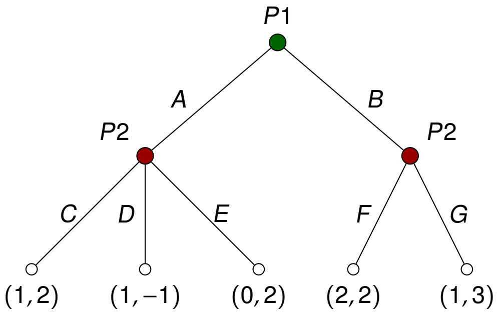
Dynamic Games

So far we have seen games in *strategic form* that are unable to capture games that unfold over time (such as chess).

Dynamic Games

So far we have seen games in *strategic form* that are unable to capture games that unfold over time (such as chess).

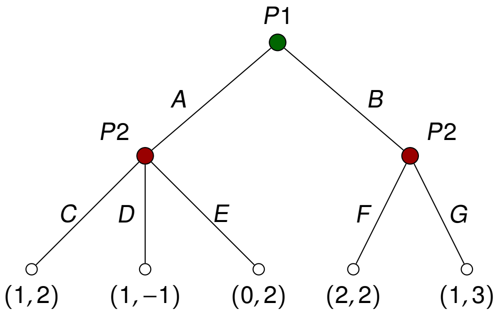
For such purpose we need to use *extensive form* games:



Dynamic Games

So far we have seen games in *strategic form* that are unable to capture games that unfold over time (such as chess).

For such purpose we need to use *extensive form* games:

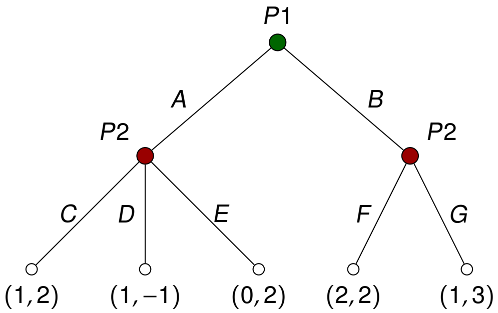


How to "solve" such games?

Dynamic Games

So far we have seen games in *strategic form* that are unable to capture games that unfold over time (such as chess).

For such purpose we need to use *extensive form* games:



How to "solve" such games?

What is their relationship to the strategic form games?

Chance and Imperfect Information

Some decisions in the game tree may be by chance and controlled by neither player (e.g. Poker, Backgammon, etc.)

Chance and Imperfect Information

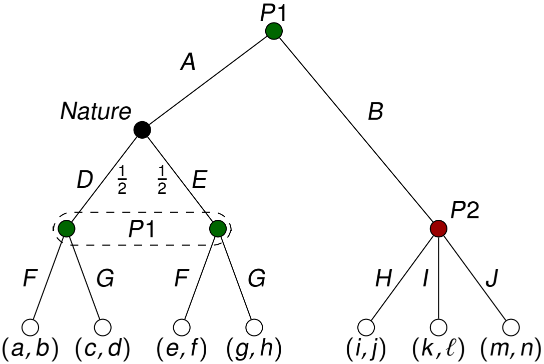
Some decisions in the game tree may be by chance and controlled by neither player (e.g. Poker, Backgammon, etc.)

Sometimes a player may not be able to distinguish between several "positions" because he does not know all the information in them (Think a card game with opponent's cards hidden).

Chance and Imperfect Information

Some decisions in the game tree may be by chance and controlled by neither player (e.g. Poker, Backgammon, etc.)

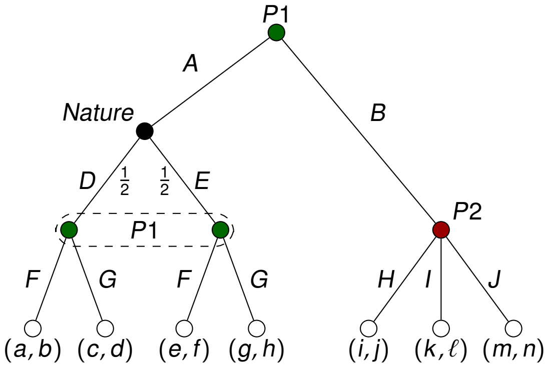
Sometimes a player may not be able to distinguish between several “positions” because he does not know all the information in them (Think a card game with opponent’s cards hidden).



Chance and Imperfect Information

Some decisions in the game tree may be by chance and controlled by neither player (e.g. Poker, Backgammon, etc.)

Sometimes a player may not be able to distinguish between several "positions" because he does not know all the information in them (Think a card game with opponent's cards hidden).



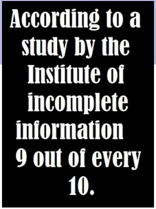
Again, how to solve such games?

Games of Incomplete Information

According to a study by the Institute of incomplete information 9 out of every 10.

In all previous games the players knew all details of the game they played, and this fact was a "common knowledge". This is not always the case.

Games of Incomplete Information

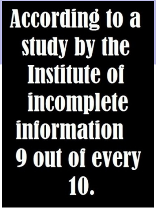


In all previous games the players knew all details of the game they played, and this fact was a “common knowledge”. This is not always the case.

Example: Sealed Bid Auction

- Two bidders are trying to purchase the same item.

Games of Incomplete Information

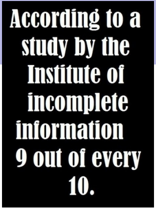


In all previous games the players knew all details of the game they played, and this fact was a “common knowledge”. This is not always the case.

Example: Sealed Bid Auction

- Two bidders are trying to purchase the same item.
- The bidders simultaneously submit bids b_1 and b_2 and the item is sold to the highest bidder at his bid price (first price auction)

Games of Incomplete Information



In all previous games the players knew all details of the game they played, and this fact was a "common knowledge". This is not always the case.

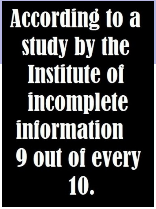
Example: Sealed Bid Auction

- Two bidders are trying to purchase the same item.
- The bidders simultaneously submit bids b_1 and b_2 and the item is sold to the highest bidder at his bid price (first price auction)
- The payoff of the player 1 (and similarly for player 2) is calculated by

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_1 & b_1 > b_2 \\ \frac{1}{2}(v_1 - b_1) & b_1 = b_2 \\ 0 & b_1 < b_2 \end{cases}$$

Here v_1 is the private value that player 1 assigns to the item and so the player 2 **does not know** u_1**.**

Games of Incomplete Information



In all previous games the players knew all details of the game they played, and this fact was a "common knowledge". This is not always the case.

Example: Sealed Bid Auction

- Two bidders are trying to purchase the same item.
- The bidders simultaneously submit bids b_1 and b_2 and the item is sold to the highest bidder at his bid price (first price auction)
- The payoff of the player 1 (and similarly for player 2) is calculated by

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_1 & b_1 > b_2 \\ \frac{1}{2}(v_1 - b_1) & b_1 = b_2 \\ 0 & b_1 < b_2 \end{cases}$$

Here v_1 is the private value that player 1 assigns to the item and so the player 2 **does not know** u_1**.**

How to deal with such a game? Assume the "worst" private value? What if we have a partial knowledge about the private values?

Inefficiency of Equilibria

In Prisoner's Dilemma, the selfish behavior of suspects (the Nash equilibrium) results in somewhat worse than ideal situation.

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

Inefficiency of Equilibria

In Prisoner's Dilemma, the selfish behavior of suspects (the Nash equilibrium) results in somewhat worse than ideal situation.

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

Defining a *welfare function* W which to every pair of strategies assigns the sum of payoffs, we get $W(C, C) = -10$ but $W(S, S) = -2$.

Inefficiency of Equilibria

In Prisoner's Dilemma, the selfish behavior of suspects (the Nash equilibrium) results in somewhat worse than ideal situation.

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

Defining a *welfare function* W which to every pair of strategies assigns the sum of payoffs, we get $W(C, C) = -10$ but $W(S, S) = -2$.

The ratio $\frac{W(C,C)}{W(S,S)} = 5$ measures the inefficiency of "selfish-behavior" (C, C) w.r.t. the optimal “centralized” solution.

Inefficiency of Equilibria

In Prisoner's Dilemma, the selfish behavior of suspects (the Nash equilibrium) results in somewhat worse than ideal situation.

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

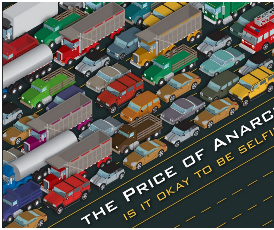
Defining a *welfare function* W which to every pair of strategies assigns the sum of payoffs, we get $W(C, C) = -10$ but $W(S, S) = -2$.

The ratio $\frac{W(C,C)}{W(S,S)} = 5$ measures the inefficiency of "selfish-behavior" (C, C) w.r.t. the optimal “centralized” solution.

Price of Anarchy is the maximum ratio between values of equilibria and the value of an optimal solution.

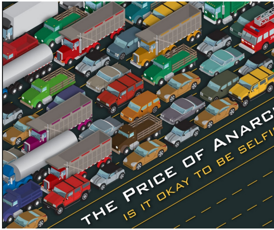
Inefficiency of Equilibria – Selfish Routing

Consider a transportation system where many agents are trying to get from some initial location to a destination. Consider the welfare to be the average time for an agent to reach the destination. There are two versions:



Inefficiency of Equilibria – Selfish Routing

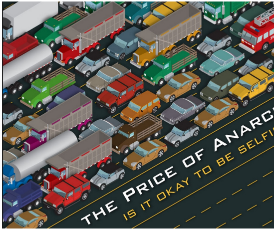
Consider a transportation system where many agents are trying to get from some initial location to a destination. Consider the welfare to be the average time for an agent to reach the destination. There are two versions:



- "Centralized": A central authority tells each agent where to go.

Inefficiency of Equilibria – Selfish Routing

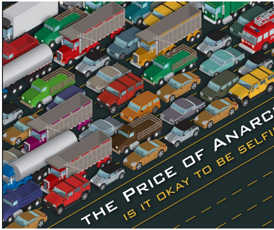
Consider a transportation system where many agents are trying to get from some initial location to a destination. Consider the welfare to be the average time for an agent to reach the destination. There are two versions:



- “Centralized”: A central authority tells each agent where to go.
- “Decentralized”: Each agent selfishly minimizes his travel time.

Inefficiency of Equilibria – Selfish Routing

Consider a transportation system where many agents are trying to get from some initial location to a destination. Consider the welfare to be the average time for an agent to reach the destination. There are two versions:

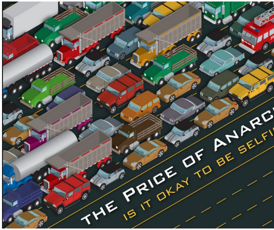


- “Centralized”: A central authority tells each agent where to go.
- “Decentralized”: Each agent selfishly minimizes his travel time.

Price of Anarchy measure the ratio between average travel time in these two cases.

Inefficiency of Equilibria – Selfish Routing

Consider a transportation system where many agents are trying to get from some initial location to a destination. Consider the welfare to be the average time for an agent to reach the destination. There are two versions:



- “Centralized”: A central authority tells each agent where to go.
- “Decentralized”: Each agent selfishly minimizes his travel time.

Price of Anarchy measure the ratio between average travel time in these two cases.

Problem: Bound the price of anarchy over all routing games?

Games in Computer Science

Game theory is a core foundation of mathematical economics. But what does it have to do with CS?

Games in Computer Science

Game theory is a core foundation of mathematical economics. But what does it have to do with CS?

