Linear Programming

Linear Optimization Problem

$$\begin{array}{rll} \text{minimize} & f(x) & \\ \text{by varying} & x \in \mathbb{R}^n & \\ \text{subject to} & g_i(x) \leq 0 & i = 1, \ldots, n_g \\ & h_j(x) = 0 & j = 1, \ldots, n_h \end{array}$$

We assume that

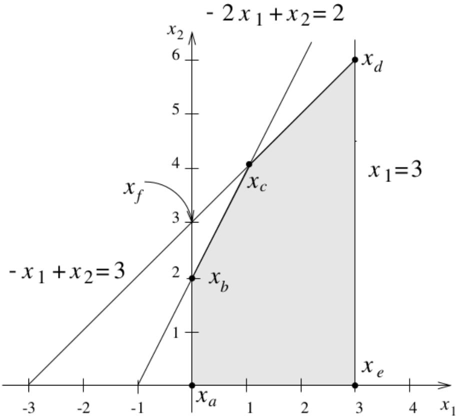
- f is linear, i.e.,

$$f(x) = c^\top x \qquad \text{here } c \in \mathbb{R}^n$$

- each g_i is linear,
- each h_j is linear.

For convenience, in what follows, we also allow constraints of the form $g_i(x) \geq 0$.

Example



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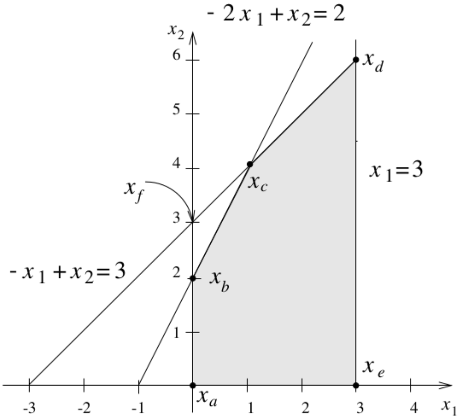

The lines define the boundaries of the feasible region

$$-2x_1 + x_2 = 2$$
$$-x_1 + x_2 = 3$$
$$x_1 = 3$$

$$x_1 = 0$$
$$x_2 = 0$$

Standard Form

The *standard form linear program*

$$\begin{array}{rl} \text{minimize} & c^\top x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Here

- $x = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n$
- $c = (c_1, \ldots, c_n)^\top \in \mathbb{R}^n$
- A is an $m \times n$ matrix of elements a_{ij} where $m < n$ and $\text{rank}(A) = m$

  That is, all rows of A are linearly independent.
- $b = (b_1, \ldots, b_m)^\top \geq 0$

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Every linear optimization problem can be transformed into a standard linear program such that there is a one-to-one correspondence between solutions of the constraints preserving values of the objective.

Transformation to Standard Form

1. For every variable x_i introduce new variables x_i', x_i'', replace every occurrence of x_i with $x_i' - x_i''$, and introduce constraints $x_i', x_i'' \geq 0$.

   Note that if a constraint is in the form $x_i + \zeta \geq 0$ we may simply replace x_i with $x_i' - \zeta$ and introduce $x_i' \geq 0$.

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6. Multiplying equations with $b_i < 0$ by -1 gives $b \geq 0$

Transformation Example

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\begin{array}{ll}
\text{maximize} & z = -5x_1 - 3x_2 \\
\text{subject to} & 3x_1 - 5x_2 - 5 \leq 0 \\
& -4x_1 - 9x_2 + 4 \leq 0
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Introduce the bounded variables:

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Check if all equations are linearly independent.

Multiply the last one with -1:

$$\begin{array}{ll} \text{maximize} & z = -5x_1' + 5x_1'' - 3x_2' + 3x_2'' \\ \text{subject to} & 3x_1' - 3x_1'' - 5x_2' + 5x_2'' + s_1 = 5 \\ & 4x_1' - 4x_1'' + 9x_2' - 9x_2'' - s_2 = 4 \\ & x_1', x_1'', x_2', x_2'', s_1, s_2 \geq 0 \end{array}$$

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In the standard form:

$$A = \begin{pmatrix} 3 & -3 & -5 & 5 & 1 & 0 \\ 4 & -4 & 9 & -9 & 0 & -1 \end{pmatrix}$$

$$x = (x_1, x_2, x_3, x_4, x_5, x_6)^\top$$

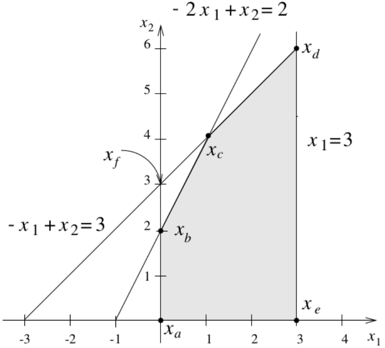
Note that we have renamed the variables.

