Linear Programming

Linear Optimization Problem

$$\begin{array}{rll} \text{minimize} & f(x) & \\ \text{by varying} & x \in \mathbb{R}^n & \\ \text{subject to} & g_i(x) \leq 0 & i = 1, \ldots, n_g \\ & h_j(x) = 0 & j = 1, \ldots, n_h \end{array}$$

We assume that

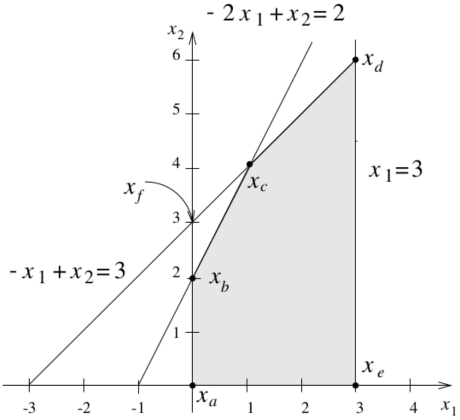
- f is linear, i.e.,

$$f(x) = c^\top x \qquad \text{here } c \in \mathbb{R}^n$$

- each g_i is linear,
- each h_j is linear.

For convenience, in what follows, we also allow constraints of the form $g_i(x) \geq 0$.

Example



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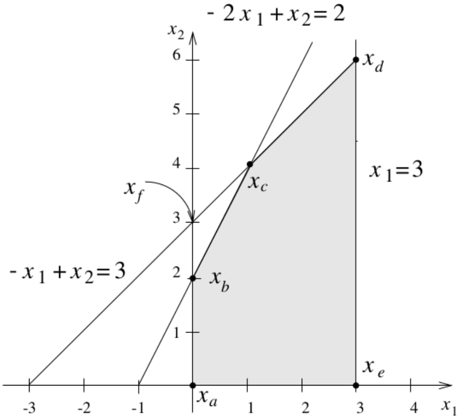

The lines define the boundaries of the feasible region

$$
\begin{aligned}
-2x_1 + x_2 &= 2 \\
-x_1 + x_2 &= 3 \\
x_1 &= 3
\end{aligned}
\qquad
\begin{aligned}
x_1 &= 0 \\
x_2 &= 0
\end{aligned}
$$

Standard Form

The *standard form linear program*

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Here

- $x = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n$
- $c = (c_1, \ldots, c_n)^\top \in \mathbb{R}^n$
- A is an $m \times n$ matrix of elements a_{ij} where $m < n$ and $\operatorname{rank}(A) = m$

  That is, all rows of A are linearly independent.
- $b = (b_1, \ldots, b_m)^\top \geq 0$

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Every linear optimization problem can be transformed into a standard linear program such that there is a one-to-one correspondence between solutions of the constraints preserving values of the objective.

Transformation to Standard Form

1. For every variable x_i introduce new variables x_i', x_i'', replace every occurrence of x_i with $x_i' - x_i''$, and introduce constraints $x_i', x_i'' \geq 0$.

   Note that if a constraint is in the form $x_i + \zeta \geq 0$ we may simply replace x_i with $x_i' - \zeta$ and introduce $x_i' \geq 0$.

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6. Multiplying equations with $b_i < 0$ by -1 gives $b \geq 0$

Transformation Example

$$\begin{array}{ll} \text{maximize} & z = -5x_1 - 3x_2 \\ \text{subject to} & 3x_1 - 5x_2 - 5 \leq 0 \\ & -4x_1 - 9x_2 + 4 \leq 0 \end{array}$$

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Introduce the bounded variables:

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Check if all equations are linearly independent.

Multiply the last one with -1:

$$\begin{array}{ll} \text{maximize} & z = -5x_1' + 5x_1'' - 3x_2' + 3x_2'' \\ \text{subject to} & 3x_1' - 3x_1'' - 5x_2' + 5x_2'' + s_1 = 5 \\ & 4x_1' - 4x_1'' + 9x_2' - 9x_2'' - s_2 = 4 \\ & x_1', x_1'', x_2', x_2'', s_1, s_2 \geq 0 \end{array}$$

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In the standard form:

$$A = \begin{pmatrix} 3 & -3 & -5 & 5 & 1 & 0 \\ 4 & -4 & 9 & -9 & 0 & -1 \end{pmatrix}$$

$$x = (x_1, x_2, x_3, x_4, x_5, x_6)^\top$$

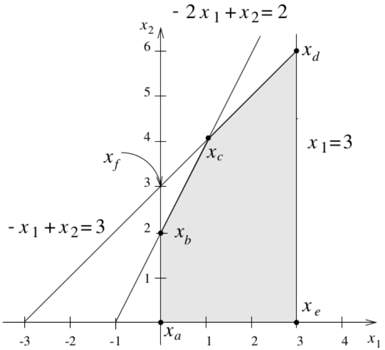
Note that we have renamed the variables.

$$b = (5, 4)^\top$$

$$Ax = b \text{ where } x \geq 0$$

$$c = (-5, 5, -3, 3)^\top$$

Example



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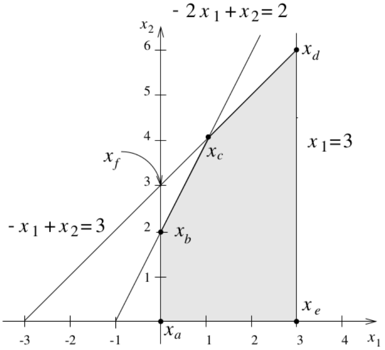

Transform to

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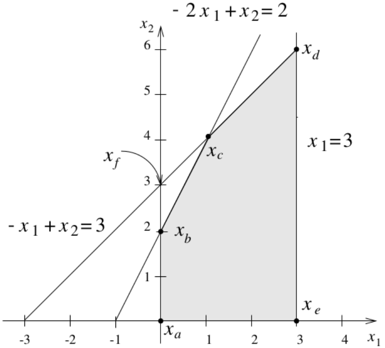

The standard form:

$$A = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$b = (2, 3, 3)^\top$$

$$Ax = b$$

$$x = (x_1, x_2, x_3, x_4, x_5)^\top$$

$$c = (-1, -2, 0, 0, 0)^\top$$

Assumptions

Consider a linear programming problem in the standard form:

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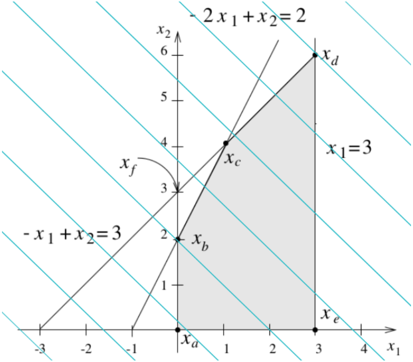
In what follows, we will use the following shorthand: Given two column vectors x, x', we write $[x, x']$ to denote the vector resulting from stacking x on top of x'.

Solutions

There are (typically) infinitely many solutions to the constraints.

Are there some distinguished ones? How do you find minimizers?



Here, the blue lines are contours of $-x_1 - x_2$.

Basic Solutions

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Given $x \in \mathbb{R}^n$, we let

- $x_B \in \mathbb{R}^m$ consist of components of x with indices in B
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Abusing notation, we denote by B and N the submatrices of A consisting of columns with indices in B and N, resp.

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Definition

Consider $x \in \mathbb{R}^n$ and a basis B, and consider the decomposition of x into $x_B \in \mathbb{R}^m$ and $x_N \in \mathbb{R}^{n-m}$.
Then x is a *basic solution w.r.t. the basis B* if $Ax = b$ and $x_N = 0$.
Components of x_B are *basic variables*.
A basic solution x is *feasible* if $x \geq 0$.