- Games in AI: modeling of “rational” agents and their interactions.

Games in Computer Science

Game theory is a core foundation of mathematical economics. But what does it have to do with CS?

- Games in AI: modeling of “rational” agents and their interactions.
- Games in machine learning: Generative adversarial networks, reinforcement learning

Games in Computer Science

Game theory is a core foundation of mathematical economics. But what does it have to do with CS?

- Games in AI: modeling of "rational" agents and their interactions.
- Games in machine learning: Generative adversarial networks, reinforcement learning
- Games in Algorithms: several game theoretic problems have a very interesting algorithmic status and are solved by interesting algorithms

Games in Computer Science

Game theory is a core foundation of mathematical economics. But what does it have to do with CS?

- Games in AI: modeling of “rational” agents and their interactions.
- Games in machine learning: Generative adversarial networks, reinforcement learning
- Games in Algorithms: several game theoretic problems have a very interesting algorithmic status and are solved by interesting algorithms
- Games in modeling and analysis of reactive systems: program inputs viewed “adversarially”, bisimulation games, etc.

Games in Computer Science

Game theory is a core foundation of mathematical economics. But what does it have to do with CS?

- Games in AI: modeling of “rational” agents and their interactions.
- Games in machine learning: Generative adversarial networks, reinforcement learning
- Games in Algorithms: several game theoretic problems have a very interesting algorithmic status and are solved by interesting algorithms
- Games in modeling and analysis of reactive systems: program inputs viewed “adversarially”, bisimulation games, etc.
- Games in computational complexity: Many complexity classes are definable in terms of games: PSPACE, polynomial hierarchy, etc.

Games in Computer Science

Game theory is a core foundation of mathematical economics. But what does it have to do with CS?

- Games in AI: modeling of "rational" agents and their interactions.
- Games in machine learning: Generative adversarial networks, reinforcement learning
- Games in Algorithms: several game theoretic problems have a very interesting algorithmic status and are solved by interesting algorithms
- Games in modeling and analysis of reactive systems: program inputs viewed "adversarially", bisimulation games, etc.
- Games in computational complexity: Many complexity classes are definable in terms of games: PSPACE, polynomial hierarchy, etc.
- Games in Logic: modal and temporal logics, Ehrenfeucht-Fraisse games, etc.

Games in Computer Science

Games, the Internet and E-commerce: An extremely active research area at the intersection of CS and Economics

Basic idea: “The internet is a HUGE experiment in interaction between agents (both human and automated)”

How do we set up the rules of this game to harness “socially optimal” results?

Summary and Brief Overview

This is a *theoretical* course aimed at some fundamental results of game theory, often related to computer science

Summary and Brief Overview

This is a *theoretical* course aimed at some fundamental results of game theory, often related to computer science

- We start with strategic form games (such as the Prisoner's dilemma), investigate several solution concepts (dominance, equilibria) and related algorithms.

Summary and Brief Overview

This is a *theoretical* course aimed at some fundamental results of game theory, often related to computer science

- We start with strategic form games (such as the Prisoner's dilemma), investigate several solution concepts (dominance, equilibria) and related algorithms.
- Then we consider repeated games which allow players to learn from history and/or to react to deviations of the other players.

Summary and Brief Overview

This is a *theoretical* course aimed at some fundamental results of game theory, often related to computer science

- We start with strategic form games (such as the Prisoner's dilemma), investigate several solution concepts (dominance, equilibria) and related algorithms.
- Then we consider repeated games which allow players to learn from history and/or to react to deviations of the other players.
- Subsequently, we move on to incomplete information games and auctions.

Summary and Brief Overview

This is a *theoretical* course aimed at some fundamental results of game theory, often related to computer science

- We start with strategic form games (such as the Prisoner's dilemma), investigate several solution concepts (dominance, equilibria) and related algorithms.
- Then we consider repeated games which allow players to learn from history and/or to react to deviations of the other players.
- Subsequently, we move on to incomplete information games and auctions.
- Finally, we consider (in)efficiency of equilibria (such as the Price of Anarchy) and its properties on important classes of routing and network formation games.

Summary and Brief Overview

This is a *theoretical* course aimed at some fundamental results of game theory, often related to computer science

- We start with strategic form games (such as the Prisoner's dilemma), investigate several solution concepts (dominance, equilibria) and related algorithms.
- Then we consider repeated games which allow players to learn from history and/or to react to deviations of the other players.
- Subsequently, we move on to incomplete information games and auctions.
- Finally, we consider (in)efficiency of equilibria (such as the Price of Anarchy) and its properties on important classes of routing and network formation games.
- Remaining time will be devoted to selected topics from extensive form games, games on graphs etc.

Static Games of Complete Information

Strategic-Form Games

Solution concepts

Static Games of Complete Information – Intuition

Proceed in two steps:

1. Players *simultaneously and independently* choose their *strategies*. This means that players play without observing strategies chosen by other players.

Static Games of Complete Information – Intuition

Proceed in two steps:

1. Players *simultaneously and independently* choose their *strategies*. This means that players play without observing strategies chosen by other players.
2. Conditional on the players' strategies, *payoffs* are distributed to all players.

Static Games of Complete Information – Intuition

Proceed in two steps:

1. Players *simultaneously and independently* choose their *strategies*. This means that players play without observing strategies chosen by other players.
2. Conditional on the players' strategies, *payoffs* are distributed to all players.

Complete information means that the following is *common knowledge* among players:

- all possible strategies of all players,
- what payoff is assigned to each combination of strategies.

Static Games of Complete Information – Intuition

Proceed in two steps:

1. Players *simultaneously and independently* choose their *strategies*. This means that players play without observing strategies chosen by other players.
2. Conditional on the players' strategies, *payoffs* are distributed to all players.

Complete information means that the following is *common knowledge* among players:

- all possible strategies of all players,
- what payoff is assigned to each combination of strategies.

Definition 1
A fact E is a *common knowledge* among players $\{1, \ldots, n\}$ if for every sequence $i_1, \ldots, i_k \in \{1, \ldots, n\}$ we have that i_1 knows that i_2 knows that ... i_{k-1} knows that i_k knows E.

Static Games of Complete Information – Intuition

Proceed in two steps:

1. Players *simultaneously and independently* choose their *strategies*. This means that players play without observing strategies chosen by other players.
2. Conditional on the players' strategies, *payoffs* are distributed to all players.

Complete information means that the following is *common knowledge* among players:

- all possible strategies of all players,
- what payoff is assigned to each combination of strategies.

Definition 1
A fact E is a *common knowledge* among players $\{1, \ldots, n\}$ if for every sequence $i_1, \ldots, i_k \in \{1, \ldots, n\}$ we have that i_1 knows that i_2 knows that ... i_{k-1} knows that i_k knows E.

The goal of each player is to maximize his payoff (and this fact is a common knowledge).

Strategic-Form Games

To formally represent static games of complete information we define *strategic-form games*.

Definition 2

A game in *strategic-form* (or normal-form) is an ordered triple $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$, in which:

- $N = \{1, 2, \ldots, n\}$ is a finite set of *players*.
- S_i is a set of *(pure) strategies* of player i, for every $i \in N$.

  A *strategy profile* is a vector of strategies of all players $(s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n$.

  We denote the set of all strategy profiles by $S = S_1 \times \cdots \times S_n$.
- $u_i : S \to \mathbb{R}$ is a function associating each strategy profile $s = (s_1, \ldots, s_n) \in S$ with the *payoff* $u_i(s)$ to player i, for every player $i \in N$.

Strategic-Form Games

To formally represent static games of complete information we define *strategic-form games*.

Definition 2

A game in *strategic-form* (or normal-form) is an ordered triple $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$, in which:

- $N = \{1, 2, \ldots, n\}$ is a finite set of *players*.
- S_i is a set of *(pure) strategies* of player i, for every $i \in N$.

  A *strategy profile* is a vector of strategies of all players $(s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n$.

  We denote the set of all strategy profiles by $S = S_1 \times \cdots \times S_n$.
- $u_i : S \to \mathbb{R}$ is a function associating each strategy profile $s = (s_1, \ldots, s_n) \in S$ with the *payoff* $u_i(s)$ to player i, for every player $i \in N$.

Definition 3

A *zero-sum* game G is one in which for all $s = (s_1, \ldots, s_n) \in S$ we have $u_1(s) + u_2(s) + \cdots + u_n(s) = 0$.

Example: Prisoner's Dilemma

- $N = \{1, 2\}$
- $S_1 = S_2 = \{S, C\}$
- u_1, u_2 are defined as follows:
  - $u_1(C, C) = -5$, $u_1(C, S) = 0$, $u_1(S, C) = -20$, $u_1(S, S) = -1$
  - $u_2(C, C) = -5$, $u_2(C, S) = -20$, $u_2(S, C) = 0$, $u_2(S, S) = -1$

(Is it zero sum?)

Example: Prisoner's Dilemma

- $N = \{1, 2\}$
- $S_1 = S_2 = \{S, C\}$
- u_1, u_2 are defined as follows:
  - $u_1(C, C) = -5$, $u_1(C, S) = 0$, $u_1(S, C) = -20$, $u_1(S, S) = -1$
  - $u_2(C, C) = -5$, $u_2(C, S) = -20$, $u_2(S, C) = 0$, $u_2(S, S) = -1$

(Is it zero sum?)

We usually write payoffs in the following form:

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

or as two matrices:

	C	S
C	-5	0
S	-20	-1

	C	S
C	-5	-20
S	0	-1

Example: Cournot Duopoly

- Two identical firms, players 1 and 2, produce some good.
  Denote by q_1 and q_2 quantities produced by firms 1 and 2, resp.

Example: Cournot Duopoly

- Two identical firms, players 1 and 2, produce some good.
  Denote by q_1 and q_2 quantities produced by firms 1 and 2, resp.
- The total quantity of products in the market is $q_1 + q_2$.

Example: Cournot Duopoly

- Two identical firms, players 1 and 2, produce some good. Denote by q_1 and q_2 quantities produced by firms 1 and 2, resp.
- The total quantity of products in the market is $q_1 + q_2$.
- The price of each item is $\kappa - q_1 - q_2$ (here κ is a positive constant)

Example: Cournot Duopoly

- Two identical firms, players 1 and 2, produce some good. Denote by q_1 and q_2 quantities produced by firms 1 and 2, resp.
- The total quantity of products in the market is $q_1 + q_2$.
- The price of each item is $\kappa - q_1 - q_2$ (here κ is a positive constant)
- Firms 1 and 2 have per item production costs c_1 and c_2, resp.

Example: Cournot Duopoly

- Two identical firms, players 1 and 2, produce some good. Denote by q_1 and q_2 quantities produced by firms 1 and 2, resp.
- The total quantity of products in the market is $q_1 + q_2$.
- The price of each item is $\kappa - q_1 - q_2$ (here κ is a positive constant)
- Firms 1 and 2 have per item production costs c_1 and c_2, resp.

Question: How these firms are going to behave?

Example: Cournot Duopoly

- Two identical firms, players 1 and 2, produce some good. Denote by q_1 and q_2 quantities produced by firms 1 and 2, resp.
- The total quantity of products in the market is $q_1 + q_2$.
- The price of each item is $\kappa - q_1 - q_2$ (here κ is a positive constant)
- Firms 1 and 2 have per item production costs c_1 and c_2, resp.

Question: How these firms are going to behave?

We may model the situation using a strategic-form game.

Example: Cournot Duopoly

- Two identical firms, players 1 and 2, produce some good. Denote by q_1 and q_2 quantities produced by firms 1 and 2, resp.
- The total quantity of products in the market is $q_1 + q_2$.
- The price of each item is $\kappa - q_1 - q_2$ (here κ is a positive constant)
- Firms 1 and 2 have per item production costs c_1 and c_2, resp.