$$b = (5, 4)^\top$$

$$Ax = b \text{ where } x \geq 0$$

$$c = (-5, 5, -3, 3)^\top$$

Example



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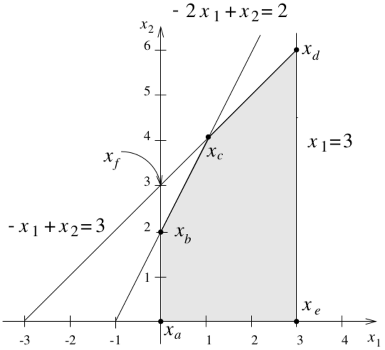

Transform to

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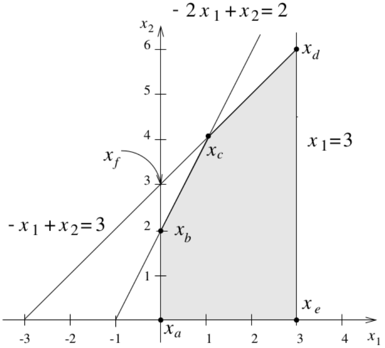

The standard form:

$$A = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$b = (2, 3, 3)^\top$$

$$Ax = b$$

$$x = (x_1, x_2, x_3, x_4, x_5)^\top$$

$$c = (-1, -2, 0, 0, 0)^\top$$

Assumptions

Consider a linear programming problem in the standard form:

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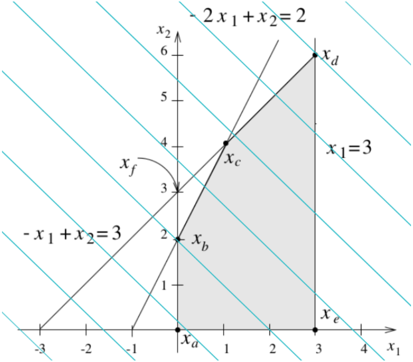
In what follows, we will use the following shorthand: Given two column vectors x, x', we write $[x, x']$ to denote the vector resulting from stacking x on top of x'.

Solutions

There are (typically) infinitely many solutions to the constraints.

Are there some distinguished ones? How do you find minimizers?



Here, the blue lines are contours of $-x_1 - x_2$.

Basic Solutions

Assume that the matrix A has full row rank (w.l.o.g).

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Given $x \in \mathbb{R}^n$, we let

- $x_B \in \mathbb{R}^m$ consist of components of x with indices in B
- $x_N \in \mathbb{R}^{n-m}$ consist of components of x with indices in N

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Abusing notation, we denote by B and N the submatrices of A consisting of columns with indices in B and N, resp.

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Definition

Consider $x \in \mathbb{R}^n$ and a basis B, and consider the decomposition of x into $x_B \in \mathbb{R}^m$ and $x_N \in \mathbb{R}^{n-m}$.
Then x is a *basic solution w.r.t. the basis B* if $Ax = b$ and $x_N = 0$.
Components of x_B are *basic variables*.
A basic solution x is *feasible* if $x \geq 0$.

Example (Whiteboard)

$$\begin{aligned} x_1 + x_2 &\leq 2 \\ x_1 &\leq 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Add slack variables x_3, x_4:

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ x_1 + x_4 &= 1 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

$$A = (u_1\ u_2\ u_3\ u_4) = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$x = (x_1, x_2, x_3, x_4)^\top$$

$$b = (2, 1)^\top$$

$$Ax = b \text{ where } x \geq 0$$

For now let us ignore the objective function and play with the polyhedron defined by the above inequalities.