Example (Whiteboard)

Add slack variables x_3, x_4:

$$\begin{aligned} x_1 + x_2 &\leq 2 \\ x_1 &\leq 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ x_1 + x_4 &= 1 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

$$A = (u_1\ u_2\ u_3\ u_4) = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$x = (x_1, x_2, x_3, x_4)^\top$$

$$b = (2, 1)^\top$$

$$Ax = b \text{ where } x \geq 0$$

For now, let us ignore the objective function and play with the polyhedron defined by the inequalities above.

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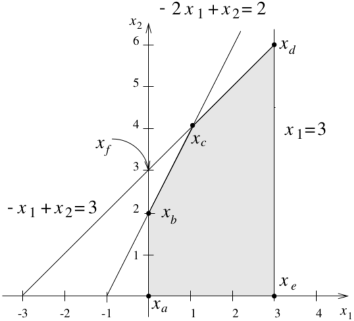

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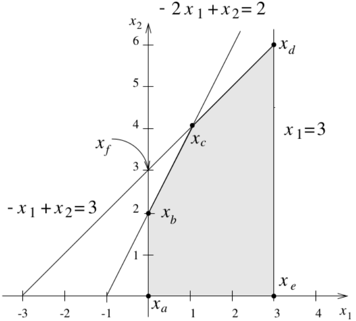

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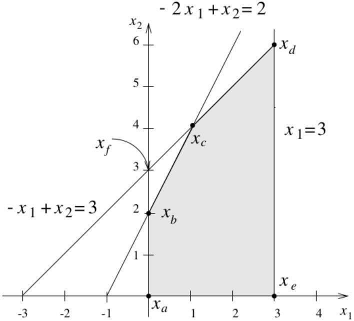

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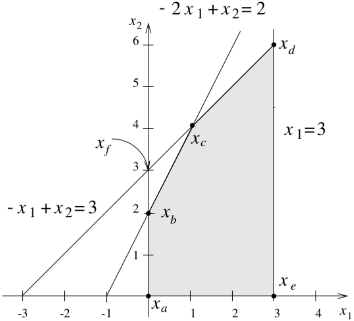

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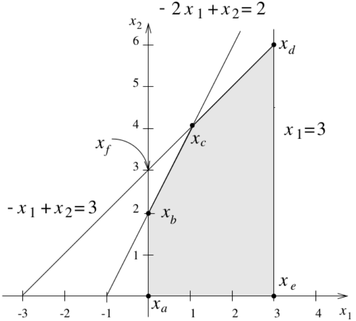

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$Ax = b$ where $x \geq 0$

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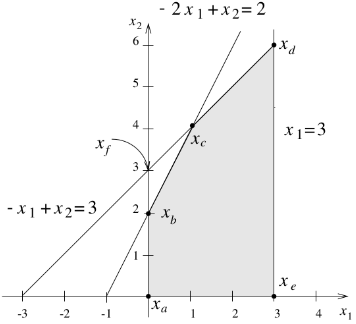

Consider a basis $\{x_3, x_4, x_5\}$ with

$$B = (u_3\ u_4\ u_5) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

What is x_B satisfying $Bx_B = b$?

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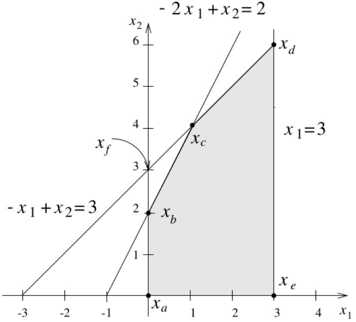

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The corresponding basic solution is

$$x = (x_1, x_2, x_3, x_4, x_5)^\top = (0, 0, 2, 3, 3)^\top = x_a \quad \text{Feasible!}$$

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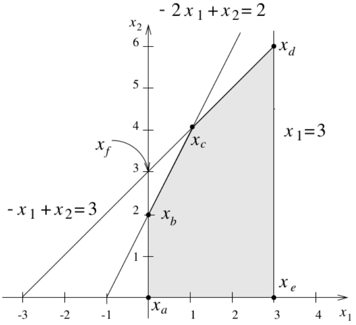

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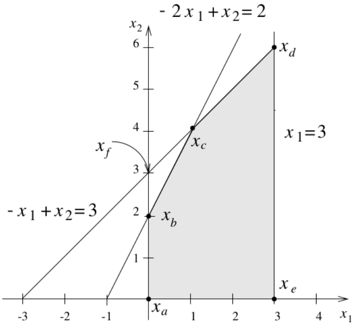

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$$x = (x_1, x_2, x_3, x_4, x_5)^\top = (0, 3, -1, 0, 3)^\top = x_f$$ Not feasible!

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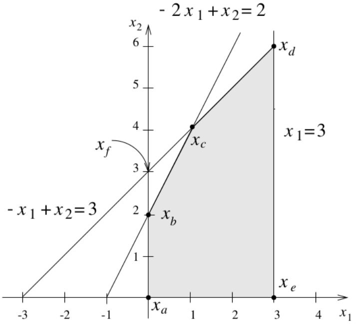

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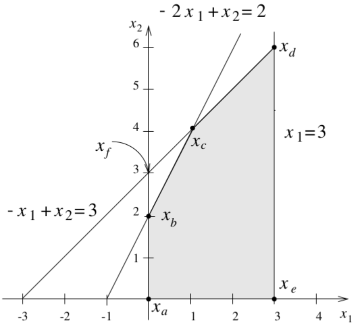

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The corresponding basic solution is

$$x = (x_1, x_2, x_3, x_4, x_5)^\top = (3, 6, 2, 0, 0)^\top = x_d \quad \text{Feasible!}$$

Existence of Basic Feasible Solutions

Theorem 1 (Fundamental Theorem of LP)

Consider a linear program in standard form.

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There are finitely many of them, which implies decidability.

However, the enumeration of all basic feasible solutions would be impractical; the number of basic feasible solutions is potentially

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

For $n = 100$ and $m = 10$, we get $535,983,370,403,809,682,970$.

Extreme Points

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Let Θ be the convex set consisting of all feasible solutions, that is, all $x \in \mathbb{R}^n$ satisfying:

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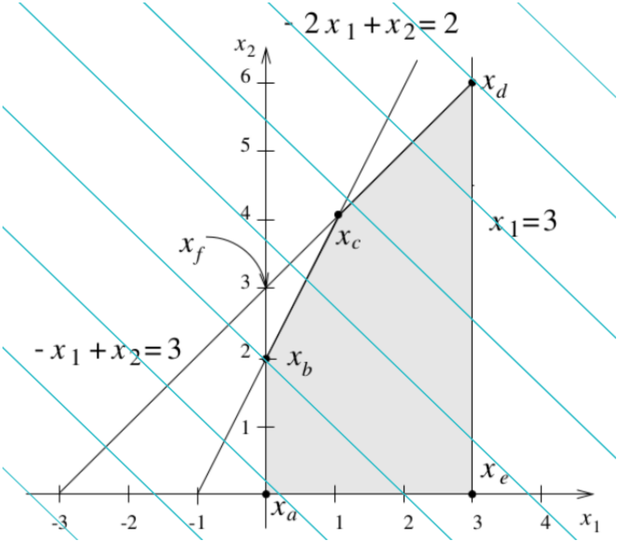
Thus, as a corollary, we obtain that to find an optimal solution to the linear optimization problem, we need to consider only extreme points of the feasibility region.

Optimal Solutions



Here, the blue lines are contours of $-x_1 - x_2$. The minimizer is x_d.

Degenerate Basic Solutions

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Two different bases can correspond to the same point. To see this, consider the constraints defined by

$$Ax = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 6 \\ 13 \\ 12 \end{pmatrix} = b.$$

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There are two bases

$\{x_1, x_2, x_3\}$ giving

$$B = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 0 & 1 \\ 4 & 0 & 0 \end{pmatrix}$$

$\{x_1, x_3, x_4\}$ giving

$$B' = \begin{pmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}$$

Each gives the same *degenerate* basic solution $x = (3, 0, 4, 0)^\top$.