Question: How these firms are going to behave?

We may model the situation using a strategic-form game.

Strategic-form game model $(N, (S_i)_{i \in N}, (u_i)_{i \in N})$

- $N = \{1, 2\}$
- $S_i = [0, \infty)$
- $u_1(q_1, q_2) = q_1(\kappa - q_1 - q_2) - q_1 c_1$
  $u_2(q_1, q_2) = q_2(\kappa - q_1 - q_2) - q_2 c_2$

Solution Concepts

A *solution concept* is a method of analyzing games with the objective of restricting the set of *all possible outcomes* to those that are *more reasonable than others.*

Solution Concepts

A *solution concept* is a method of analyzing games with the objective of restricting the set of *all possible outcomes* to those that are *more reasonable than others.*

We will use term *equilibrium* for any one of the strategy profiles that emerges as one of the solution concepts' predictions.
(I follow the approach of Steven Tadelis here, it is not completely standard)

Solution Concepts

A *solution concept* is a method of analyzing games with the objective of restricting the set of *all possible outcomes* to those that are *more reasonable than others.*

We will use term *equilibrium* for any one of the strategy profiles that emerges as one of the solution concepts' predictions.
(I follow the approach of Steven Tadelis here, it is not completely standard)

Example 4

Nash equilibrium is a solution concept. That is, we "solve" games by finding Nash equilibria and declare them to be reasonable outcomes.

Assumptions

Throughout the lecture we assume that:

1. Players are **rational**: a *rational* player is one who chooses his strategy to maximize his payoff.

Assumptions

Throughout the lecture we assume that:

1. Players are **rational**: a *rational* player is one who chooses his strategy to maximize his payoff.
2. Players are **intelligent**: An *intelligent* player knows everything about the game (actions and payoffs) and can make any inferences about the situation that we can make.

Assumptions

Throughout the lecture we assume that:

1. Players are **rational**: a *rational* player is one who chooses his strategy to maximize his payoff.
2. Players are **intelligent**: An *intelligent* player knows everything about the game (actions and payoffs) and can make any inferences about the situation that we can make.
3. **Common knowledge**: The fact that players are rational and intelligent is a common knowledge among them.

Assumptions

Throughout the lecture we assume that:

1. Players are **rational**: a *rational* player is one who chooses his strategy to maximize his payoff.
2. Players are **intelligent**: An *intelligent* player knows everything about the game (actions and payoffs) and can make any inferences about the situation that we can make.
3. **Common knowledge**: The fact that players are rational and intelligent is a common knowledge among them.
4. **Self-enforcement**: Any prediction (or equilibrium) of a solution concept must be *self-enforcing*.

Assumptions

Throughout the lecture we assume that:

1. Players are **rational**: a *rational* player is one who chooses his strategy to maximize his payoff.
2. Players are **intelligent**: An *intelligent* player knows everything about the game (actions and payoffs) and can make any inferences about the situation that we can make.
3. **Common knowledge**: The fact that players are rational and intelligent is a common knowledge among them.
4. **Self-enforcement**: Any prediction (or equilibrium) of a solution concept must be *self-enforcing*.

Here 4. implies non-cooperative game theory: Each player is in control of his actions, and he will stick to an action only if he finds it to be in his best interest.

Evaluating Solution Concepts

In order to evaluate our theory as a methodological tool we use the following criteria:

Evaluating Solution Concepts

In order to evaluate our theory as a methodological tool we use the following criteria:

1. **Existence** (i.e., how often does it apply?): Solution concept should apply to a wide variety of games.

   E.g. We shall see that mixed Nash equilibria exist in all two player finite strategic-form games.

Evaluating Solution Concepts

In order to evaluate our theory as a methodological tool we use the following criteria:

1. **Existence** (i.e., how often does it apply?): Solution concept should apply to a wide variety of games.

   E.g. We shall see that mixed Nash equilibria exist in all two player finite strategic-form games.

2. **Uniqueness** (How much does it restrict behavior?): We demand our solution concept to restrict the behavior as much as possible.

   E.g. So called strictly dominant strategy equilibria are always unique as opposed to Nash eq.

Evaluating Solution Concepts

In order to evaluate our theory as a methodological tool we use the following criteria:

1. **Existence** (i.e., how often does it apply?): Solution concept should apply to a wide variety of games.

   E.g. We shall see that mixed Nash equilibria exist in all two player finite strategic-form games.

2. **Uniqueness** (How much does it restrict behavior?): We demand our solution concept to restrict the behavior as much as possible.

   E.g. So called strictly dominant strategy equilibria are always unique as opposed to Nash eq.

Solution Concepts – Pure Strategies

We will consider the following solution concepts:

- strict dominant strategy equilibrium
- iterated elimination of strictly dominated strategies (IESDS)
- rationalizability
- Nash equilibria

Solution Concepts – Pure Strategies

We will consider the following solution concepts:

- strict dominant strategy equilibrium
- iterated elimination of strictly dominated strategies (IESDS)
- rationalizability
- Nash equilibria

For now, let us concentrate on

pure strategies only!

I.e., no mixed strategies are allowed. We will generalize to mixed setting later.

Notation

- Let $N = \{1, \ldots, n\}$ be a finite set and for each $i \in N$ let X_i be a set. Let $X := \prod_{i \in N} X_i = \{(x_1, \ldots, x_n) \mid x_j \in X_j, j \in N\}$.
  - For $i \in N$ we define $X_{-i} := \prod_{j \neq i} X_j$, i.e.,

    $$X_{-i} = \{(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \mid x_j \in X_j, \forall j \neq i\}$$

  - An element of X_{-i} will be denoted by

    $$x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

    We slightly abuse notation and write (x_i, x_{-i}) to denote $(x_1, \ldots, x_i, \ldots, x_n) \in X$.

Strict Dominance in Pure Strategies

Definition 5

Let $s_i, s_i' \in S_i$ be strategies of player i. Then s_i' is *strictly dominated* by s_i (write $s_i \succ s_i'$) if for any possible combination of the other players' strategies, $s_{-i} \in S_{-i}$, we have

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}) \quad \text{for all } s_{-i} \in S_{-i}$$

Strict Dominance in Pure Strategies

Definition 5

Let $s_i, s_i' \in S_i$ be strategies of player i. Then s_i' is *strictly dominated* by s_i (write $s_i \succ s_i'$) if for any possible combination of the other players' strategies, $s_{-i} \in S_{-i}$, we have

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}) \quad \text{for all } s_{-i} \in S_{-i}$$

Is there a strictly dominated strategy in the Prisoner's dilemma?

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

Strict Dominance in Pure Strategies

Definition 5

Let $s_i, s_i' \in S_i$ be strategies of player i. Then s_i' is *strictly dominated* by s_i (write $s_i \succ s_i'$) if for any possible combination of the other players' strategies, $s_{-i} \in S_{-i}$, we have

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}) \quad \text{for all } s_{-i} \in S_{-i}$$

Is there a strictly dominated strategy in the Prisoner's dilemma?

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

Claim 1

An intelligent and rational player will never play a strictly dominated strategy.

Clearly, intelligence implies that the player should recognize dominated strategies, rationality implies that the player will avoid playing them.

Strictly Dominant Strategy Equilibrium in Pure Str.

Definition 6
$s_i \in S_i$ is *strictly dominant* if every other pure strategy of player i is strictly dominated by s_i.

Strictly Dominant Strategy Equilibrium in Pure Str.

Definition 6

$s_i \in S_i$ is *strictly dominant* if every other pure strategy of player i is strictly dominated by s_i.

Observe that every player has at most one strictly dominant strategy, and that strictly dominant strategies do not have to exist.

Strictly Dominant Strategy Equilibrium in Pure Str.

Definition 6
$s_i \in S_i$ is *strictly dominant* if every other pure strategy of player i is strictly dominated by s_i.

Observe that every player has at most one strictly dominant strategy, and that strictly dominant strategies do not have to exist.

Claim 2
Any rational player will play the strictly dominant strategy (if it exists).

Strictly Dominant Strategy Equilibrium in Pure Str.

Definition 6

$s_i \in S_i$ is *strictly dominant* if every other pure strategy of player i is strictly dominated by s_i.

Observe that every player has at most one strictly dominant strategy, and that strictly dominant strategies do not have to exist.

Claim 2

Any rational player will play the strictly dominant strategy (if it exists).

Definition 7

A strategy profile $s \in S$ is a *strictly dominant strategy equilibrium* if $s_i \in S_i$ is strictly dominant for all $i \in N$.

Strictly Dominant Strategy Equilibrium in Pure Str.

Definition 6

$s_i \in S_i$ is *strictly dominant* if every other pure strategy of player i is strictly dominated by s_i.

Observe that every player has at most one strictly dominant strategy, and that strictly dominant strategies do not have to exist.

Claim 2

Any rational player will play the strictly dominant strategy (if it exists).

Definition 7

A strategy profile $s \in S$ is a *strictly dominant strategy equilibrium* if $s_i \in S_i$ is strictly dominant for all $i \in N$.

Corollary 8

If the strictly dominant strategy equilibrium exists, it is unique and rational players will play it.

Examples

In the Prisoner's dilemma:

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

Examples

In the Prisoner's dilemma:

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

(C, C) is the strictly dominant strategy equilibrium.

Examples

In the Prisoner's dilemma:

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

(C, C) is the strictly dominant strategy equilibrium.

In the Battle of Sexes:

	O	F
O	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

Examples

In the Prisoner's dilemma:

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

(C, C) is the strictly dominant strategy equilibrium.

In the Battle of Sexes:

	O	F
O	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

no strictly dominant strategies exist.

Examples

In the Prisoner's dilemma:

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

(C, C) is the strictly dominant strategy equilibrium.

In the Battle of Sexes:

	O	F
O	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

no strictly dominant strategies exist.

Indiana Jones and the Last Crusade

(Taken from Dixit & Nalebuff’s "The Art of Strategy" and a lecture of Robert Marks)

Indiana Jones, his father, and the Nazis have all converged at the site of the Holy Grail. The two Joneses refuse to help the Nazis reach the last step. So the Nazis shoot Indiana’s dad. Only the healing power of the Holy Grail can save the senior Dr. Jones from his mortal wound. Suitably motivated, Indiana leads the way to the Holy Grail. But there is one final challenge. He must choose between literally scores of chalices, only one of which is the cup of Christ. While the right cup brings eternal life, the wrong choice is fatal. The Nazi leader impatiently chooses a beautiful gold chalice, drinks the holy water, and dies from the sudden death that follows from the wrong choice. Indiana picks a wooden chalice, the cup of a carpenter. Exclaiming "There’s only one way to find out" he dips the chalice into the font and drinks what he hopes is the cup of life. Upon discovering that he has chosen wisely, Indiana brings the cup to his father and the water heals the mortal wound.

Indiana Jones and the Last Crusade (cont.)

Indy Goofed

- Although this scene adds excitement, it is somewhat embarrassing that such a distinguished professor as Dr. Indiana Jones would overlook his dominant strategy.
- He should have given the water to his father without testing it first.
  - If Indiana has chosen the right cup, his father is still saved.
  - If Indiana has chosen the wrong cup, then his father dies but Indiana is spared.
- Testing the cup before giving it to his father doesn't help, since if Indiana has made the wrong choice, there is no second chance – Indiana dies from the water and his father dies from the wound.

Iterated Strict Dominance in Pure Strategies

We know that no rational player ever plays strictly dominated strategies.