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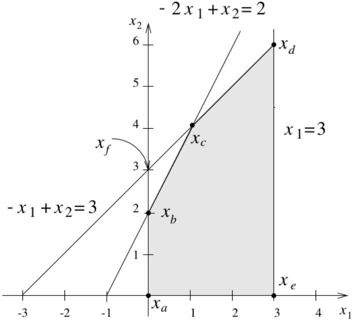

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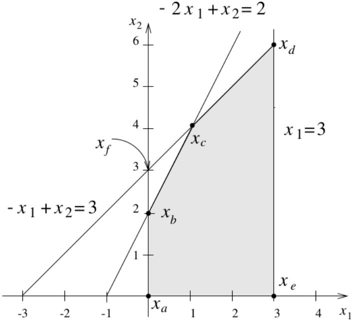

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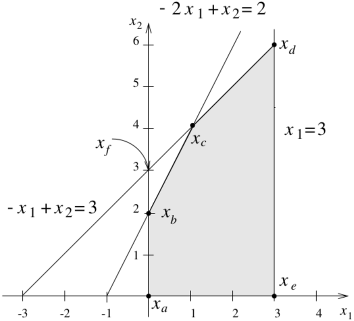

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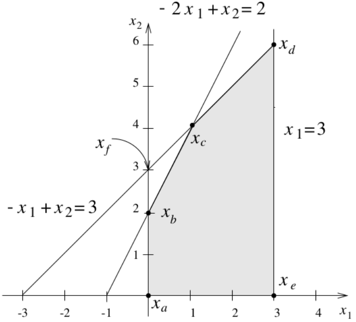

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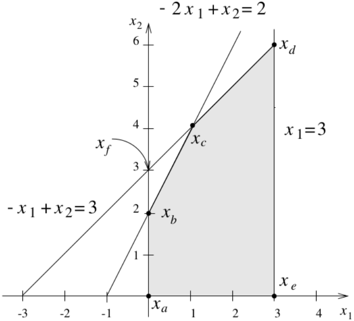

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$Ax = b$ where $x \geq 0$

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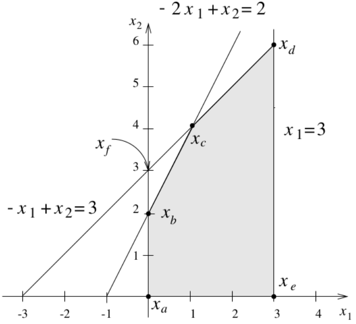

Consider a basis $\{x_3, x_4, x_5\}$ with

$$B = (u_3\ u_4\ u_5) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

What is x_B satisfying $Bx_B = b$?

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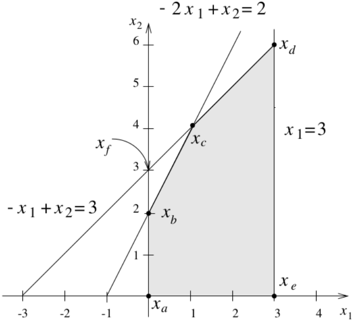

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The corresponding basic solution is

$$x = (x_1, x_2, x_3, x_4, x_5)^\top = (0, 0, 2, 3, 3)^\top = x_a \quad \text{Feasible!}$$

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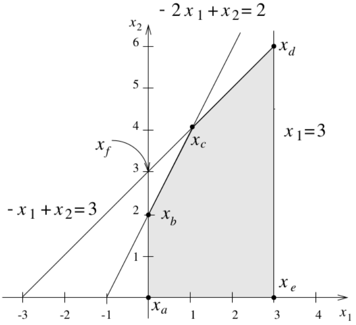

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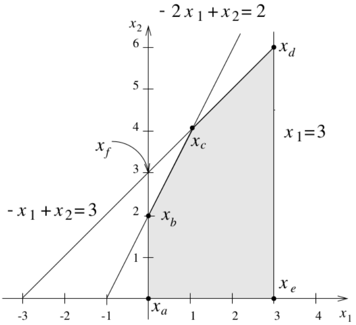

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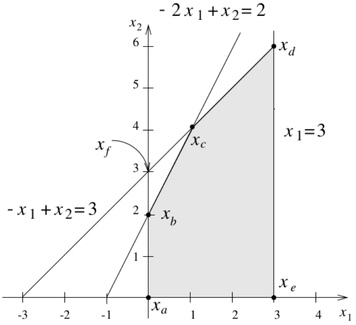

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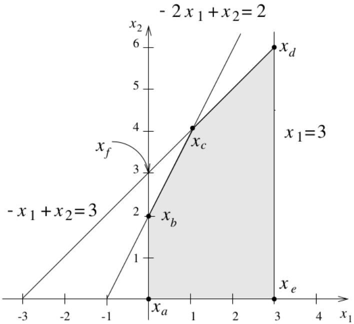

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What is x_B satisfying $Bx_B = b$? $x_B = (x_1, x_2, x_3)^\top = (3, 6, 2)^\top$.