Simplex Algorithm

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Now, how do you move from one vertex to another one algebraically?

First, we consider LP problems where each basic solution is non-degenerate.

Later we drop this assumption.

Changing Basis (Non-Degenerate Case)

Consider a basis B and write $A = (B\ N) = (u_1 \ldots u_m\, u_{m+1} \ldots u_n)$ where $B = (u_1 \ldots u_m)$ and $N = (u_{m+1} \ldots u_n)$.

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Consider a basic feasible solution $x = [x_B \ x_N]$ where $x_N = 0$. Then

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For a non-degenerate case, we have $x_j > 0$ for all $j = 1, \dots, m$.

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Now consider maximum $\alpha > 0$ such that $x_j - \alpha y_j \geq 0$ for all j.

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Obtain a basis $B_{j\to i} = B \setminus \{j\} \cup \{i\}$ and a basic feasible solution

$$x_{j\to i} = (x'_1, \ldots, x'_{j-1}, 0, x'_{j+1}, \ldots, x'_m, 0, \ldots, 0, \alpha, 0, \ldots, 0)^\top$$

Here $x'_k = x_k - \alpha y_k$ for each $k \in \{1, \ldots, j-1, j+1, \ldots, m\}$.

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Here $x'_k = x_k - \alpha y_k$ for each $k \in \{1, \ldots, j-1, j+1, \ldots, m\}$.
We say that we *pivot about* (j, i).

Algorithm 1 Simplex - **Non-degenerate**

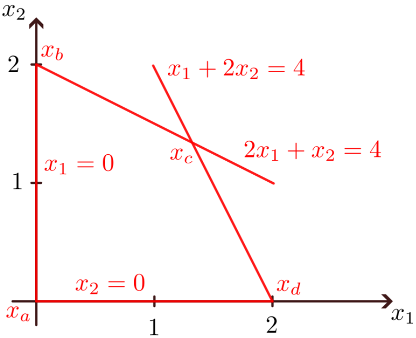
Choose a starting basis $B = (u_1 \dots u_m)$ (here $A = (B\ N)$)
repeat
  Compute the basic solution x for the basis B
  for $i \in \{m+1, \dots, n\}$ **do**
    Solve $B(y_1, \dots, y_m)^\top = u_i$
    if $y_k \leq 0$ for all $k \in \{1, \dots, m\}$ **then**
      Stop, unbounded problem.
    end if
    Select $j = \text{argmin}\{x_k/y_k \mid y_k > 0 \wedge k = 1, \dots, m\}$
    Compute $x_{j \to i}$
  end for
  if $c^\top(x_{j \to i} - x) \geq 0$ for all $i \in \{m+1, \dots, n\}$ **then**
    Stop, we have an optimal solution.
  end if
  Select $i \in \{m+1, \dots, n\}$ such that $c^\top(x_{j \to i} - x) < 0$
  $B \leftarrow B_{j \to i}$
until convergence

$$A = (u_1\ u_2\ u_3\ u_4) = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}$$

$$x = (x_1, x_2, x_3, x_4)^\top$$

$$b = (4, 4)^\top$$

$$c = (-1, -1, 0, 0)^\top$$



minimize $c^\top x$ subject to $Ax = b$ where $x \geq 0$

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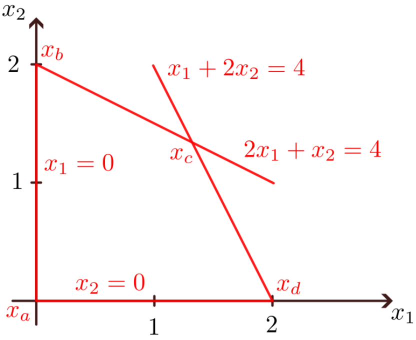

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Consider a basis

$$B = (a_3\ a_4) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The basic solution is $x = (x_1, x_2, x_3, x_4)^\top = (0, 0, 4, 4)^\top$

Non-Degenerate Example

$$c = (-1, -1, 0, 0) \quad A = (u_1\ u_2\ u_3\ u_4) = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

Start with the basis $\{x_3, x_4\}$ giving $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the basic solution $x = (x_1, x_2, x_3, x_4) = (0, 0, 4, 4)$.

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$$c = (-1, -1, 0, 0) \quad A = (u_1\, u_2\, u_3\, u_4) = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

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Consider x_1 as a candidate to the basis, i.e., consider the first column u_1 of A expressed in the basis B:

$$u_1 = (1, 2)^\top = B\ (1, 2)^\top \text{ thus } y = (y_3, y_4) = (1, 2)$$

Non-Degenerate Example

$$c = (-1, -1, 0, 0) \quad A = (u_1\, u_2\, u_3\, u_4) = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

Start with the basis $\{x_3, x_4\}$ giving $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the basic solution $x = (x_1, x_2, x_3, x_4) = (0, 0, 4, 4)$.

Consider x_1 as a candidate to the basis, i.e., consider the first column u_1 of A expressed in the basis B:

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Similarly, we may also put x_2 into the basis instead of x_3 and obtain the basis $\{x_2, x_4\}$ and the basic solution $(0, 2, 0, 2)$.

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We have $c^\top (x_{4\to 1} - x) = -2 < 0$
So let us move to the basis $\{x_1, x_3\}$.

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$$u_2 = (2, 1)^\top = B\ (1/2, 3/2)^\top \text{ thus } y = (y_1, y_3) = (1/2, 3/2)$$

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$$c^\top (x_{3\to 2} - x) = c(-2/3, 4/3)^\top = -2/3 < 0$$

We have reached a minimizer. All changes would lead to a higher objective value.

We may exchange x_1 with x_4, but this would give us the initial basis with a higher objective value.

Non-Degenerate Case Convergence

Theorem 3

Suppose the simplex method is applied to a linear program, and every basic variable is strictly positive at every iteration. Then, in a finite number of iterations, the method either terminates at an optimal basic feasible solution or determines that the problem is unbounded.

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However, what happens if we meet a degenerate solution?

Non-Degenerate Case Convergence

Theorem 3

Suppose the simplex method is applied to a linear program, and every basic variable is strictly positive at every iteration. Then, in a finite number of iterations, the method either terminates at an optimal basic feasible solution or determines that the problem is unbounded.

However, what happens if we meet a degenerate solution?

So, let us drop the non-degeneracy assumption.

Changing Basis (Degenerate Case)

Consider a basis B and write $A = (B\ N) = (u_1 \dots u_m\, u_{m+1} \dots u_n)$ where $B = (u_1 \dots u_m)$ and $N = (u_{m+1} \dots u_n)$.

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Note that each u_i is a column vector of dimension m.

Consider a basic feasible solution $x = [x_B\ x_N]$ where $x_N = 0$. Then

$$x_1 u_1 + \cdots + x_m u_m = b$$

For a degenerate case, we have $x_j \geq 0$ for all $j \in \{1, \ldots, m\}$, and *may have* $x_j = 0$ *for some* $j \in \{1, \ldots, m\}$.

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Now as B is a basis, we have that for each $i \in \{m+1, \ldots, n\}$ there are coefficients $y_1, \ldots, y_m$ such that $y_1 u_1 + \cdots + y_m u_m = u_i$.

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$$\begin{aligned}
b &= x_1 u_1 + \cdots + x_m u_m \\
&= x_1 u_1 + \cdots + x_m u_m - \alpha u_i + \alpha u_i \\
&= x_1 u_1 + \cdots + x_m u_m - \alpha(y_1 u_1 + \cdots + y_m u_m) + \alpha u_i \\
&= (x_1 - \alpha y_1) u_1 + \cdots + (x_m - \alpha y_m) u_m + \alpha u_i
\end{aligned}$$

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Now consider maximum $\alpha \geq 0$ such that $x_j - \alpha y_j \geq 0$ for all j.

