Chapter 2

Random variables

Exercise 2.1

Let X be uniformly distributed on $0, 1, \ldots, 99$. Calculate $\mathcal{P}(X \geq 25)$.

Exercise 2.2

Suppose that X has a geometric probability distribution with $p = 4/5$. Compute the probability that $4 \leq X \leq 7$ or $X > 9$.

Exercise 2.3

Let $n \in \mathbb{N}$ and let

$$f(x) = \begin{cases} c2^x, & x = 0, 1, 2, \ldots, n \\ 0, & \text{otherwise} \end{cases}$$

Find the value of c such that f is a probability distribution.

Exercise 2.4

Prove that $\binom{n-1}{r-1} p^r (1-p)^{n-r} = \binom{-r}{n-r} (-1)^{n-r} p^r (1-p)^{n-r}$.

Exercise 2.5

Professor R. A. Bertlmann (*http://homepage.univie.ac.at/reinhold.bertlmann/*) is going to a attend a conference in Erice (Italy) and wants to pack 10 socks. He draws them randomly from a box with 20 socks. However, prof. Bertlmann likes to wear a sock of different color on each leg. What is the probability that he draws out 3 red socks given that there are 4 red socks in the box?

Exercise 2.6

Consider a probabilistic space over a set $\mathbf{S}$. Show that for every event $A \subseteq \mathbf{S}$ and its indicator I_A it holds $E(I_A) = \mathcal{P}(A)$. (An indicator is defined as $I_A(w) = 1$

for all $w \in A$ and $I_A(w) = 0$ for all $w \notin A$.)

Exercise 2.7

In the same space as in Exercise **??** consider two random variables X, Y such that $\forall w \in$
$SsS : X(w) \leq Y(w)$. Prove that $E(X) \leq E(Y)$.

Exercise 2.8 (Banach's matchbox problem)

Suppose a mathematician carries two matchboxes in his pocket. He chooses either of them with the probability 0.5 when taking a match. Consider the moment when he reaches an empty box in his pocket. Assume there were N matches intially in each matchbox. What is the probability that there are exactly r matches in the nonempty matchbox?

Exercise 2.9

Random variables $X_1, X_2, \ldots, X_r$ with probability distributions $p_{X_1}, p_{X_2}, \ldots, p_{X_r}$ are mutually independent if for all $x_1 \in Im(X_1)$, $x_2 \in Im(X_2)$, ..., $x_r \in Im(X_r)$

$$p_{X_1,X_2,\ldots,X_r}(x_1, x_2, \ldots, x_r) = p_{X_1}(x_1)p_{X_2}(x_2)\cdots, p_{X_r}(x_r).$$

Does this imply that for any set $i_1, i_2 \ldots i_q \in \{1, 2 \ldots r\}$ of distinct indices we have

$$p_{X_{i_1},X_{i_2},\ldots,X_{i_q}}(x_{i_1}, x_{i_2}, \ldots, x_{i_q}) = p_{X_{i_1}}(x_{i_1})p_{X_{i_2}}(x_{i_2})\cdots, p_{X_{i_q}}(x_{i_q})?$$

Exercise 2.10

Let $A_1, A_2 \ldots A_r$ be events such that we have

$$\mathcal{P}(A_1 \cap A_2 \cap \cdots \cap A_r) = \mathcal{P}(A_1)\mathcal{P}(A_2)\cdots\mathcal{P}(A_r)$$

Does this imply that for all $i_1, i_2, \ldots i_q$ distinct indices we have

$$\mathcal{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_q}) = \mathcal{P}(A_{i_1})\mathcal{P}(A_{i_2})\cdots\mathcal{P}(A_{i_q})$$