The corresponding basic solution is

$$x = (x_1, x_2, x_3, x_4, x_5)^\top = (3, 6, 2, 0, 0)^\top = x_d \quad \text{Feasible!}$$

Existence of Basic Feasible Solutions

Theorem 1 (Fundamental Theorem of LP)

Consider a linear program in standard form.

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There are finitely many of them, which implies decidability.

However, the enumeration of all basic feasible solutions would be impractical; the number of basic feasible solutions is potentially

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

For $n = 100$ and $m = 10$, we get $535,983,370,403,809,682,970$.

Extreme Points

Note that the set Θ of points x satisfying $Ax = b, x \geq 0$ is *convex polyhedron*.

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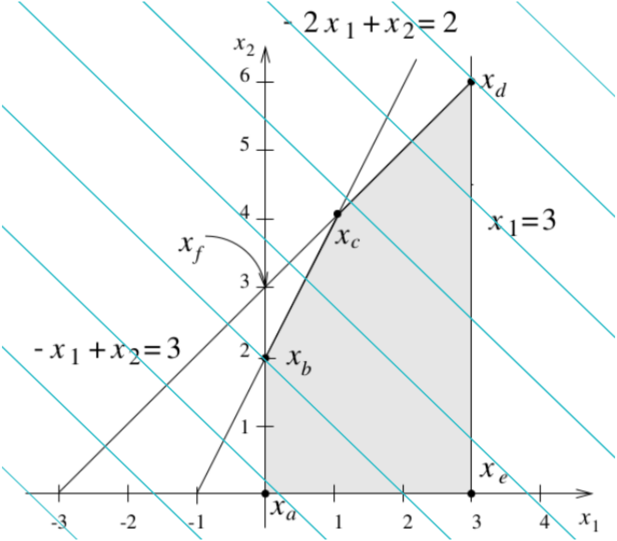
Thus, as a corollary, we obtain that to find an optimal solution to the linear optimization problem, we need to consider only extreme points of the feasibility region.

Optimal Solutions



Here, the blue lines are contours of $-x_1 - x_2$. The minimizer is x_d.

Degenerate Basic Solutions

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Two different bases can correspond to the same point. To see this, consider the constraints defined by

$$Ax = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 6 \\ 13 \\ 12 \end{pmatrix} = b.$$

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There are two bases

$\{x_1, x_2, x_3\}$ giving

$$B = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 0 & 1 \\ 4 & 0 & 0 \end{pmatrix}$$

$\{x_1, x_3, x_4\}$ giving

$$B' = \begin{pmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}$$

Each gives the same *degenerate* basic solution $x = (3, 0, 4, 0)^\top$.

Simplex Algorithm

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Now, how do you move from one vertex to another one algebraically?

First, we consider LP problems where each basic solution is non-degenerate.

Later we drop this assumption.

Changing Basis (Non-Degenerate Case)

Consider a basis B and write $A = (B\ N) = (u_1 \dots u_m\ u_{m+1} \dots u_n)$ where $B = (u_1 \dots u_m)$ and $N = (u_{m+1} \dots u_n)$.

Note that each u_i is a column vector of dimension m.

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Note that each u_i is a column vector of dimension m.

Consider a basic feasible solution $x = [x_B \ x_N]$ where $x_N = 0$. Then

$$x_1 u_1 + \cdots x_m u_m = b$$

For a non-degenerate case, we have $x_j > 0$ for all $j = 1, \ldots, m$.

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$$\begin{aligned} b &= x_1 u_1 + \cdots x_m u_m \\ &= x_1 u_1 + \cdots x_m u_m - \alpha u_i + \alpha u_i \\ &= x_1 u_1 + \cdots x_m u_m - \alpha(y_1 u_1 + \cdots + y_m u_m) + \alpha u_i \\ &= (x_1 - \alpha y_1) u_1 + \cdots + (x_m - \alpha y_m) u_m + \alpha u_i \end{aligned}$$

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Now consider maximum $\alpha > 0$ such that $x_j - \alpha y_j \geq 0$ for all j.

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Obtain a basis $B_{j \to i} = B \smallsetminus \{j\} \cup \{i\}$ and a basic feasible solution

$$x_{j \to i} = (x'_1, \ldots, x'_{j-1}, 0, x'_{j+1}, \ldots, x'_m, 0, \ldots, 0, \alpha, 0, \ldots, 0)^\top$$

Here $x'_k = x_k - \alpha y_k$ for each $k \in \{1, \ldots, j-1, j+1, \ldots, m\}$.