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Otherwise, we put

$$\alpha = \min\{x_k / y_k \mid y_k > 0 \wedge k = 1, \ldots, m\}$$

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j DOES NOT have to be unique in a degenerate case.

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Obtain a basis $B_{j \to i} = B \smallsetminus \{j\} \cup \{i\}$ and a basic feasible solution

$$x_{j \to i} = (x'_1, \ldots, x'_{j-1}, 0, x'_{j+1}, \ldots, x'_m, 0, \ldots, 0, \alpha, 0, \ldots, 0)^\top$$

Here $x'_k = x_k - \alpha y_k$ for each $k \in \{1, \ldots, j-1, j+1, \ldots, m\}$.

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We say that we *pivot about* (j, i).

Degenerate Example

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Consider x_4 as a candidate for the basis:

$$u_4 = (0, 1)^\top = B\,(1, -1)^\top \text{ thus } y = (y_2, y_3) = (1, -1)$$

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$$x_{2\to 4} = (0, (x_2 - \alpha y_2), (x_3 - \alpha y_3), \alpha)^\top = (0,0,1,1)^\top$$

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$$x_{2\to4} = (0, (x_2 - \alpha y_2), (x_3 - \alpha y_3), \alpha)^\top = (0, 0, 1, 1)^\top$$

Note that $c^\top x_{2\to4} = 0$.

Thus **no effect on the objective value!**

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Consider x_1 as a candidate for the basis:

$$u_1 = (1, -1)^\top = B(-1, 2)^\top \text{ thus } y = (y_2, y_3) = (-1, 2)$$

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Pivot about $(3, 1)$, that is x_3 exchanges with x_1 and $\alpha = x_3/y_3 = 0$.

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Thus **no effect on the objective value either!**

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No change in the basic solution, and thus $c^\top x_{3\to1} = c^\top x = 0$.

Thus **no effect on the objective value either!**

Which variable should go to the basis?!

Reduced Cost

Given a basis B, we denote by c_B the vector of components of c that correspond to the variables of B.

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One can prove that for every $i \in \{m+1, \ldots, n\}$ we have

$$c^\top x_{j \to i} - c^\top x = (c_i - c_B^\top y)\alpha$$

Here $y = (y_1, \ldots, y_m)^\top$ where $By = u_i$.

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For non-degenerate case, we have $\alpha > 0$ and thus

$$c^\top x_{j \to i} < c^\top x \quad \text{iff} \quad c_i - c_B^\top y < 0$$

For the degenerate case, we may have $\alpha = 0$ and $c_i - c_B y < 0$.

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Define the *reduced cost* by

$$r_i = c_i - c_B^\top y$$

Intuitively, c_i is the cost of x_i in the new basis and $c_B^\top y$ in the old one.

Derivation of Reduced Cost

$$
\begin{aligned}
c^\top x_{j\to i} &= c^\top(x'_1, \ldots, x'_{j-1}, 0, x'_{j+1}, \ldots, x'_m, 0, \ldots, 0, \alpha, 0, \ldots, 0)^\top \\
&= c^\top(x'_1, \ldots, x'_{j-1}, x'_j, x'_{j+1}, \ldots, x'_m, 0, \ldots, 0, \alpha, 0, \ldots, 0)^\top \\
&= c_1 x'_1 + \cdots + c_m x'_m + c_i\alpha \\
&= c_1(x_1 - \alpha y_1) + \cdots c_m(x_m - \alpha y_m) + c_i\alpha \\
&= (c_1 x_1 + \cdots + c_m x_m) - (c_1 y_1 + \cdots + c_m y_m - c_i)\alpha \\
&= c^\top x - (-c_i + c_B y)\alpha
\end{aligned}
$$

Here we use the fact that $x'_k = x_k - \alpha y_k$ for each $k \in \{1, \ldots, j-1, j+1, \ldots, m\}$ and that $x_j - \alpha y_j = 0$.

Then clearly

$$c^\top x_{j\to i} - c^\top x = (c_i - c_B y)\alpha$$

$$\alpha = \min\{x_k/y_k \mid y_k > 0 \wedge k = 1, \ldots, m\}$$

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$$r_4 = c_4 - (c_2 y_2 + c_3 y_3) = 0 - (0 \cdot 1 + 0 \cdot (-1)) = 0$$

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So, we should put x_1 into the basis (the reduced cost gets smaller).

Algorithm 2 Simplex

Choose a starting basis $B = (u_1 \dots u_m)$ (here $A = (B\ N)$)
repeat
    Compute the basic solution x for the basis B
    for $i \in \{m+1, \dots, n\}$ **do**
        Solve $B\,(y_1, \dots, y_m)^\top = u_i$
        if $y_k \leq 0$ for all $k \in \{1, \dots, m\}$ **then**
            Stop, unbounded problem.
        end if
        Select $j \in \operatorname{argmin}\{x_k / y_k \mid y_k > 0 \wedge k = 1, \dots, m\}$
        Compute $r_i = c_i - c_B^\top y$ where $y = (y_1, \dots, y_m)^\top$
    end for
    if $r_i \geq 0$ for all $i \in \{m+1, \dots, n\}$ **then**
        Stop, we have an optimal solution.
    end if
    Select $i \in \{m+1, \dots, n\}$ such that $r_i < 0$
    $B \leftarrow B_{j \to i}$
until convergence

Degenerate Example (Cont.)

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After following the reduced cost from the basis $\{x_2, x_3\}$, we end up in the basis $\{x_1, x_2\}$ giving $B = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ and the basic solution $x = (x_1, x_2, x_3, x_4) = (0, 1, 0, 0)$ with $c^\top x = 0$.

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$$u_4 = (0, 1)^\top = B\,(-1/2, 1/2)^\top \text{ thus } y = (y_1, y_2) = (-1/2, 1/2)$$

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Pivot about $(2, 4)$, that is x_2 exchanges with x_4 and $\alpha = x_2/y_2 = 2$

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This is the minimizer!

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This is the minimizer!

Does this always work? Unfortunately, NO!

Degenerate Case - Looping

Consider the following linear program:

$$
\begin{array}{ll}
\text{minimize} & z = -\frac{3}{4}x_1 + 150x_2 - \frac{1}{50}x_3 + 6x_4 \\
\text{subject to} & \frac{1}{4}x_1 - 60x_2 - \frac{1}{25}x_3 + 9x_4 + x_5 = 0 \\
& \frac{1}{2}x_1 - 90x_2 - \frac{1}{50}x_3 + 3x_4 + x_6 = 0 \\
& x_3 + x_7 = 1 \\
& x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0
\end{array}
$$

Executing the simplex method on this program starting with the basis $\{x_5, x_6, x_7\}$ and always choosing i minimizing the reduced cost at line 15, eventually ends up back in the basis $\{x_5, x_6, x_7\}$.

In other words, even though the reduced cost is always negative, the overall effect on the objective is 0.

Convergence of Simplex Method

A solution is to use Bland's rule:

- Select the smallest index j at line 9.
- Select the smallest index i at line 15.

Theorem 4

If the simplex method is implemented using Bland's rule to select the entering and leaving variables, then the simplex method is guaranteed to terminate.

Simplex Convergence Summary

In a non-degenerate case:

- There is always a unique j to be selected at line 9.
- The objective of the basic solution decreases with each step.

Thus, a deterministic algorithm always terminates in a non-degenerate case.

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- The objective of the basic solution decreases with each step.

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In a degenerate case:

- We may have several j from which to select at line 9.
- Even though the reduced cost is negative, the basic solution may remain the same.

The simplex algorithm may cycle!