Iterated Strict Dominance in Pure Strategies

We know that no rational player ever plays strictly dominated strategies.

As each player knows that each player is rational, each player knows that his opponents will not play strictly dominated strategies and thus all opponents know that *effectively* they are facing a "smaller" game.

Iterated Strict Dominance in Pure Strategies

We know that no rational player ever plays strictly dominated strategies.

As each player knows that each player is rational, each player knows that his opponents will not play strictly dominated strategies and thus all opponents know that *effectively* they are facing a "smaller" game.

As rationality is a common knowledge, everyone knows that everyone knows that the game is effectively smaller.

Iterated Strict Dominance in Pure Strategies

We know that no rational player ever plays strictly dominated strategies.

As each player knows that each player is rational, each player knows that his opponents will not play strictly dominated strategies and thus all opponents know that *effectively* they are facing a "smaller" game.

As rationality is a common knowledge, everyone knows that everyone knows that the game is effectively smaller.

Thus everyone knows, that nobody will play strictly dominated strategies in the smaller game (and such strategies may indeed exist).

Iterated Strict Dominance in Pure Strategies

We know that no rational player ever plays strictly dominated strategies.

As each player knows that each player is rational, each player knows that his opponents will not play strictly dominated strategies and thus all opponents know that *effectively* they are facing a "smaller" game.

As rationality is a common knowledge, everyone knows that everyone knows that the game is effectively smaller.

Thus everyone knows, that nobody will play strictly dominated strategies in the smaller game (and such strategies may indeed exist).

Because it is a common knowledge that all players will perform this kind of reasoning again, the process can continue until no more strictly dominated strategies can be eliminated.

IESDS

The previous reasoning yields the **Iterated Elimination of Strictly Dominated Strategies (IESDS)**:

Define a sequence $D_i^0, D_i^1, D_i^2, \dots$ of strategy sets of player i.
(Denote by G_{DS}^k the game obtained from G by restricting to D_i^k, $i \in N$.)

IESDS

The previous reasoning yields the **Iterated Elimination of Strictly Dominated Strategies (IESDS)**:

Define a sequence $D_i^0, D_i^1, D_i^2, \ldots$ of strategy sets of player i.
(Denote by G_{DS}^k the game obtained from G by restricting to D_i^k, $i \in N$.)

1. Initialize $k = 0$ and $D_i^0 = S_i$ for each $i \in N$.

IESDS

The previous reasoning yields the **Iterated Elimination of Strictly Dominated Strategies (IESDS)**:

Define a sequence $D_i^0, D_i^1, D_i^2, \ldots$ of strategy sets of player i.
(Denote by G_{DS}^k the game obtained from G by restricting to D_i^k, $i \in N$.)

1. Initialize $k = 0$ and $D_i^0 = S_i$ for each $i \in N$.
2. For all players $i \in N$: Let D_i^{k+1} be the set of all pure strategies of D_i^k that are **not** strictly dominated in G_{DS}^k.

IESDS

The previous reasoning yields the **Iterated Elimination of Strictly Dominated Strategies (IESDS)**:

Define a sequence $D_i^0, D_i^1, D_i^2, \ldots$ of strategy sets of player i.
(Denote by G_{DS}^k the game obtained from G by restricting to D_i^k, $i \in N$.)

1. Initialize $k = 0$ and $D_i^0 = S_i$ for each $i \in N$.
2. For all players $i \in N$: Let D_i^{k+1} be the set of all pure strategies of D_i^k that are **not** strictly dominated in G_{DS}^k.
3. Let $k := k + 1$ and go to 2.

IESDS

The previous reasoning yields the **Iterated Elimination of Strictly Dominated Strategies (IESDS)**:

Define a sequence $D_i^0, D_i^1, D_i^2, \ldots$ of strategy sets of player i.
(Denote by G_{DS}^k the game obtained from G by restricting to D_i^k, $i \in N$.)

1. Initialize $k = 0$ and $D_i^0 = S_i$ for each $i \in N$.
2. For all players $i \in N$: Let D_i^{k+1} be the set of all pure strategies of D_i^k that are **not** strictly dominated in G_{DS}^k.
3. Let $k := k + 1$ and go to 2.

We say that $s_i \in S_i$ *survives IESDS* if $s_i \in D_i^k$ for all $k = 0, 1, 2, \ldots$

IESDS

The previous reasoning yields the **Iterated Elimination of Strictly Dominated Strategies (IESDS)**:

Define a sequence $D_i^0, D_i^1, D_i^2, \ldots$ of strategy sets of player i.
(Denote by G_{DS}^k the game obtained from G by restricting to D_i^k, $i \in N$.)

1. Initialize $k = 0$ and $D_i^0 = S_i$ for each $i \in N$.
2. For all players $i \in N$: Let D_i^{k+1} be the set of all pure strategies of D_i^k that are **not** strictly dominated in G_{DS}^k.
3. Let $k := k + 1$ and go to 2.

We say that $s_i \in S_i$ *survives IESDS* if $s_i \in D_i^k$ for all $k = 0, 1, 2, \ldots$

Definition 9
A strategy profile $s = (s_1, \ldots, s_n) \in S$ is an *IESDS equilibrium* if each s_i survives IESDS.

IESDS

The previous reasoning yields the **Iterated Elimination of Strictly Dominated Strategies (IESDS)**:

Define a sequence $D_i^0, D_i^1, D_i^2, \ldots$ of strategy sets of player i.
(Denote by G_{DS}^k the game obtained from G by restricting to D_i^k, $i \in N$.)

1. Initialize $k = 0$ and $D_i^0 = S_i$ for each $i \in N$.
2. For all players $i \in N$: Let D_i^{k+1} be the set of all pure strategies of D_i^k that are **not** strictly dominated in G_{DS}^k.
3. Let $k := k + 1$ and go to 2.

We say that $s_i \in S_i$ *survives IESDS* if $s_i \in D_i^k$ for all $k = 0, 1, 2, \ldots$

Definition 9

A strategy profile $s = (s_1, \ldots, s_n) \in S$ is an *IESDS equilibrium* if each s_i survives IESDS.

A game is *IESDS solvable* if it has a unique IESDS equilibrium.

IESDS

The previous reasoning yields the **Iterated Elimination of Strictly Dominated Strategies (IESDS)**:

Define a sequence $D_i^0, D_i^1, D_i^2, \ldots$ of strategy sets of player i.
(Denote by G_{DS}^k the game obtained from G by restricting to D_i^k, $i \in N$.)

1. Initialize $k = 0$ and $D_i^0 = S_i$ for each $i \in N$.
2. For all players $i \in N$: Let D_i^{k+1} be the set of all pure strategies of D_i^k that are **not** strictly dominated in G_{DS}^k.
3. Let $k := k + 1$ and go to 2.

We say that $s_i \in S_i$ *survives IESDS* if $s_i \in D_i^k$ for all $k = 0, 1, 2, \ldots$

Definition 9

A strategy profile $s = (s_1, \ldots, s_n) \in S$ is an *IESDS equilibrium* if each s_i survives IESDS.

A game is *IESDS solvable* if it has a unique IESDS equilibrium.

Remark: If all S_i are *finite*, then in 2. we may remove only some of the strictly dominated strategies (not necessarily all). The result is *not* affected by the order of elimination since strictly dominated strategies remain strictly dominated even after removing some other strictly dominated strategies.

IESDS Examples

In the Prisoner's dilemma:

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

IESDS Examples

In the Prisoner's dilemma:

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

(C, C) is the only one surviving the first round of IESDS.

IESDS Examples

In the Prisoner's dilemma:

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

(C, C) is the only one surviving the first round of IESDS.

In the Battle of Sexes:

	O	F
O	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

IESDS Examples

In the Prisoner's dilemma:

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

(C, C) is the only one surviving the first round of IESDS.

In the Battle of Sexes:

	O	F
O	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

all strategies survive all rounds (i.e. IESDS $\equiv$ anything may happen, sorry)

A Bit More Interesting Example

	L	C	R
L	$4,3$	$5,1$	$6,2$
C	$2,1$	$8,4$	$3,6$
R	$3,0$	$9,6$	$2,8$

IESDS on greenboard!

Political Science Example: Median Voter Theorem

Hotelling (1929) and Downs (1957)

- $N = \{1, 2\}$

Political Science Example: Median Voter Theorem

Hotelling (1929) and Downs (1957)

- $N = \{1, 2\}$
- $S_i = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ (political and ideological spectrum)

Political Science Example: Median Voter Theorem

Hotelling (1929) and Downs (1957)

- $N = \{1, 2\}$
- $S_i = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ (political and ideological spectrum)
- 10 voters belong to each position

  (Here 10 means ten percent in the real-world)

Political Science Example: Median Voter Theorem

Hotelling (1929) and Downs (1957)

- $N = \{1, 2\}$
- $S_i = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ (political and ideological spectrum)
- 10 voters belong to each position

  (Here 10 means ten percent in the real-world)
- Voters vote for the closest candidate. If there is a tie, then $\frac{1}{2}$ got to each candidate

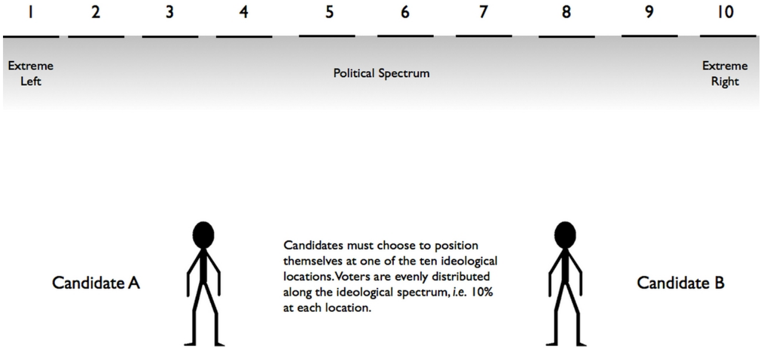
Political Science Example: Median Voter Theorem

Hotelling (1929) and Downs (1957)

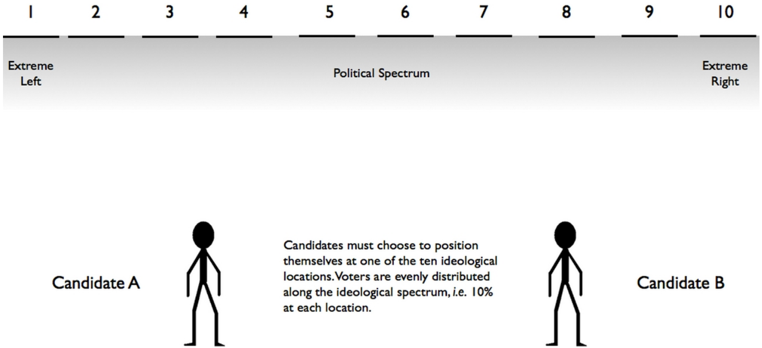
- $N = \{1, 2\}$
- $S_i = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ (political and ideological spectrum)
- 10 voters belong to each position

  (Here 10 means ten percent in the real-world)
- Voters vote for the closest candidate. If there is a tie, then $\frac{1}{2}$ got to each candidate
- Payoff: The number of voters for the candidate, each candidate (selfishly) strives to maximize this number