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Here $x'_k = x_k - \alpha y_k$ for each $k \in \{1, \dots, j-1, j+1, \dots, m\}$.
We say that we *pivot about (j, i)*.

Algorithm 1 Simplex - **Non-degenerate**

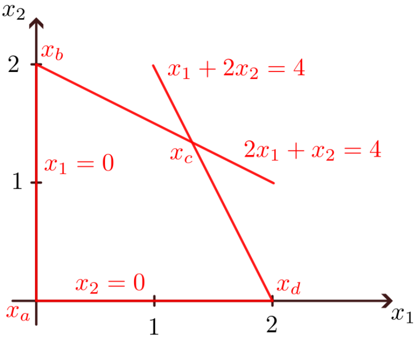
Choose a starting basis $B = (u_1 \dots u_m)$ (here $A = (B\ N)$)
repeat
    Compute the basic solution x for the basis B
    for $i \in \{m+1, \dots, n\}$ **do**
        Solve $B(y_1, \dots, y_m)^\top = u_i$
        if $y_k \leq 0$ for all $k \in \{1, \dots, m\}$ **then**
            Stop, unbounded problem.
        end if
        Select $j = \text{argmin}\{x_k/y_k \mid y_k > 0 \wedge k = 1, \dots, m\}$
        Compute $x_{j \to i}$
    end for
    if $c^\top(x_{j \to i} - x) \geq 0$ for all $i \in \{m+1, \dots, n\}$ **then**
        Stop, we have an optimal solution.
    end if
    Select $i \in \{m+1, \dots, n\}$ such that $c^\top(x_{j \to i} - x) < 0$
    $B \leftarrow B_{j \to i}$
until convergence

$$A = (u_1\ u_2\ u_3\ u_4) = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}$$

$$x = (x_1, x_2, x_3, x_4)^\top$$

$$b = (4, 4)^\top$$

$$c = (-1, -1, 0, 0)^\top$$



minimize $c^\top x$ subject to $Ax = b$ where $x \geq 0$

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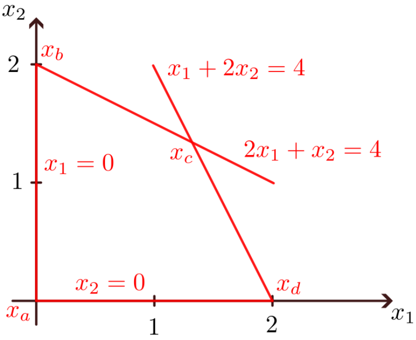

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Consider a basis

$$B = (a_3\ a_4) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The basic solution is $x = (x_1, x_2, x_3, x_4)^\top = (0, 0, 4, 4)^\top$

Non-Degenerate Example

$$c = (-1, -1, 0, 0) \quad A = (u_1\ u_2\ u_3\ u_4) = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

Start with the basis $\{x_3, x_4\}$ giving $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the basic solution $x = (x_1, x_2, x_3, x_4) = (0, 0, 4, 4)$.

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$$c = (-1, -1, 0, 0) \quad A = (u_1\, u_2\, u_3\, u_4) = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

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Consider x_1 as a candidate to the basis, i.e., consider the first column u_1 of A expressed in the basis B:

$$u_1 = (1, 2)^\top = B\ (1, 2)^\top \text{ thus } y = (y_3, y_4) = (1, 2)$$

Non-Degenerate Example

$$c = (-1, -1, 0, 0) \quad A = (u_1\, u_2\, u_3\, u_4) = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

Start with the basis $\{x_3, x_4\}$ giving $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the basic solution $x = (x_1, x_2, x_3, x_4) = (0, 0, 4, 4)$.

Consider x_1 as a candidate to the basis, i.e., consider the first column u_1 of A expressed in the basis B:

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We have $c^\top (x_{4\to1} - x) = -2 < 0$

So let us move to the basis $\{x_1, x_3\}$.

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$$u_2 = (2, 1)^\top = B\ (1/2, 3/2)^\top \text{ thus } y = (y_1, y_3) = (1/2, 3/2)$$

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We have reached a minimizer. All changes would lead to a higher objective value.

We may exchange x_1 with x_4, but this would give us the initial basis with a higher objective value.

Non-Degenerate Case Convergence

Theorem 3

Suppose that the simplex method is applied to a linear program and that every basic variable is strictly positive at every iteration. Then, in a finite number of iterations, the method either terminates at an optimal basic feasible solution or determines that the problem is unbounded.

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However, what happens if we meet a degenerate solution?