Using Bland's rule, the simplex method always converges to a minimizer or detects an unbounded LP.

Two-Phase Simplex Algorithm

A Simplex algorithm is initialized with a basic feasible solution.

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How do we obtain such a solution? Given a standard form LP

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

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$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

We construct an artificial LP problem.

$$\begin{aligned} \text{minimize} \quad & y_1 + y_2 + \cdots + y_m \\ \text{subject to} \quad & (A \ I_m) \begin{pmatrix} x \\ y \end{pmatrix} = b \\ & \begin{pmatrix} x \\ y \end{pmatrix} \geq 0 \end{aligned}$$

Here $y = (y_1, \ldots, y_m)^\top$ is a vector of artificial variables, I_m is the identity matrix of dimensions $m \times m$.

Two-Phase Simplex Algorithm

Solve the *artificial LP problem*:

$$\begin{aligned} \text{minimize} \quad & y_1 + y_2 + \cdots + y_m \\ \text{subject to} \quad & [A \ I_m] \begin{pmatrix} x \\ y \end{pmatrix} = b \\ & \begin{pmatrix} x \\ y \end{pmatrix} \geq 0 \end{aligned}$$

Proposition 1

The original LP problem has a basic feasible solution iff the associated artificial LP problem has an optimal feasible solution with the objective function 0.

If we solve the artificial problem with $y = 0$, we obtain x such that $Ax = b, x \geq 0$ is a basic feasible solution for the original problem.

If there is no such a solution to the artificial problem, there is no basic feasible solution, and hence no feasible solution, to the original problem.

Two-Phase Simplex Algorithm

The procedure for solving a given LP problem using the Two-Phase Simplex algorithm is following:

- Solve the artificial LP problem using the Simplex algorithm: Initialize with the basic solution of the form:

  $$(0, \ldots, 0, b_1, \ldots, b_m)^\top$$

- If the algorithm reaches a basic feasible solution $(x_1, \ldots, x_n, y_1, \ldots, y_m)^\top$ of the artificial LP problem where

  $$y_1 = \cdots = y_m = 0$$

  Use $(x_1, \ldots, x_n)^\top$ as the initial basic feasible solution to the original LP problem.
- Else stop, there is no feasible solution for the original LP problem.

Linear Programming
Properties

LP Complexity

Iterations of the simplex algorithm can be implemented to compute the first step using $\mathcal{O}(m^2 n)$ arithmetic operations and each next step $\mathcal{O}(mn)$.

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There are as many as $\binom{n}{m}$ basic solutions (many of them likely infeasible). How large are these numbers?

m	$\binom{2m}{m}$
1	2
5	252
10	184756
20	1×10^{11}
50	1×10^{29}
100	9×10^{58}
200	1×10^{119}
300	1×10^{179}
400	2×10^{239}
500	3×10^{299}

The number of iterations may be proportional to $\binom{n}{m}$ that is EXPTIME.

Linear Programming Complexity

Complexity of the simplex algorithm:

- In the worst case, the time complexity of the simplex algorithm is exponential. This holds for any deterministic pivoting rule.
  For details, see "How good is the simplex algorithm?" by Klee, Victor, and Minty, George J. Inequalities 1972.

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- There is a theory that shows that examples with exponential complexity are rare. More precisely (but still very imprecisely)
  - Consider small random perturbations of the coefficients in the LP (use Gaussian noise with a small variance)

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  For details, see "How good is the simplex algorithm?" by Klee, Victor, and Minty, George J. Inequalities 1972.

- There is a theory that shows that examples with exponential complexity are rare. More precisely (but still very imprecisely)
  - Consider small random perturbations of the coefficients in the LP (use Gaussian noise with a small variance)
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Is there a deterministic polynomial time algorithm for solving LP?

Linear Programming Complexity

We assume that all coefficients are encoded in binary (more precisely, as fractions of two integers encoded in binary).

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There is also a polynomial time algorithm (by Karmarkar) that has lower complexity upper bounds than the Khachiyan's and sometimes works even better than the simplex.

Linear Programming in Practice

Heavily used tools for solving practical problems.

Several advanced linear programming solvers (usually parts of larger optimization packages) implement various heuristics for solving large-scale problems, such as sensitivity analysis.

See an overview of tools here:

http://en.wikipedia.org/wiki/Linear_programming#Solvers_and_scripting_.28programming.29_languages

For example, the well-known Gurobi solver uses the simplex algorithm to solve LP problems.

Linear Programming - Tableaus

Tableau

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$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

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Tableaus provide all information about the current state of the simplex algorithm and can be used to streamline the process.

Keep in mind that we are not developing a new algorithm. Tableau just provides another view of the same simplex algorithm as presented before.

Tableau (Matrix Form)

Consider LP with a matrix A and vectors b, c. Assume $A = (B\ N)$ where B consists of basic columns and N of the non-basic ones.

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Apply elementary row operations so that the matrix B is turned into I_m (preserving the last row for now). That is, multiply with

$$\begin{pmatrix} B^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

The result is

$$\begin{pmatrix} B^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} B & N & b \\ c_B^\top & c_N^\top & 0 \end{pmatrix} = \begin{pmatrix} I_m & B^{-1}N & B^{-1}b \\ c_B^\top & c_N^\top & 0 \end{pmatrix}$$

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$$\begin{pmatrix} I_m & 0 \\ -c_B^\top & 1 \end{pmatrix} \begin{pmatrix} I_m & B^{-1}N & B^{-1}b \\ c_B^\top & c_N^\top & 0 \end{pmatrix} = \begin{pmatrix} I_m & B^{-1}N & B^{-1}b \\ 0 & c_N^\top - c_B^\top B^{-1}N & -c_B^\top B^{-1}b \end{pmatrix}$$

This is the *canonical form tableau for the basis B*.

Tableau (Components)

Let $A = (u_1 \ \ldots, u_n)$, the basis $\{x_1, \ldots, x_m\}$, $B = (u_1 \ \ldots, u_m)$.

Assume $u_k = (u_{1k}, \ldots, u_{nk})$. Then the initial tableau is

$$\begin{pmatrix} B & N & b \\ c_B^\top & c_N^\top & 0 \end{pmatrix} = \begin{pmatrix} u_{11} & \cdots & u_{1m} & u_{1(m+1)} & \cdots & u_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ u_{m1} & \cdots & u_{mm} & u_{m(m+1)} & \cdots & u_{mn} & b_m \\ c_1 & \cdots & c_m & c_{m+1} & \cdots & c_n & 0 \end{pmatrix}$$

Tableau (Components)

... the canonical form for the basis $\{x_1, \ldots, x_m\}$:

$$\begin{pmatrix} 1 & \cdots & 0 & y_{1(m+1)} & \cdots & y_{1n} & b'_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & y_{m(m+1)} & \cdots & y_{mn} & b'_m \\ 0 & \cdots & 0 & c'_{m+1} & \cdots & c'_n & -z \end{pmatrix}$$

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Here, $(b'_1, \ldots, b'_m)^\top = B^{-1}b$ is the vector b transformed to the basis B, and for $k = m+1, \ldots, n$ we have

$$c'_k = c_k - (y_{1k}c_1 + \cdots + y_{mk}c_m)$$

the reduced cost for the k-th column (non-basic).

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$$c'_k = c_k - (y_{1k}c_1 + \cdots + y_{mk}c_m)$$

the reduced cost for the k-th column (non-basic). Also, note that the basic solution is $x = (b'_1, \ldots, b'_m, 0, \ldots, 0)$ and

$$-z = (-c_1)b'_1 + \cdots + (-c_m)b'_m$$

is the negative of the value of the objective for the basic solution corresponding to the basis $\{x_1, \ldots, x_m\}$.

Recall that, by definition, the basic solution x satisfies $x_{m+1} = \cdots = x_n = 0$.