Political Science Example: Median Voter Theorem

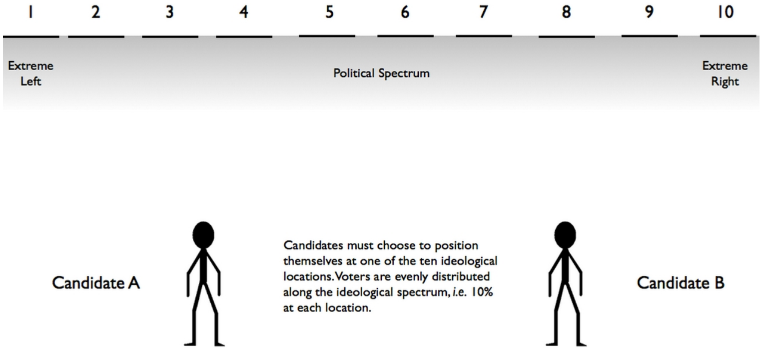


Political Science Example: Median Voter Theorem



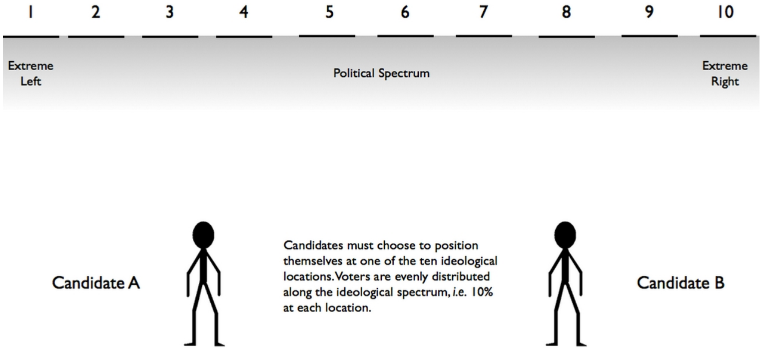
- 1 and 10 are the (only) strictly dominated strategies $\Rightarrow$ $D_1^1 = D_2^1 = \{2, \ldots, 9\}$

Political Science Example: Median Voter Theorem



- 1 and 10 are the (only) strictly dominated strategies $\Rightarrow$ $D_1^1 = D_2^1 = \{2, \ldots, 9\}$
- in G_{DS}^1, 2 and 9 are the (only) strictly dominated strategies $\Rightarrow$ $D_1^2 = D_2^2 = \{3, \ldots, 8\}$

Political Science Example: Median Voter Theorem



- 1 and 10 are the (only) strictly dominated strategies $\Rightarrow$ $D_1^1 = D_2^1 = \{2, \ldots, 9\}$
- in G_{DS}^1, 2 and 9 are the (only) strictly dominated strategies $\Rightarrow$ $D_1^2 = D_2^2 = \{3, \ldots, 8\}$
- $\ldots$
- only $5, 6$ survive IESDS

Belief & Best Response

IESDS eliminated apparently unreasonable behavior (leaving "reasonable" behavior implicitly untouched).

Belief & Best Response

IESDS eliminated apparently unreasonable behavior (leaving "reasonable" behavior implicitly untouched).

What if we rather want to actively preserve reasonable behavior? What is reasonable? .... what we believe is reasonable :-).

Belief & Best Response

IESDS eliminated apparently unreasonable behavior (leaving "reasonable" behavior implicitly untouched).

What if we rather want to actively preserve reasonable behavior? What is reasonable? .... what we believe is reasonable :-).

Intuition:

Belief & Best Response

IESDS eliminated apparently unreasonable behavior (leaving "reasonable" behavior implicitly untouched).

What if we rather want to actively preserve reasonable behavior? What is reasonable? .... what we believe is reasonable :-).

Intuition:

- Imagine that your colleague did something stupid

Belief & Best Response

IESDS eliminated apparently unreasonable behavior (leaving "reasonable" behavior implicitly untouched).

What if we rather want to actively preserve reasonable behavior? What is reasonable? .... what we believe is reasonable :-).

Intuition:

- Imagine that your colleague did something stupid
- What would you ask him? Usually something like "What were you thinking?"

Belief & Best Response

IESDS eliminated apparently unreasonable behavior (leaving "reasonable" behavior implicitly untouched).

What if we rather want to actively preserve reasonable behavior? What is reasonable? .... what we believe is reasonable :-).

Intuition:

- Imagine that your colleague did something stupid
- What would you ask him? Usually something like "What were you thinking?"
- The colleague may respond with a reasonable description of his *belief* in which his action was (one of) the best he could do

  (You may of course question reasonableness of the belief)

Belief & Best Response

IESDS eliminated apparently unreasonable behavior (leaving "reasonable" behavior implicitly untouched).

What if we rather want to actively preserve reasonable behavior? What is reasonable? .... what we believe is reasonable :-).

Intuition:

- Imagine that your colleague did something stupid
- What would you ask him? Usually something like "What were you thinking?"
- The colleague may respond with a reasonable description of his *belief* in which his action was (one of) the best he could do

  (You may of course question reasonableness of the belief)

Let us formalize this type of reasoning ....

Belief & Best Response

Definition 10

A *belief* of player i is a pure strategy profile $s_{-i} \in S_{-i}$ of his opponents.

Belief & Best Response

Definition 10
A *belief* of player i is a pure strategy profile $s_{-i} \in S_{-i}$ of his opponents.

Definition 11
A strategy $s_i \in S_i$ of player i is a *best response* to a belief $s_{-i} \in S_{-i}$ if

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \text{ for all } s_i' \in S_i$$

Belief & Best Response

Definition 10
A *belief* of player i is a pure strategy profile $s_{-i} \in S_{-i}$ of his opponents.

Definition 11
A strategy $s_i \in S_i$ of player i is a *best response* to a belief $s_{-i} \in S_{-i}$ if

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \text{ for all } s_i' \in S_i$$

Claim 3
A rational player who believes that his opponents will play $s_{-i} \in S_{-i}$ always chooses a best response to $s_{-i} \in S_{-i}$.

Belief & Best Response

Definition 10
A *belief* of player i is a pure strategy profile $s_{-i} \in S_{-i}$ of his opponents.

Definition 11
A strategy $s_i \in S_i$ of player i is a *best response* to a belief $s_{-i} \in S_{-i}$ if

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \text{ for all } s_i' \in S_i$$

Claim 3
A rational player who believes that his opponents will play $s_{-i} \in S_{-i}$ always chooses a best response to $s_{-i} \in S_{-i}$.

Definition 12
A strategy $s_i \in S_i$ is *never best response* if it is not a best response to any belief $s_{-i} \in S_{-i}$.

Belief & Best Response

Definition 10
A *belief* of player i is a pure strategy profile $s_{-i} \in S_{-i}$ of his opponents.

Definition 11
A strategy $s_i \in S_i$ of player i is a *best response* to a belief $s_{-i} \in S_{-i}$ if

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \text{ for all } s_i' \in S_i$$

Claim 3
A rational player who believes that his opponents will play $s_{-i} \in S_{-i}$ always chooses a best response to $s_{-i} \in S_{-i}$.

Definition 12
A strategy $s_i \in S_i$ is *never best response* if it is not a best response to any belief $s_{-i} \in S_{-i}$.

A rational player never plays any strategy that is never best response.

Best Response vs Strict Dominance

Proposition 1

If s_i is strictly dominated for player i, then it is never best response.

Best Response vs Strict Dominance

Proposition 1

If s_i is strictly dominated for player i, then it is never best response.

The opposite does not have to be true in pure strategies:

	X	Y
A	$1, 1$	$1, 1$
B	$2, 1$	$0, 1$
C	$0, 1$	$2, 1$

Here A is never best response but is strictly dominated neither by B, nor by C.

Elimination of Stupid Strategies = Rationalizability

Using similar iterated reasoning as for IESDS, strategies that are never best response can be iteratively eliminated.

Define a sequence $R_i^0, R_i^1, R_i^2, \ldots$ of strategy sets of player i.
(Denote by G_{Rat}^k the game obtained from G by restricting to R_i^k, $i \in N$.)

Elimination of Stupid Strategies = Rationalizability

Using similar iterated reasoning as for IESDS, strategies that are never best response can be iteratively eliminated.

Define a sequence $R_i^0, R_i^1, R_i^2, \ldots$ of strategy sets of player i.
(Denote by G_{Rat}^k the game obtained from G by restricting to R_i^k, $i \in N$.)

1. Initialize $k = 0$ and $R_i^0 = S_i$ for each $i \in N$.

Elimination of Stupid Strategies = Rationalizability

Using similar iterated reasoning as for IESDS, strategies that are never best response can be iteratively eliminated.

Define a sequence $R_i^0, R_i^1, R_i^2, \ldots$ of strategy sets of player i.
(Denote by G_{Rat}^k the game obtained from G by restricting to R_i^k, $i \in N$.)

1. Initialize $k = 0$ and $R_i^0 = S_i$ for each $i \in N$.
2. For all players $i \in N$: Let R_i^{k+1} be the set of all strategies of R_i^k that are best responses to some beliefs in G_{Rat}^k.

Elimination of Stupid Strategies = Rationalizability

Using similar iterated reasoning as for IESDS, strategies that are never best response can be iteratively eliminated.

Define a sequence $R_i^0, R_i^1, R_i^2, \ldots$ of strategy sets of player i.
(Denote by G_{Rat}^k the game obtained from G by restricting to R_i^k, $i \in N$.)

1. Initialize $k = 0$ and $R_i^0 = S_i$ for each $i \in N$.
2. For all players $i \in N$: Let R_i^{k+1} be the set of all strategies of R_i^k that are best responses to some beliefs in G_{Rat}^k.
3. Let $k := k + 1$ and go to 2.

We say that $s_i \in S_i$ is *rationalizable* if $s_i \in R_i^k$ for all $k = 0, 1, 2, \ldots$

Elimination of Stupid Strategies = Rationalizability

Using similar iterated reasoning as for IESDS, strategies that are never best response can be iteratively eliminated.

Define a sequence $R_i^0, R_i^1, R_i^2, \ldots$ of strategy sets of player i.
(Denote by G_{Rat}^k the game obtained from G by restricting to R_i^k, $i \in N$.)

1. Initialize $k = 0$ and $R_i^0 = S_i$ for each $i \in N$.
2. For all players $i \in N$: Let R_i^{k+1} be the set of all strategies of R_i^k that are best responses to some beliefs in G_{Rat}^k.
3. Let $k := k + 1$ and go to 2.

We say that $s_i \in S_i$ is *rationalizable* if $s_i \in R_i^k$ for all $k = 0, 1, 2, \ldots$

Definition 13
A strategy profile $s = (s_1, \ldots, s_n) \in S$ is a *rationalizable equilibrium* if each s_i is rationalizable.

Elimination of Stupid Strategies = Rationalizability

Using similar iterated reasoning as for IESDS, strategies that are never best response can be iteratively eliminated.

Define a sequence $R_i^0, R_i^1, R_i^2, \ldots$ of strategy sets of player i.
(Denote by G_{Rat}^k the game obtained from G by restricting to R_i^k, $i \in N$.)

1. Initialize $k = 0$ and $R_i^0 = S_i$ for each $i \in N$.
2. For all players $i \in N$: Let R_i^{k+1} be the set of all strategies of R_i^k that are best responses to some beliefs in G_{Rat}^k.
3. Let $k := k + 1$ and go to 2.

We say that $s_i \in S_i$ is *rationalizable* if $s_i \in R_i^k$ for all $k = 0, 1, 2, \ldots$

Definition 13
A strategy profile $s = (s_1, \ldots, s_n) \in S$ is a *rationalizable equilibrium* if each s_i is rationalizable.

We say that a game is *solvable by rationalizability* if it has a unique rationalizable equilibrium.

Elimination of Stupid Strategies = Rationalizability

Using similar iterated reasoning as for IESDS, strategies that are never best response can be iteratively eliminated.