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Theorem 3
Suppose that the simplex method is applied to a linear program and that every basic variable is strictly positive at every iteration. Then, in a finite number of iterations, the method either terminates at an optimal basic feasible solution or determines that the problem is unbounded.

However, what happens if we meet a degenerate solution?

So, let us drop the non-degeneracy assumption.

Changing Basis (Degenerate Case)

Consider a basis B and write $A = (B\ N) = (u_1 \ldots u_m\, u_{m+1} \ldots u_n)$ where $B = (u_1 \ldots u_m)$ and $N = (u_{m+1} \ldots u_n)$.

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Note that each u_i is a column vector of dimension m.

Consider a basic feasible solution $x = [x_B\ x_N]$ where $x_N = 0$. Then

$$x_1 u_1 + \cdots + x_m u_m = b$$

For a degenerate case, we have $x_j \geq 0$ for all $j \in \{1, \dots, m\}$, and *may have* $x_i = 0$ *for some* $j \in \{1, \dots, m\}$.

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Now as B is a basis, we have that for each $i \in \{m+1, \dots, n\}$ there are coefficients $y_1, \dots, y_m$ such that $y_1 u_1 + \cdots + y_m u_m = u_i$.

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$$\begin{aligned} b &= x_1 u_1 + \cdots + x_m u_m \\ &= x_1 u_1 + \cdots + x_m u_m - \alpha u_i + \alpha u_i \\ &= x_1 u_1 + \cdots + x_m u_m - \alpha(y_1 u_1 + \cdots + y_m u_m) + \alpha u_i \\ &= (x_1 - \alpha y_1) u_1 + \cdots + (x_m - \alpha y_m) u_m + \alpha u_i \end{aligned}$$

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Now consider maximum $\alpha \geq 0$ such that $x_j - \alpha y_j \geq 0$ for all j.

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Obtain a basis $B_{j \to i} = B \setminus \{j\} \cup \{i\}$ and a basic feasible solution

$$x_{j \to i} = (x'_1, \ldots, x'_{j-1}, 0, x'_{j+1}, \ldots, x'_m, 0, \ldots, 0, \alpha, 0, \ldots, 0)^\top$$

Here $x'_k = x_k - \alpha y_k$ for each $k \in \{1, \ldots, j-1, j+1, \ldots, m\}$.
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We say that we *pivot about* (j, i).

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$$x_{2\to 4} = (0, (x_2 - \alpha y_2), (x_3 - \alpha y_3), \alpha)^\top = (0, 0, 1, 1)^\top$$

Note that $c^\top x_{2\to 4} = 0$.

Thus **no effect on the objective value!**

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Consider x_1 as a candidate for the basis:

$$u_1 = (1, -1)^\top = B(-1, 2)^\top \text{ thus } y = (y_2, y_3) = (-1, 2)$$

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$$c = (-1, 0, 0, 0)^\top \quad A = (u_1\ u_2\ u_3\ u_4) = \begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

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Pivot about $(3, 1)$, that is x_3 exchanges with x_1 and $\alpha = x_3/y_3 = 0$.

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Thus **no effect on the objective value either!**

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No change in the basic solution, and thus $c^\top x_{3\to1} = c^\top x = 0$.

Thus **no effect on the objective value either!**

Which variable should go to the basis?!

Reduced Cost

Given a basis B, we denote by c_B the vector of components of c that correspond to the variables of B.

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One can prove that for every $i \in \{m+1, \ldots, n\}$ we have

$$c^\top x_{j \to i} - c^\top x = (c_i - c_B^\top y)\alpha$$

Here $y = (y_1, \ldots, y_m)^\top$ where $By = u_i$.

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For non-degenerate case, we have $\alpha > 0$ and thus

$$c^\top x_{j \to i} < c^\top x \quad \text{iff} \quad c_i - c_B^\top y < 0$$

For the degenerate case, we may have $\alpha = 0$ and $c_i - c_B y < 0$.

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Define the *reduced cost* by

$$r_i = c_i - c_B^\top y$$

Intuitively, c_i is the cost of x_i in the new basis and $c_B^\top y$ in the old one.