Tableau Simplex

Assume that for a basis B we have obtained the canonical tableau:

$$\begin{pmatrix} 1 & \cdots & 0 & y_{1(m+1)} & \cdots & y_{1n} & b'_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & y_{m(m+1)} & \cdots & y_{mn} & b'_m \\ 0 & \cdots & 0 & c'_{m+1} & \cdots & c'_n & -z \end{pmatrix}$$

The simplex algorithm then proceeds as follows:

1. Choose $i \in \{m+1, \ldots, n\}$ such that $c'_i < 0$.
2. Choose $j \in \{1, \ldots, m\}$ minimizing b'_j / y_{ji} over all j satisfying $y_{ji} > 0$.
   Note that $b'_j = x_j$ for the basic solution x w.r.t. B.
3. Move the i-the column into the basis and the j-th column out of the basis.
4. Use elementary row operations to transform the tableau into the canonical form for the new basis.
5. Repeat until $c'_{m+1}, \ldots, c'_n \geq 0$,

Example

Add slack variables x_3, x_4:

$$\begin{aligned} x_1 + x_2 &\le 2 \\ x_1 &\le 1 \\ x_1, x_2 &\ge 0 \end{aligned}$$

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ x_1 + x_4 &= 1 \\ x_1, x_2, x_3, x_4 &\ge 0 \end{aligned}$$

$$A = (u_1\ u_2\ u_3\ u_4) = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$x = (x_1, x_2, x_3, x_4)^\top$$

$$b = (2, 1)^\top$$

$Ax = b$ where $x \ge 0$

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Tableau for the basis $\{x_3, x_4\}$:

$$\left[\begin{array}{c|cccc|c} x_3 & 1 & 1 & 1 & 0 & 2 \\ x_4 & 1 & 0 & 0 & 1 & 1 \\ \hline -z & -3 & -2 & 0 & 0 & 0 \end{array}\right]$$

is already in the canonical form.

Note that the last row of the tableau corresponds to writing the objective as $-z + c^\top x = 0$ where z is a new variable and x is the basic solution for $\{x_3, x_4\}$.

Start with the basis $\{x_3, x_4\}$ and consider the canonical form:

$$\left[\begin{array}{c|cccc|c} x_3 & y_{31} & y_{32} & 1 & 0 & b_1 \\ x_4 & y_{41} & y_{42} & 0 & 1 & b_2 \\ \hline -z & c_1 & c_2 & c_3 & c_4 & 0 \end{array}\right] = \left[\begin{array}{c|cccc|c} x_3 & 1 & 1 & 1 & 0 & 2 \\ x_4 & 1 & 0 & 0 & 1 & 1 \\ \hline -z & -3 & -2 & 0 & 0 & 0 \end{array}\right]$$

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$$\left[\begin{array}{c|cccc|c} x_1 & 1 & y_{12} & 0 & y_{14} & b'_1 \\ x_3 & 0 & y_{32} & 1 & y_{34} & b'_2 \\ \hline -z & c'_1 & c'_2 & c'_3 & c'_4 & 3 \end{array}\right] = \left[\begin{array}{c|cccc|c} x_1 & 1 & 0 & 0 & 1 & 1 \\ x_3 & 0 & 1 & 1 & -1 & 1 \\ \hline -z & 0 & -2 & 0 & 3 & 3 \end{array}\right]$$

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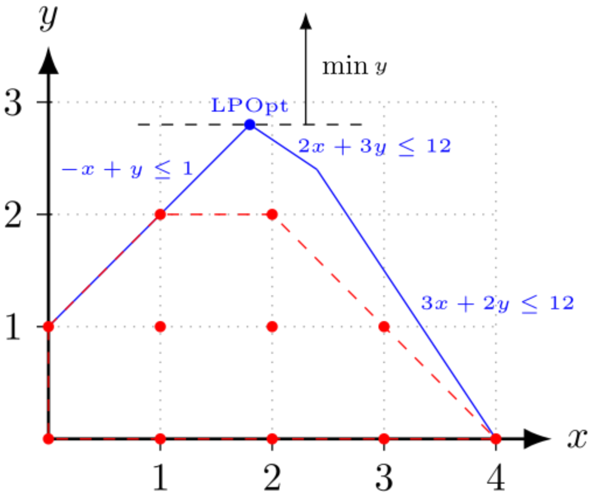
Integer Linear Programming

Integer Linear Programming



ILP $=$ LP $+$ variables constrained to integer values

Integer Linear Programming

We consider several variants of integer programming:

- 0-1 integer linear programming
- Mixed 0-1 integer linear programming
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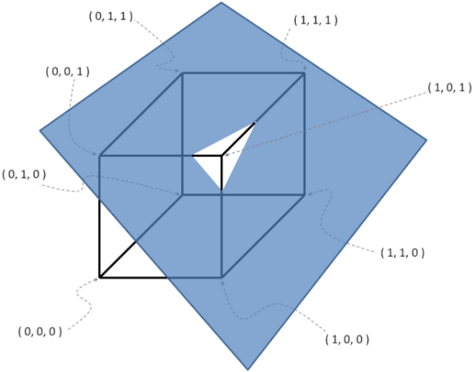
Integer linear programming is a huge subject; we shall only scratch its surface slightly.

0-1 Integer Linear Programming

Let us start with a special case where variables are constrained to values from $\{0, 1\}$.

0-1 integer linear program (0-1 ILP) is

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & Ax \leq b \\ & x_i \in \{0, 1\} \end{aligned}$$



0-1 Integer Linear Programming

Consider the following example:

$$\begin{array}{rl} \text{minimize} & c^\top x \\ \text{subject to} & a^\top x \leq b \\ & x \geq 0 \\ & x_i \in \{0, 1\} \end{array}$$

Here $c, a \in \mathbb{R}^n$ and $b \in \mathbb{R}$.

Do you recognize the problem?

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Do you recognize the problem? It is the 0-1 knapsack problem.

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Do you recognize the problem? It is the 0-1 knapsack problem.

Theorem 6

Finding $x \in \{0, 1\}^n$ satisfying the constraints of a given 0-1 integer linear program is NP-complete.

It is one of Karp's 21 NP-complete problems.

0-1 Mixed Integer Linear Programming

0-1 mixed integer linear program (0-1 MILP) is

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \\ & x_i \in \{0, 1\} \text{ for } x_i \in \mathcal{D} \end{aligned}$$

Here $\mathcal{D} \subseteq \{x_1, \ldots, x_n\}$ is a set of *binary variables*.

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The problem can be solved by searching for possible values 0 and 1 in the binary variables and solving the linear programs with binary variables fixed to concrete values.

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The problem can be solved by searching for possible values 0 and 1 in the binary variables and solving the linear programs with binary variables fixed to concrete values.

An exhaustive search through all possible binary assignments would be infeasible for many variables.

Usually, a sequential search that fixes only some of the binary variables and leaves the rest unrestricted to 0 or 1 is used.

Notation

In what follows, *LP relaxation* is the linear program obtained from 0-1 MILP by removing the constraints $x_i \in \{0, 1\}$ for $x_i \in \mathcal{D}$ and adding constraints $x_i \geq 0$ and $x \leq 1$ for all $x_i \in \mathcal{D}$.

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Keep a pool of 0-1 MILP problems $\mathcal{P}$ initialized with $\mathcal{P} = \{P\}$ where P is the original 0-1 MILP to be solved.