Define a sequence $R_i^0, R_i^1, R_i^2, \ldots$ of strategy sets of player i.
(Denote by G_{Rat}^k the game obtained from G by restricting to R_i^k, $i \in N$.)

1. Initialize $k = 0$ and $R_i^0 = S_i$ for each $i \in N$.
2. For all players $i \in N$: Let R_i^{k+1} be the set of all strategies of R_i^k that are best responses to some beliefs in G_{Rat}^k.
3. Let $k := k + 1$ and go to 2.

We say that $s_i \in S_i$ is *rationalizable* if $s_i \in R_i^k$ for all $k = 0, 1, 2, \ldots$

Definition 13
A strategy profile $s = (s_1, \ldots, s_n) \in S$ is a *rationalizable equilibrium* if each s_i is rationalizable.

We say that a game is *solvable by rationalizability* if it has a unique rationalizable equilibrium.

(Warning: For some reasons, rationalizable strategies are almost always defined using mixed strategies!)

Rationalizability Examples

In the Prisoner's dilemma:

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

Rationalizability Examples

In the Prisoner's dilemma:

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

(C, C) is the only rationalizable equilibrium.

Rationalizability Examples

In the Prisoner's dilemma:

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

(C, C) is the only rationalizable equilibrium.

In the Battle of Sexes:

	O	F
O	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

Rationalizability Examples

In the Prisoner's dilemma:

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

(C, C) is the only rationalizable equilibrium.

In the Battle of Sexes:

	O	F
O	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

all strategies are rationalizable.

Cournot Duopoly

$G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$

- $N = \{1, 2\}$
- $S_i = [0, \infty)$
- $u_1(q_1, q_2) = q_1(\kappa - q_1 - q_2) - q_1 c_1 = (\kappa - c_1)q_1 - q_1^2 - q_1 q_2$
  $u_2(q_1, q_2) = q_2(\kappa - q_2 - q_1) - q_2 c_2 = (\kappa - c_2)q_2 - q_2^2 - q_2 q_1$

Assume for simplicity that $c_1 = c_2 = c$ and denote $\theta = \kappa - c$.

What is a best response of player 1 to a given q_2 ?

Cournot Duopoly

$G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$

- $N = \{1, 2\}$
- $S_i = [0, \infty)$
- $u_1(q_1, q_2) = q_1(\kappa - q_1 - q_2) - q_1 c_1 = (\kappa - c_1)q_1 - q_1^2 - q_1 q_2$
  $u_2(q_1, q_2) = q_2(\kappa - q_2 - q_1) - q_2 c_2 = (\kappa - c_2)q_2 - q_2^2 - q_2 q_1$

Assume for simplicity that $c_1 = c_2 = c$ and denote $\theta = \kappa - c$.

What is a best response of player 1 to a given q_2 ?

Solve $\frac{\delta u_1}{\delta q_1} = \theta - 2q_1 - q_2 = 0$, which gives that $q_1 = (\theta - q_2)/2$ is the only best response of player 1 to q_2.
Similarly, $q_2 = (\theta - q_1)/2$ is the only best response of player 2 to q_1.

Cournot Duopoly

$G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$

- $N = \{1, 2\}$
- $S_i = [0, \infty)$
- $u_1(q_1, q_2) = q_1(\kappa - q_1 - q_2) - q_1 c_1 = (\kappa - c_1)q_1 - q_1^2 - q_1 q_2$
  $u_2(q_1, q_2) = q_2(\kappa - q_2 - q_1) - q_2 c_2 = (\kappa - c_2)q_2 - q_2^2 - q_2 q_1$

Assume for simplicity that $c_1 = c_2 = c$ and denote $\theta = \kappa - c$.

What is a best response of player 1 to a given q_2 ?

Solve $\frac{\delta u_1}{\delta q_1} = \theta - 2q_1 - q_2 = 0$, which gives that $q_1 = (\theta - q_2)/2$ is the only best response of player 1 to q_2.
Similarly, $q_2 = (\theta - q_1)/2$ is the only best response of player 2 to q_1.

Since $q_2 \geq 0$, we obtain that q_1 is never best response iff $q_1 > \theta/2$.
Similarly q_2 is never best response iff $q_2 > \theta/2$.

Cournot Duopoly

$G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$

- $N = \{1, 2\}$
- $S_i = [0, \infty)$
- $u_1(q_1, q_2) = q_1(\kappa - q_1 - q_2) - q_1 c_1 = (\kappa - c_1)q_1 - q_1^2 - q_1 q_2$
  $u_2(q_1, q_2) = q_2(\kappa - q_2 - q_1) - q_2 c_2 = (\kappa - c_2)q_2 - q_2^2 - q_2 q_1$

Assume for simplicity that $c_1 = c_2 = c$ and denote $\theta = \kappa - c$.

What is a best response of player 1 to a given q_2 ?

Solve $\frac{\delta u_1}{\delta q_1} = \theta - 2q_1 - q_2 = 0$, which gives that $q_1 = (\theta - q_2)/2$ is the only best response of player 1 to q_2.
Similarly, $q_2 = (\theta - q_1)/2$ is the only best response of player 2 to q_1.

Since $q_2 \geq 0$, we obtain that q_1 is never best response iff $q_1 > \theta/2$.
Similarly q_2 is never best response iff $q_2 > \theta/2$.

Thus $R_1^1 = R_2^1 = [0, \theta/2]$.

Cournot Duopoly

$G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$

- $N = \{1, 2\}$
- $S_i = [0, \infty)$
- $u_1(q_1, q_2) = q_1(\kappa - q_1 - q_2) - q_1 c_1 = (\kappa - c_1)q_1 - q_1^2 - q_1 q_2$
  $u_2(q_1, q_2) = q_2(\kappa - q_2 - q_1) - q_2 c_2 = (\kappa - c_2)q_2 - q_2^2 - q_2 q_1$

Assume for simplicity that $c_1 = c_2 = c$ and denote $\theta = \kappa - c$.

Now, in G^1_{Rat}, we still have that $q_1 = (\theta - q_2)/2$ is the best response to q_2, and $q_2 = (\theta - q_1)/2$ the best resp. to q_1

Cournot Duopoly

$G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$

- $N = \{1, 2\}$
- $S_i = [0, \infty)$
- $u_1(q_1, q_2) = q_1(\kappa - q_1 - q_2) - q_1 c_1 = (\kappa - c_1)q_1 - q_1^2 - q_1 q_2$
  $u_2(q_1, q_2) = q_2(\kappa - q_2 - q_1) - q_2 c_2 = (\kappa - c_2)q_2 - q_2^2 - q_2 q_1$

Assume for simplicity that $c_1 = c_2 = c$ and denote $\theta = \kappa - c$.

Now, in G^1_{Rat}, we still have that $q_1 = (\theta - q_2)/2$ is the best response to q_2, and $q_2 = (\theta - q_1)/2$ the best resp. to q_1

Since $q_2 \in R^1_2 = [0, \theta/2]$, we obtain that q_1 is never best response iff $q_1 \in [0, \theta/4)$
Similarly q_2 is never best response iff $q_2 \in [0, \theta/4)$

Cournot Duopoly

$G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$

- $N = \{1, 2\}$
- $S_i = [0, \infty)$
- $u_1(q_1, q_2) = q_1(\kappa - q_1 - q_2) - q_1 c_1 = (\kappa - c_1)q_1 - q_1^2 - q_1 q_2$
  $u_2(q_1, q_2) = q_2(\kappa - q_2 - q_1) - q_2 c_2 = (\kappa - c_2)q_2 - q_2^2 - q_2 q_1$

Assume for simplicity that $c_1 = c_2 = c$ and denote $\theta = \kappa - c$.

Now, in G^1_{Rat}, we still have that $q_1 = (\theta - q_2)/2$ is the best response to q_2, and $q_2 = (\theta - q_1)/2$ the best resp. to q_1

Since $q_2 \in R^1_2 = [0, \theta/2]$, we obtain that q_1 is never best response iff $q_1 \in [0, \theta/4)$
Similarly q_2 is never best response iff $q_2 \in [0, \theta/4)$

Thus $R^2_1 = R^2_2 = [\theta/4, \theta/2]$.

....

Cournot Duopoly (cont.)

$G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$

- $N = \{1, 2\}$
- $S_i = [0, \infty)$
- $u_1(q_1, q_2) = q_1(\kappa - q_1 - q_2) - q_1 c_1 = (\kappa - c_1)q_1 - q_1^2 - q_1 q_2$
  $u_2(q_1, q_2) = q_2(\kappa - q_2 - q_1) - q_2 c_2 = (\kappa - c_2)q_2 - q_2^2 - q_2 q_1$

Assume for simplicity that $c_1 = c_2 = c$ and denote $\theta = \kappa - c$.

In general, after $2k$ iterations we have $R_i^{2k} = R_i^{2k} = [\ell_k, r_k]$ where

- $r_k = (\theta - \ell_{k-1})/2$ for $k \geq 1$
- $\ell_k = (\theta - r_k)/2$ for $k \geq 1$ and $\ell_0 = 0$

Cournot Duopoly (cont.)

$G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$

- $N = \{1, 2\}$
- $S_i = [0, \infty)$
- $u_1(q_1, q_2) = q_1(\kappa - q_1 - q_2) - q_1 c_1 = (\kappa - c_1)q_1 - q_1^2 - q_1 q_2$
  $u_2(q_1, q_2) = q_2(\kappa - q_2 - q_1) - q_2 c_2 = (\kappa - c_2)q_2 - q_2^2 - q_2 q_1$

Assume for simplicity that $c_1 = c_2 = c$ and denote $\theta = \kappa - c$.

In general, after $2k$ iterations we have $R_i^{2k} = R_i^{2k} = [\ell_k, r_k]$ where

- $r_k = (\theta - \ell_{k-1})/2$ for $k \geq 1$
- $\ell_k = (\theta - r_k)/2$ for $k \geq 1$ and $\ell_0 = 0$

Solving the recurrence we obtain

- $\ell_k = \theta/3 - \left(\frac{1}{4}\right)^k \theta/3$
- $r_k = \theta/3 + \left(\frac{1}{4}\right)^{k-1} \theta/6$

Cournot Duopoly (cont.)

$G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$

- $N = \{1, 2\}$
- $S_i = [0, \infty)$
- $u_1(q_1, q_2) = q_1(\kappa - q_1 - q_2) - q_1 c_1 = (\kappa - c_1)q_1 - q_1^2 - q_1 q_2$
  $u_2(q_1, q_2) = q_2(\kappa - q_2 - q_1) - q_2 c_2 = (\kappa - c_2)q_2 - q_2^2 - q_2 q_1$

Assume for simplicity that $c_1 = c_2 = c$ and denote $\theta = \kappa - c$.

In general, after $2k$ iterations we have $R_i^{2k} = R_i^{2k} = [\ell_k, r_k]$ where

- $r_k = (\theta - \ell_{k-1})/2$ for $k \geq 1$
- $\ell_k = (\theta - r_k)/2$ for $k \geq 1$ and $\ell_0 = 0$

Solving the recurrence we obtain

- $\ell_k = \theta/3 - \left(\frac{1}{4}\right)^k \theta/3$
- $r_k = \theta/3 + \left(\frac{1}{4}\right)^{k-1} \theta/6$

Hence, $\lim_{k\to\infty} \ell_k = \lim_{k\to\infty} r_k = \theta/3$ and thus $(\theta/3, \theta/3)$ is the only rationalizable equilibrium.

Cournot Duopoly (cont.)