Derivation of Reduced Cost

$$
\begin{aligned}
c^\top x_{j\to i} &= c^\top(x'_1,\ldots,x'_{j-1},0,x'_{j+1},\ldots,x'_m,0,\ldots,0,\alpha,0,\ldots,0)^\top \\
&= c^\top(x'_1,\ldots,x'_{j-1},x'_j,x'_{j+1},\ldots,x'_m,0,\ldots,0,\alpha,0,\ldots,0)^\top \\
&= c_1x'_1+\cdots+c_mx'_m+c_i\alpha \\
&= c_1(x_1-\alpha y_1)+\cdots c_m(x_m-\alpha y_m)+c_i\alpha \\
&= (c_1x_1+\cdots+c_mx_m)-(c_1y_1+\cdots+c_my_m-c_i)\alpha \\
&= c^\top x-(-c_i+c_By)\alpha
\end{aligned}
$$

Here we use the fact that $x'_k = x_k - \alpha y_k$ for each $k \in \{1,\ldots,j-1,j+1,\ldots,m\}$ and that $x_j - \alpha y_j = 0$.

Then clearly

$$c^\top x_{j\to i} - c^\top x = (c_i - c_B y)\alpha$$

$$\alpha = \min\{x_k/y_k \mid y_k > 0 \wedge k = 1,\ldots,m\}$$

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$$r_4 = c_4 - (c_2 y_2 + c_3 y_3) = 0 - (0 \cdot (-1) + 0 \cdot 2) = 0$$

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So we should put x_1 into the basis (the reduced cost gets smaller).

Algorithm 2 Simplex

Choose a starting basis $B = (u_1 \dots u_m)$ (here $A = (B \;\; N)$)
repeat
  Compute the basic solution x for the basis B
  for $i \in \{m+1, \dots, n\}$ **do**
    Solve $B\,(y_1, \dots, y_m)^\top = u_i$
    if $y_k \leq 0$ for all $k \in \{1, \dots, m\}$ **then**
      Stop, unbounded problem.
    end if
    Select $j \in \operatorname{argmin}\{x_k/y_k \mid y_k > 0 \wedge k = 1, \dots, m\}$
    Compute $r_i = c_i - c_B^\top y$ where $y = (y_1, \dots, y_m)^\top$
  end for
  if $r_i \geq 0$ for all $i \in \{m+1, \dots, n\}$ **then**
    Stop, we have an optimal solution.
  end if
  Select $i \in \{m+1, \dots, n\}$ such that $r_i < 0$
  $B \leftarrow B_{j \to i}$
until convergence

Degenerate Example (Cont.)

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After following the reduced cost from the basis $\{x_2, x_3\}$, we end up in the basis $\{x_1, x_2\}$ giving $B = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ and the basic solution $x = (x_1, x_2, x_3, x_4) = (0, 1, 0, 0)$ with $c^\top x = 0$.

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$$u_4 = (0, 1)^\top = B\,(-1/2, 1/2)^\top \text{ thus } y = (y_1, y_2) = (-1/2, 1/2)$$

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Pivot about $(2, 4)$, that is x_2 exchanges with x_4 and $\alpha = x_2/y_2 = 2$

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This is the minimizer!

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Does this always work?

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This is the minimizer!

Does this always work? Unfortunately, NO!

Degenerate Case - Looping

Consider the following linear program:

$$
\begin{array}{ll}
\text{minimize} & z = -\frac{3}{4}x_1 + 150x_2 - \frac{1}{50}x_3 + 6x_4 \\
\text{subject to} & \frac{1}{4}x_1 - 60x_2 - \frac{1}{25}x_3 + 9x_4 + x_5 = 0 \\
& \frac{1}{2}x_1 - 90x_2 - \frac{1}{50}x_3 + 3x_4 + x_6 = 0 \\
& x_3 + x_7 = 1 \\
& x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0
\end{array}
$$

Executing the simplex method on this program starting with the basis $\{x_5, x_6, x_7\}$ and always choosing i minimizing the reduced cost at line 15, eventually ends up back in the basis $\{x_5, x_6, x_7\}$.

In other words, even though the reduced cost is always negative, the overall effect on the objective is 0.

Convergence of Simplex Method

A solution is to use Bland's rule:

- Select the smallest index j at line 9.
- Select the smallest index i at line 15.

Theorem 4

If the simplex method is implemented using Bland's rule to select the entering and leaving variables, then the simplex method is guaranteed to terminate.

Simplex Convergence Summary

In a non-degenerate case:

- There is always a unique j to be selected at line 9.
- The objective of the basic solution decreases with each step.

Thus, we have a deterministic algorithm that always terminates in a non-degenerate case.

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In a non-degenerate case:

- There is always a unique j to be selected at line 9.
- The objective of the basic solution decreases with each step.

Thus, we have a deterministic algorithm that always terminates in a non-degenerate case.