Algorithm 3 Branch and Bound (Non-Deterministic)

repeat
    Choose $P \in \mathcal{P}$
    if LP relaxation of P is feasible **then**
        Find a solution x of the LP relaxation of P
        if $c^\top x < f^*$ **then**
            if $x_i \in \{0, 1\}$ for all $x_i \in \mathcal{D}$ **then**
                $x^* \leftarrow x$
                $f^* \leftarrow c^\top x$
            else
                Choose $x_i \in \mathcal{D}$ such that $x_i \notin \{0, 1\}$
                Generate LP P_0 by adding $x_i = 0$ to P
                Generate LP P_1 by adding $x_i = 1$ to P
                Add P_0 and P_1 to $\mathcal{P}$.
            end if
        end if
    end if
    $\mathcal{P} \leftarrow \mathcal{P} \setminus \{P\}$
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Strategies

There are many possible strategies for choosing the problem to be solved next:

- DFS, BFS, etc.
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The solutions to the LP relaxations can be reused. Some methods (dual simplex) exploit that we are just adding a single constraint $x_i = 0$ or $x_i = 1$.

The procedure may be stopped when we find a solution x, which gives a small enough value of the objective.

(Mixed) Integer Programming

Integer linear program (ILP) is

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & Ax \leq b \\ & x \geq 0 \\ & x \in \mathbb{Z}^n \end{aligned}$$

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Here $\mathcal{D} \subseteq \{x_1, \ldots, x_n\}$ is a set of integer variables.

We may use a similar branch and bound approach as for the binary variables. The problem is that now, each integer variable has an infinite domain.

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In what follows, we temporarily cease to abuse notation and use $\bar{x}$ to denote the vector of values of the vector of variables x. Then $\bar{x}_i$ will denote the concrete value of the variable x_i.

Algorithm 4 Branch and Bound (Non-Deterministic)

repeat
    Choose $P \in \mathcal{P}$
    if LP relaxation of P is feasible **then**
        Find a solution $\bar{x}$ of the LP relaxation of P
        if $c^\top \bar{x} < f^*$ **then**
            if $\bar{x}_i \in \mathbb{Z}$ for all $x_i \in \mathcal{D}$ **then**
                $x^* \leftarrow \bar{x}$
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            else
                Choose $x_i \in \mathcal{D}$ such that $\bar{x}_i \notin \mathbb{Z}$
                Generate LP P_- by adding $x_i \leq \lfloor \bar{x}_i \rfloor$ to P
                Generate LP P_+ by adding $x_i \geq \lceil \bar{x}_i \rceil$ to P
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    $\mathcal{P} \leftarrow \mathcal{P} \setminus \{P\}$
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Example

Consider the following MILP P:

$$\begin{array}{ll} \text{minimize} & -x_1 - 2x_2 - 3x_3 - 1.5x_4 \\ \text{subject to} & x_1 + x_2 + 2x_3 + 2x_4 \leq 10 \\ & 7x_1 + 8x_2 + 5x_3 + x_4 = 31.5 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

and assume $\mathcal{D} = \{x_1, x_2, x_3\}$. That is, $x_1, x_2, x_3 \in \mathbb{Z}$.

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The solution to the LP relaxation of P is:

$$x = [0, 1.1818, 4.4091, 0], \quad \text{the objective value is } -15.59$$

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and assume $\mathcal{D} = \{x_1, x_2, x_3\}$. That is, $x_1, x_2, x_3 \in \mathbb{Z}$.

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Let us choose x_3. So, consider two programs:

- P_- where we add $x_3 \leq 4$ to P
- P_+ where we add $x_3 \geq 5$ to P

Now $\mathcal{P} = \{P_-, P_+\}$.

Consider first P_+.

P_+ is P with the added constraint $x_3 \geq 5$. The LP relaxation of P_+ is infeasible. We get $\mathcal{P} = \{P_-\}$.

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We still have $f^* = \infty$ so we split P_- by constraining x_2:

- P_{--} is obtained from P_- by adding $x_2 \leq 1$
- P_{-+} is obtained from P_- by adding $x_2 \geq 2$

and we continue with $\mathcal{P} = \{P_{--}, P_{-+}\}$.

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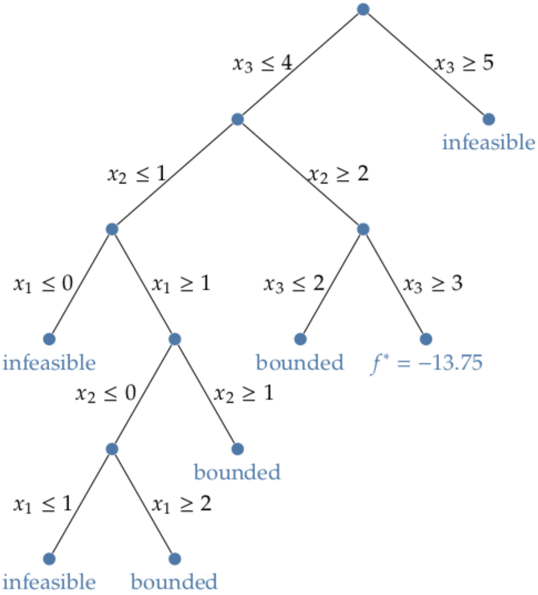
Adding one more constraint $x_3 \geq 3$ to P_{-+} would yield a MILP solution $(0, 2, 3, 0.5)$ to the LP relaxation with the objective value equal to -13.75.

The algorithm assigns $f^* = -13.75$ and $x^* = (0, 2, 3, 0.5)$.

The remaining search always leads either to an infeasible relaxation or to a relaxation with an objective value worse than f^*.



The final solution: $x^* = (0, 2, 3, 0.5)$ and $f^* = -13.75$.

Cutting Planes

Removing Non-Integer Solutions

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Another strategy might be to successively cut out non-integer optimal solutions and preserve the integer ones until an integer optimal solution is computed by the LP relaxation

We consider a concrete method for obtaining such cuts from the ILP constraints called *Gomory cuts*.

Gomory Cuts

Consider an ILP and transform it into a MILP by adding slack variables:

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \\ & x \in \mathbb{Z} \text{ for } x \in \mathcal{D} \end{aligned}$$

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Here, $\mathcal{D}$ contains the original (i.e., non-slack) variables of the ILP.

We demand the integer solution only for the original $\mathcal{D}$ variables.

However, one can prove that if all constants in the ILP are integer, then there is an optimal solution where all variables (including the slacks) are integer-valued.

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Let $A = (u_1 \; \ldots, u_n)$, the basis $\{x_1, \ldots, x_n\}$, $B = (u_1 \; \ldots, u_m)$.

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$$A' = \begin{pmatrix} 1 & \cdots & 0 & y_{1(m+1)} & \cdots & y_{1n} & b'_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & y_{m(m+1)} & \cdots & y_{mn} & b'_m \end{pmatrix}$$

The $-z$ row is omitted as it is unnecessary for the discussion.

$$u_k = B(y_{1k}, \ldots, y_{mk})^\top \text{ for } k = 1, \ldots, n \text{ and } b' = B^{-1}b$$

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$$u_k = B(y_{1k}, \ldots, y_{mk})^\top \text{ for } k = 1, \ldots, n \text{ and } b' = B^{-1}b$$

Consider a basic solution $x = (b'_1, \ldots, b'_m, 0, \ldots, 0)$.

If all $b'_1, \ldots, b'_m$ are integers, then also x solves the ILP.

Otherwise, assume that b'_i is not an integer.

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But, subtracting the inequalities, integer feasible solutions x satisfy:

$$(y_{i(m+1)} - \lfloor y_{i(m+1)} \rfloor)x_{m+1} + \cdots + (y_{in} - \lfloor y_{in} \rfloor)x_n \geq b'_i - \lfloor b'_i \rfloor$$

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But note that the *basic feasible solution* $x = (b'_1, \ldots, b'_m, 0, \ldots, 0)$ *does not* satisfy the last inequality because $b'_i > \lfloor b'_i \rfloor$ and $x_{m+1} = \cdots = x_n = 0$.