$G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$

- $N = \{1, 2\}$
- $S_i = [0, \infty)$
- $u_1(q_1, q_2) = q_1(\kappa - q_1 - q_2) - q_1 c_1 = (\kappa - c_1)q_1 - q_1^2 - q_1 q_2$
  $u_2(q_1, q_2) = q_2(\kappa - q_2 - q_1) - q_2 c_2 = (\kappa - c_2)q_2 - q_2^2 - q_2 q_1$

Assume for simplicity that $c_1 = c_2 = c$ and denote $\theta = \kappa - c$.

Are $q_i = \theta/3$ the best outcomes possible?

Cournot Duopoly (cont.)

$G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$

- $N = \{1, 2\}$
- $S_i = [0, \infty)$
- $u_1(q_1, q_2) = q_1(\kappa - q_1 - q_2) - q_1 c_1 = (\kappa - c_1)q_1 - q_1^2 - q_1 q_2$
  $u_2(q_1, q_2) = q_2(\kappa - q_2 - q_1) - q_2 c_2 = (\kappa - c_2)q_2 - q_2^2 - q_2 q_1$

Assume for simplicity that $c_1 = c_2 = c$ and denote $\theta = \kappa - c$.

Are $q_i = \theta/3$ the best outcomes possible? NO!

$$u_1(\theta/3, \theta/3) = u_2(\theta/3, \theta/3) = \theta^2/9$$

but

$$u_1(\theta/4, \theta/4) = u_2(\theta/4, \theta/4) = \theta^2/8$$

IESDS vs Rationalizability in Pure Strategies

Theorem 14
Assume that S is finite. Then for all k we have that $R_i^k \subseteq D_i^k$. That is, in particular, all rationalizable strategies survive IESDS.

IESDS vs Rationalizability in Pure Strategies

Theorem 14
Assume that S is finite. Then for all k we have that $R_i^k \subseteq D_i^k$. That is, in particular, all rationalizable strategies survive IESDS.

The opposite inclusion does not have to be true in pure strategies:

	X	Y
A	$1, 1$	$1, 1$
B	$2, 1$	$0, 1$
C	$0, 1$	$2, 1$

IESDS vs Rationalizability in Pure Strategies

Theorem 14
Assume that S is finite. Then for all k we have that $R_i^k \subseteq D_i^k$. That is, in particular, all rationalizable strategies survive IESDS.

The opposite inclusion does not have to be true in pure strategies:

	X	Y
A	$1, 1$	$1, 1$
B	$2, 1$	$0, 1$
C	$0, 1$	$2, 1$

Recall that A is never best response but is strictly dominated by neither B, nor C. That is, A survives IESDS but is not rationalizable.

IESDS vs Rationalizability in Pure Strategies

Theorem 14
Assume that S is finite. Then for all k we have that $R_i^k \subseteq D_i^k$. That is, in particular, all rationalizable strategies survive IESDS.

The opposite inclusion does not have to be true in pure strategies:

	X	Y
A	$1, 1$	$1, 1$
B	$2, 1$	$0, 1$
C	$0, 1$	$2, 1$

Recall that A is never best response but is strictly dominated by neither B, nor C. That is, A survives IESDS but is not rationalizable.

Proof of Theorem 14

Claim
If s_i is a best response to s_{-i} in G^k_{Rat}, then s_i is a best response to s_{-i} in G.

Proof of Theorem 14

Claim
If s_i is a best response to s_{-i} in G^k_{Rat}, then s_i is a best response to s_{-i} in G.

Proof of the Claim. By induction on k. For $k = 0$ we have $G^k_{Rat} = G^0_{Rat} = G$ and the claim holds trivially.

Proof of Theorem 14

Claim
If s_i is a best response to s_{-i} in G^k_{Rat}, then s_i is a best response to s_{-i} in G.

Proof of the Claim. By induction on k. For $k = 0$ we have $G^k_{Rat} = G^0_{Rat} = G$ and the claim holds trivially.

Assume that the claim is true for some k and that s_i is a best response to s_{-i} in G^{k+1}_{Rat}.

Proof of Theorem 14

Claim
If s_i is a best response to s_{-i} in G^k_{Rat}, then s_i is a best response to s_{-i} in G.

Proof of the Claim. By induction on k. For $k = 0$ we have $G^k_{Rat} = G^0_{Rat} = G$ and the claim holds trivially.

Assume that the claim is true for some k and that s_i is a best response to s_{-i} in G^{k+1}_{Rat}. Let s'_i be a best response to s_{-i} in G^k_{Rat}. Then $s'_i \in G^{k+1}_{Rat}$ since s'_i is *not* eliminated from G^k_{Rat}.

Proof of Theorem 14

Claim

If s_i is a best response to s_{-i} in G^k_{Rat}, then s_i is a best response to s_{-i} in G.

Proof of the Claim. By induction on k. For $k = 0$ we have $G^k_{Rat} = G^0_{Rat} = G$ and the claim holds trivially.

Assume that the claim is true for some k and that s_i is a best response to s_{-i} in G^{k+1}_{Rat}. Let s'_i be a best response to s_{-i} in G^k_{Rat}.
Then $s'_i \in G^{k+1}_{Rat}$ since s'_i is *not* eliminated from G^k_{Rat}.
However, since s_i is a best response to s_{-i} in G^{k+1}_{Rat}, we get $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$.
Thus s_i is a best response to s_{-i} in G^k_{Rat}.

Proof of Theorem 14

Claim
If s_i is a best response to s_{-i} in G^k_{Rat}, then s_i is a best response to s_{-i} in G.

Proof of the Claim. By induction on k. For $k = 0$ we have $G^k_{Rat} = G^0_{Rat} = G$ and the claim holds trivially.

Assume that the claim is true for some k and that s_i is a best response to s_{-i} in G^{k+1}_{Rat}. Let s'_i be a best response to s_{-i} in G^k_{Rat}.
Then $s'_i \in G^{k+1}_{Rat}$ since s'_i is *not* eliminated from G^k_{Rat}.
However, since s_i is a best response to s_{-i} in G^{k+1}_{Rat}, we get $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$.
Thus s_i is a best response to s_{-i} in G^k_{Rat}.

By induction hypothesis, s_i is a best response to s_{-i} in G and the claim has been proved.

Proof of Theorem 14

Keep in mind: If s_i is a best response to s_{-i} in G^k_{Rat}, then s_i is a best response to s_{-i} in G.

Now we prove $R_i^k \subseteq D_i^k$ for all players i by induction on k.

Proof of Theorem 14

Keep in mind: If s_i is a best response to s_{-i} in G^k_{Rat}, then s_i is a best response to s_{-i} in G.

Now we prove $R_i^k \subseteq D_i^k$ for all players i by induction on k.
For $k = 0$ we have that $R_i^0 = S_i = D_i^0$ by definition.

Proof of Theorem 14

Keep in mind: If s_i is a best response to s_{-i} in G^k_{Rat}, then s_i is a best response to s_{-i} in G.

Now we prove $R_i^k \subseteq D_i^k$ for all players i by induction on k.
For $k = 0$ we have that $R_i^0 = S_i = D_i^0$ by definition.
Assume that $R_i^k \subseteq D_i^k$ for some $k \geq 0$ and prove that $R_i^{k+1} \subseteq D_i^{k+1}$.

Proof of Theorem 14

Keep in mind: If s_i is a best response to s_{-i} in G^k_{Rat}, then s_i is a best response to s_{-i} in G.

Now we prove $R^k_i \subseteq D^k_i$ for all players i by induction on k.
For $k = 0$ we have that $R^0_i = S_i = D^0_i$ by definition.
Assume that $R^k_i \subseteq D^k_i$ for some $k \geq 0$ and prove that $R^{k+1}_i \subseteq D^{k+1}_i$.
Let $s_i \in R^{k+1}_i$. Then there must be $s_{-i} \in R^k_{-i}$ such that

s_i is a best response to s_{-i} in G^k_{Rat}

(This follows from the fact that s_i has not been eliminated in G^k_{Rat}.)

Proof of Theorem 14

Keep in mind: If s_i is a best response to s_{-i} in G^k_{Rat}, then s_i is a best response to s_{-i} in G.

Now we prove $R_i^k \subseteq D_i^k$ for all players i by induction on k.
For $k = 0$ we have that $R_i^0 = S_i = D_i^0$ by definition.
Assume that $R_i^k \subseteq D_i^k$ for some $k \geq 0$ and prove that $R_i^{k+1} \subseteq D_i^{k+1}$.
Let $s_i \in R_i^{k+1}$. Then there must be $s_{-i} \in R_{-i}^k$ such that

$$s_i \text{ is a best response to } s_{-i} \text{ in } G^k_{Rat}$$

(This follows from the fact that s_i has not been eliminated in G^k_{Rat}.)
By the claim, s_i is a best response to s_{-i} in G as well!
By induction hypothesis, $s_i \in R_i^{k+1} \subseteq R_i^k \subseteq D_i^k$ and $s_{-i} \in R_{-i}^k \subseteq D_{-i}^k$.

Proof of Theorem 14

Keep in mind: If s_i is a best response to s_{-i} in G^k_{Rat}, then s_i is a best response to s_{-i} in G.

Now we prove $R^k_i \subseteq D^k_i$ for all players i by induction on k.
For $k = 0$ we have that $R^0_i = S_i = D^0_i$ by definition.
Assume that $R^k_i \subseteq D^k_i$ for some $k \geq 0$ and prove that $R^{k+1}_i \subseteq D^{k+1}_i$.
Let $s_i \in R^{k+1}_i$. Then there must be $s_{-i} \in R^k_{-i}$ such that

s_i is a best response to s_{-i} in G^k_{Rat}

(This follows from the fact that s_i has not been eliminated in G^k_{Rat}.)
By the claim, s_i is a best response to s_{-i} in G as well!
By induction hypothesis, $s_i \in R^{k+1}_i \subseteq R^k_i \subseteq D^k_i$ and $s_{-i} \in R^k_{-i} \subseteq D^k_{-i}$.
However, then s_i is a best response to s_{-i} in G^k_{DS}.
(This follows from the fact that the "best response" relationship of s_i and s_{-i} is preserved by removing arbitrarily many other strategies.)

Proof of Theorem 14

Keep in mind: If s_i is a best response to s_{-i} in G^k_{Rat}, then s_i is a best response to s_{-i} in G.

Now we prove $R^k_i \subseteq D^k_i$ for all players i by induction on k.
For $k = 0$ we have that $R^0_i = S_i = D^0_i$ by definition.
Assume that $R^k_i \subseteq D^k_i$ for some $k \geq 0$ and prove that $R^{k+1}_i \subseteq D^{k+1}_i$.
Let $s_i \in R^{k+1}_i$. Then there must be $s_{-i} \in R^k_{-i}$ such that

$$s_i \text{ is a best response to } s_{-i} \text{ in } G^k_{Rat}$$

(This follows from the fact that s_i has not been eliminated in G^k_{Rat}.)
By the claim, s_i is a best response to s_{-i} in G as well!
By induction hypothesis, $s_i \in R^{k+1}_i \subseteq R^k_i \subseteq D^k_i$ and $s_{-i} \in R^k_{-i} \subseteq D^k_{-i}$.
However, then s_i is a best response to s_{-i} in G^k_{DS}.
(This follows from the fact that the "best response" relationship of s_i and s_{-i} is preserved by removing arbitrarily many other strategies.)
Thus s_i is not strictly dominated in G^k_{DS} and $s_i \in D^{k+1}_i$. □

Pinning Down Beliefs – Nash Equilibria

Criticism of previous approaches:

- Strictly dominant strategy equilibria often do not exist
- IESDS and rationalizability may not remove any strategies

Pinning Down Beliefs – Nash Equilibria

Criticism of previous approaches:

- Strictly dominant strategy equilibria often do not exist
- IESDS and rationalizability may not remove any strategies

Typical example is Battle of Sexes:

	O	F
O	$2,1$	$0,0$
F	$0,0$	$1,2$

Here all strategies are equally reasonable according to the above concepts.