In a degenerate case:

- We may have several j from which to select at line 9.
- Even though the reduced cost is negative, the basic solution may remain the same.

The simplex algorithm may cycle!

Using Bland's rule, the simplex method always converges to a minimizer or detects an unbounded LP.

Two-Phase Simplex Algorithm

A Simplex algorithm is initialized with a basic feasible solution.

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How do we obtain such a solution? Given a standard form LP

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$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

We construct an artificial LP problem.

$$\begin{aligned} \text{minimize} \quad & y_1 + y_2 + \cdots + y_m \\ \text{subject to} \quad & (A \ I_m) \begin{pmatrix} x \\ y \end{pmatrix} = b \\ & \begin{pmatrix} x \\ y \end{pmatrix} \geq 0 \end{aligned}$$

Here $y = (y_1, \ldots, y_m)^\top$ is a vector of artificial variables, I_m is the identity matrix of dimensions $m \times m$.

Two-Phase Simplex Algorithm

Solve the artificial LP problem:

$$\begin{aligned} \text{minimize} \quad & y_1 + y_2 + \cdots + y_m \\ \text{subject to} \quad & [A \; I_m] \begin{pmatrix} x \\ y \end{pmatrix} = b \\ & \begin{pmatrix} x \\ y \end{pmatrix} \geq 0 \end{aligned}$$

Proposition 1

The original LP problem has a basic feasible solution iff the associated artificial LP problem has an optimal feasible solution with the objective function 0.

If we solve the artificial problem with $y = 0$, we obtain x such that $Ax = b, x \geq 0$ is a basic feasible solution for the original problem.

If there is no such a solution to the artificial problem, there is no basic feasible solution, and hence no feasible solution, to the original problem.

Linear Programming
Properties

LP Complexity

Iterations of the simplex algorithm can be implemented to compute the first step using $\mathcal{O}(m^2 n)$ arithmetic operations and each next step $\mathcal{O}(mn)$.

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There are as many as $\binom{n}{m}$ basic solutions (many of them likely infeasible). How large are these numbers?

m	$\binom{2m}{m}$
1	2
5	252
10	184756
20	1×10^{11}
50	1×10^{29}
100	9×10^{58}
200	1×10^{119}
300	1×10^{179}
400	2×10^{239}
500	3×10^{299}

The number of iterations may be proportional to $\binom{n}{m}$ that is EXPTIME.

Linear Programming Complexity

Complexity of the simplex algorithm:

- In the worst case, the time complexity of the simplex algorithm is exponential. This holds for any deterministic pivoting rule.
  For details, see "How good is the simplex algorithm?" by Klee, Victor, and Minty, George J. Inequalities 1972.

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- There is a theory that shows that examples with exponential complexity are rare. More precisely (but still very imprecisely)
  - Consider small random perturbations of the coefficients in the LP (use Gaussian noise with a small variance)
  - Then, the expected computation time for the resulting instances of LP is polynomial.

  For details, see "Smoothed analysis of algorithms: Why the simplex algorithm usually takes polynomial time" by Daniel A. Spielman and Shang-Hua Teng in JACM 2004.

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Complexity of the simplex algorithm:

- In the worst case, the time complexity of the simplex algorithm is exponential. This holds for any deterministic pivoting rule.

  For details, see "How good is the simplex algorithm?" by Klee, Victor, and Minty, George J. Inequalities 1972.

- There is a theory that shows that examples with exponential complexity are rare. More precisely (but still very imprecisely)
  - Consider small random perturbations of the coefficients in the LP (use Gaussian noise with a small variance)
  - Then, the expected computation time for the resulting instances of LP is polynomial.

  For details, see "Smoothed analysis of algorithms: Why the simplex algorithm usually takes polynomial time" by Daniel A. Spielman and Shang-Hua Teng in JACM 2004.

Is there a deterministic polynomial time algorithm for solving LP?

Linear Programming Complexity

We assume that all coefficients are encoded in binary (more precisely, as fractions of two integers encoded in binary).

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There is also a polynomial time algorithm (by Karmarkar) that has lower complexity upper bounds than the Khachiyan's and sometimes works even better than the simplex.

Linear Programming in Practice

Heavily used tools for solving practical problems.

Several advanced linear programming solvers (usually parts of larger optimization packages) implement various heuristics for solving large-scale problems, such as sensitivity analysis.

See an overview of tools here:

http://en.wikipedia.org/wiki/Linear_programming#Solvers_and_scripting_.28programming.29_languages

For example, the well-known Gurobi solver uses the simplex algorithm to solve LP problems.