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$$(y_{i(m+1)} - \lfloor y_{i(m+1)} \rfloor)x_{m+1} + \cdots + (y_{in} - \lfloor y_{in} \rfloor)x_n \geq b'_i - \lfloor b'_i \rfloor$$

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Transform the above inequality into equality by introducing a new variable x_{n+1} and obtain the following constraint (*Gomory cut*)

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Add the Gomory cut and the constraint $x_{n+1} \geq 0$ to the program.

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$$A' = \begin{pmatrix} 1 & \cdots & 0 & y_{1(m+1)} & \cdots & y_{1n} & b'_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & y_{m(m+1)} & \cdots & y_{mn} & b'_m \end{pmatrix}$$

Choose a non-integer component $x_i = b'_i$ of the basic feasible solution w.r.t. B and consider the constraint

$$(y_{i(m+1)} - \lfloor y_{i(m+1)} \rfloor)x_{m+1} + \cdots + (y_{in} - \lfloor y_{in} \rfloor)x_n \geq b'_i - \lfloor b'_i \rfloor$$

Transform the above inequality into equality by introducing a new variable x_{n+1} and obtain the following constraint (*Gomory cut*)

$$(y_{i(m+1)} - \lfloor y_{i(m+1)} \rfloor)x_{m+1} + \cdots + (y_{in} - \lfloor y_{in} \rfloor)x_n - x_{n+1} = b'_i - \lfloor b'_i \rfloor$$

Add the Gomory cut and the constraint $x_{n+1} \geq 0$ to the program.

Repeat until an integer solution is reached.

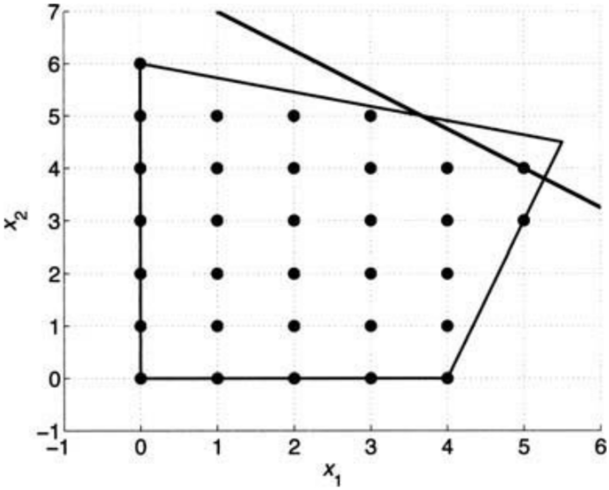
Example

Consider ILP:

$$\begin{array}{rl} \text{minimize} & -3x_1 - 4x_2 \\ \text{subject to} & 3x_1 - x_2 \leq 12 \\ & 3x_1 + 11x_2 \leq 66 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 \in \mathbb{Z} \end{array}$$

Adding slack variables x_3, x_4 we obtain the following MILP:

$$\begin{array}{rl} \text{minimize} & -3x_1 - 4x_2 \\ \text{subject to} & 3x_1 - x_2 + x_3 = 12 \\ & 3x_1 + 11x_2 + x_4 = 66 \\ & x_1, x_2, x_3, x_4 \geq 0 \\ & x_1, x_2 \in \mathbb{Z} \end{array}$$



7
6
5
4
3
2
1
0
−1
−1
0
1
2
3
4
5
6
x_1
x_2

We have

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An optimal basic solution to the LP relaxation is

$$\left(\frac{11}{2}, \frac{9}{2}, 0, 0\right)^\top$$

and the canonical tableau w.r.t. the basis $\{x_1, x_2\}$ is

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & b' \\ 1 & 0 & \frac{11}{36} & \frac{1}{36} & \frac{11}{2} \\ 0 & 1 & -\frac{1}{12} & \frac{1}{12} & \frac{9}{2} \end{pmatrix}$$

Let us introduce the Gomory cut corresponding to the variable x_1.

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & b' \\ 1 & 0 & \frac{11}{36} & \frac{1}{36} & \frac{11}{2} \\ 0 & 1 & -\frac{1}{12} & \frac{1}{12} & \frac{9}{2} \end{pmatrix}$$

Then

$$(y_{i(m+1)} - \lfloor y_{i(m+1)} \rfloor) x_{m+1} + \cdots + (y_{in} - \lfloor y_{in} \rfloor) x_n - x_{n+1} = b'_i - \lfloor b'_i \rfloor$$

with $i = 1$ and $m = 2$ turns into

$$\left(\frac{11}{36} - 0 \right) x_3 + \left(\frac{1}{36} - 0 \right) x_4 - x_5 = \frac{1}{2} \quad (= \frac{11}{2} - 5)$$

We add this constraint to our MILP.

$$
\begin{array}{rl}
\text{minimize} & -3x_1 - 4x_2 \\
\text{subject to} & 3x_1 - x_2 + x_3 = 12 \\
& 3x_1 + 11x_2 + x_4 = 66 \\
& \frac{11}{36}x_3 + \frac{1}{36}x_4 - x_5 = \frac{1}{2} \\
& x_1, x_2, x_3, x_4 \geq 0 \\
& x_1, x_2 \in \mathbb{Z}
\end{array}
$$

Solving the LP relaxation yields

$$
\left(5, \frac{51}{11}, \frac{18}{11}, 0, 0\right)^\top
$$

The canonical tableau for the solution is

$$
\begin{pmatrix}
x_1 & x_2 & x_3 & x_4 & x_5 & b' \\
1 & 0 & 0 & 0 & 1 & 5 \\
0 & 1 & 0 & \frac{1}{11} & -\frac{3}{11} & \frac{51}{11} \\
0 & 0 & 1 & \frac{1}{11} & -\frac{36}{11} & \frac{18}{11}
\end{pmatrix}
$$

Introduce the Gomory cut for x_2.

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & b' \\ 1 & 0 & 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & \frac{1}{11} & -\frac{3}{11} & \frac{51}{11} \\ 0 & 0 & 1 & \frac{1}{11} & -\frac{36}{11} & \frac{18}{11} \end{pmatrix}$$

Then

$$(y_{i(m+1)} - \lfloor y_{i(m+1)} \rfloor) x_{m+1} + \cdots + (y_{in} - \lfloor y_{in} \rfloor) x_n - x_{n+1} = b'_i - \lfloor b'_i \rfloor$$

with $i = 2$ and $m = 3$ turns into

$$\left(\frac{1}{11} - 0\right) x_4 + \left(-\frac{3}{11} + \frac{11}{11}\right) x_5 - x_6 = \frac{7}{11} \quad (= \frac{51}{11} - \frac{44}{11})$$

We add this to our MILP.

$$
\begin{aligned}
\text{minimize} \quad & -3x_1 - 4x_2 \\
\text{subject to} \quad & 3x_1 - x_2 + x_3 = 12 \\
& 3x_1 + 11x_2 + x_4 = 66 \\
& \tfrac{11}{36}x_3 + \tfrac{1}{36}x_4 - x_5 = \tfrac{1}{2} \\
& {\color{red}\tfrac{1}{11}x_4 + \tfrac{8}{11}x_5 - x_6 = \tfrac{7}{11}} \\
& x_1, x_2, x_3, x_4 \geq 0 \\
& x_1, x_2 \in \mathbb{Z}
\end{aligned}
$$

Once more the solution of the above is non-integer. However, introducing another Gomory cut (and a variable x_7) would yield a solution:

$$(5, 4, 1, 7, 0, 0, 0)^\top$$

Which gives the point $(x_1, x_2) = (5, 4)$ corresponding to the graphical solution.

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Cutting planes are also used in other non-linear, non-smooth optimization methods.

Most importantly, cutting plane techniques are combined with branch and bound methods. The constraints are introduced before branching to eliminate some solutions before the split.

The resulting method is called *branch and cut*.

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  - Does not branch but is usually combined with branch and bound (branch and cut).