Pinning Down Beliefs – Nash Equilibria

Criticism of previous approaches:

- Strictly dominant strategy equilibria often do not exist
- IESDS and rationalizability may not remove any strategies

Typical example is Battle of Sexes:

	O	F
O	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

Here all strategies are equally reasonable according to the above concepts.

But are all strategy profiles really equally reasonable?

Pinning Down Beliefs – Nash Equilibria

	O	F
O	$2,1$	$0,0$
F	$0,0$	$1,2$

Assume that each player has a belief about strategies of other players.

Pinning Down Beliefs – Nash Equilibria

	O	F
O	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

Assume that each player has a belief about strategies of other players.

By Claim 3, each player plays a best response to his beliefs.

Pinning Down Beliefs – Nash Equilibria

	O	F
O	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

Assume that each player has a belief about strategies of other players.

By Claim 3, each player plays a best response to his beliefs.

Is (O, F) as reasonable as (O, O) in this respect?

Pinning Down Beliefs – Nash Equilibria

	O	F
O	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

Assume that each player has a belief about strategies of other players.

By Claim 3, each player plays a best response to his beliefs.

Is (O, F) as reasonable as (O, O) in this respect?

Note that if player 1 believes that player 2 plays O, then playing O is reasonable, and if player 2 believes that player 1 plays F, then playing F is reasonable. But such **beliefs cannot be correct together**!

Pinning Down Beliefs – Nash Equilibria

	O	F
O	$2,1$	$0,0$
F	$0,0$	$1,2$

Assume that each player has a belief about strategies of other players.

By Claim 3, each player plays a best response to his beliefs.

Is (O, F) as reasonable as (O, O) in this respect?

Note that if player 1 believes that player 2 plays O, then playing O is reasonable, and if player 2 believes that player 1 plays F, then playing F is reasonable. But such **beliefs cannot be correct together**!

(O, O) can be obtained as a profile where each player plays the best response to his belief and the **beliefs are correct**.

Nash Equilibrium

Nash equilibrium can be defined as a set of beliefs (one for each player) and a strategy profile in which every player plays a best response to his belief and each strategy of each player is consistent with beliefs of his opponents.

Nash Equilibrium

Nash equilibrium can be defined as a set of beliefs (one for each player) and a strategy profile in which every player plays a best response to his belief and each strategy of each player is consistent with beliefs of his opponents.

A usual definition is following:

Definition 15
A pure-strategy profile $s^* = (s_1^*, \ldots, s_n^*) \in S$ is a (pure) Nash equilibrium if s_i^* is a best response to s_{-i}^* for each $i \in N$, that is

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \quad \text{for all } s_i \in S_i \text{ and all } i \in N$$

Nash Equilibrium

Nash equilibrium can be defined as a set of beliefs (one for each player) and a strategy profile in which every player plays a best response to his belief and each strategy of each player is consistent with beliefs of his opponents.

A usual definition is following:

Definition 15

A pure-strategy profile $s^* = (s_1^*, \ldots, s_n^*) \in S$ is a (pure) Nash equilibrium if s_i^* is a best response to s_{-i}^* for each $i \in N$, that is

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \quad \text{for all } s_i \in S_i \text{ and all } i \in N$$

Note that this definition is equivalent to the previous one in the sense that s_{-i}^* may be considered as the (consistent) belief of player i to which he plays a best response s_i^*

Nash Equilibria Examples

In the Prisoner's dilemma:

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

Nash Equilibria Examples

In the Prisoner's dilemma:

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

(C, C) is the only Nash equilibrium.

Nash Equilibria Examples

In the Prisoner's dilemma:

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

(C, C) is the only Nash equilibrium.

In the Battle of Sexes:

	O	F
O	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

Nash Equilibria Examples

In the Prisoner's dilemma:

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

(C, C) is the only Nash equilibrium.

In the Battle of Sexes:

	O	F
O	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

only (O, O) and (F, F) are Nash equilibria.

Nash Equilibria Examples

In the Prisoner's dilemma:

	C	S
C	$-5, -5$	$0, -20$
S	$-20, 0$	$-1, -1$

(C, C) is the only Nash equilibrium.

In the Battle of Sexes:

	O	F
O	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

only (O, O) and (F, F) are Nash equilibria.

In Cournot Duopoly, $(\theta/3, \theta/3)$ is the only Nash equilibrium.

(Best response relations: $q_1 = (\theta - q_2)/2$ and $q_2 = (\theta - q_1)/2$ are both satisfied only by $q_1 = q_2 = \theta/3$)

Example: Stag Hunt

Story:

- Two (in some versions more than two) hunters, players 1 and 2, can each choose to hunt

  - stag (S) = a large tasty meal
  - hare (H) = also tasty but small

Example: Stag Hunt

Story:

- Two (in some versions more than two) hunters, players 1 and 2, can each choose to hunt

  - stag (S) = a large tasty meal
  - hare (H) = also tasty but small

- Hunting stag is much more demanding and forces of both players need to be joined (hare can be hunted individually)

Example: Stag Hunt

Story:

- Two (in some versions more than two) hunters, players 1 and 2, can each choose to hunt

  - stag (S) = a large tasty meal
  - hare (H) = also tasty but small

- Hunting stag is much more demanding and forces of both players need to be joined (hare can be hunted individually)

Strategy-form game model: $N = \{1, 2\}$, $S_1 = S_2 = \{S, H\}$, the payoff:

	S	H
S	$5, 5$	$0, 3$
H	$3, 0$	$3, 3$

Example: Stag Hunt

Story:

- Two (in some versions more than two) hunters, players 1 and 2, can each choose to hunt

  - stag (S) = a large tasty meal
  - hare (H) = also tasty but small

- Hunting stag is much more demanding and forces of both players need to be joined (hare can be hunted individually)

Strategy-form game model: $N = \{1, 2\}$, $S_1 = S_2 = \{S, H\}$, the payoff:

	S	H
S	$5, 5$	$0, 3$
H	$3, 0$	$3, 3$

Two NE: (S, S), and (H, H), where the former is strictly better for each player than the latter! Which one is more reasonable?

Example: Stag Hunt

Strategy-form game model: $N = \{1, 2\}$, $S_1 = S_2 = \{S, H\}$, the payoff:

	S	H
S	$5, 5$	$0, 3$
H	$3, 0$	$3, 3$

Two NE: (S, S), and (H, H), where the former is strictly better for each player than the latter! Which one is more reasonable?

If each player believes that the other one will go for hare, then (H, H) is a reasonable outcome $\Rightarrow$ a society of individualists who do not cooperate at all.

Example: Stag Hunt

Strategy-form game model: $N = \{1, 2\}$, $S_1 = S_2 = \{S, H\}$, the payoff:

	S	H
S	$5, 5$	$0, 3$
H	$3, 0$	$3, 3$

Two NE: (S, S), and (H, H), where the former is strictly better for each player than the latter! Which one is more reasonable?

If each player believes that the other one will go for hare, then (H, H) is a reasonable outcome $\Rightarrow$ a society of individualists who do not cooperate at all.

If each player believes that the other will cooperate, then this anticipation is self-fulfilling and results in what can be called a cooperative society.

Example: Stag Hunt

Strategy-form game model: $N = \{1, 2\}$, $S_1 = S_2 = \{S, H\}$, the payoff:

	S	H
S	$5, 5$	$0, 3$
H	$3, 0$	$3, 3$

Two NE: (S, S), and (H, H), where the former is strictly better for each player than the latter! Which one is more reasonable?

If each player believes that the other one will go for hare, then (H, H) is a reasonable outcome $\Rightarrow$ a society of individualists who do not cooperate at all.

If each player believes that the other will cooperate, then this anticipation is self-fulfilling and results in what can be called a cooperative society.

This is supposed to explain that in real world there are societies that have similar endowments, access to technology and physical environment but have very different achievements, all because of self-fulfilling beliefs (or *norms* of behavior).

Example: Stag Hunt

Strategy-form game model: $N = \{1, 2\}$, $S_1 = S_2 = \{S, H\}$, the payoff:

	S	H
S	$5, 5$	$0, 3$
H	$3, 0$	$3, 3$

Two NE: (S, S), and (H, H), where the former is strictly better for each player than the latter! Which one is more reasonable?

Another point of view: (H, H) is less risky

Example: Stag Hunt

Strategy-form game model: $N = \{1, 2\}$, $S_1 = S_2 = \{S, H\}$, the payoff:

	S	H
S	$5, 5$	$0, 3$
H	$3, 0$	$3, 3$

Two NE: (S, S), and (H, H), where the former is strictly better for each player than the latter! Which one is more reasonable?

Another point of view: (H, H) is less risky

Minimum secured by playing S is 0 as opposed to 3 by playing H
(We will get to this *minimax* principle later)

Example: Stag Hunt

Strategy-form game model: $N = \{1, 2\}$, $S_1 = S_2 = \{S, H\}$, the payoff:

	S	H
S	$5, 5$	$0, 3$
H	$3, 0$	$3, 3$

Two NE: (S, S), and (H, H), where the former is strictly better for each player than the latter! Which one is more reasonable?

Another point of view: (H, H) is less risky

Minimum secured by playing S is 0 as opposed to 3 by playing H
(We will get to this *minimax* principle later)

So it seems to be rational to expect (H, H) (?)

Nash Equilibria vs Previous Concepts

Theorem 16

1. *If s^* is a strictly dominant strategy equilibrium, then it is the unique Nash equilibrium.*
2. *Each Nash equilibrium is rationalizable and survives IESDS.*
3. *If S is finite, neither rationalizability, nor IESDS creates new Nash equilibria.*

Proof: Homework!

Nash Equilibria vs Previous Concepts

Theorem 16

1. *If s^* is a strictly dominant strategy equilibrium, then it is the unique Nash equilibrium.*
2. *Each Nash equilibrium is rationalizable and survives IESDS.*
3. *If S is finite, neither rationalizability, nor IESDS creates new Nash equilibria.*

Proof: Homework!

Corollary 17

Assume that S is finite. If rationalizability or IESDS result in a unique strategy profile, then this profile is a Nash equilibrium.

Interpretations of Nash Equilibria

Except the two definitions, usual interpretations are following:

- When the goal is to give advice to all of the players in a game (i.e., to advise each player what strategy to choose), any advice that was not an equilibrium would have the unsettling property that there would always be some player for whom the advice was bad, in the sense that, if all other players followed the parts of the advice directed to them, it would be better for some player to do differently than he was advised. If the advice is an equilibrium, however, this will not be the case, because the advice to each player is the best response to the advice given to the other players.

Interpretations of Nash Equilibria

Except the two definitions, usual interpretations are following:

- When the goal is to give advice to all of the players in a game (i.e., to advise each player what strategy to choose), any advice that was not an equilibrium would have the unsettling property that there would always be some player for whom the advice was bad, in the sense that, if all other players followed the parts of the advice directed to them, it would be better for some player to do differently than he was advised. If the advice is an equilibrium, however, this will not be the case, because the advice to each player is the best response to the advice given to the other players.
- When the goal is prediction rather than prescription, a Nash equilibrium can also be interpreted as a potential stable point of a dynamic adjustment process in which individuals adjust their behavior to that of the other players in the game, searching for strategy choices that will give them better results.