Chapter 4

The probabilistic method

Exercise 1

Consider an instance of SAT with m clauses, where every clause has exactly k literals.

(a) Give a Las Vegas algorithm that finds an assignment satisfying at least $m(1-2^{-k})$ clauses and analyze its expected running time.

(b) Give a derandomization of the randomized algorithm using the method of conditional expectations.

Answer of exercise 1

(a) Assign values independently and uniformly at random to the variables. The probability that the i^{th}-claus with k literals is satisfied is $(1-2^{-k})$. Let N_c be the random variable indicating the number of satisfied clauses. Then

$$E[N_c] = \sum_{i=1}^{m} (1-2^{-k}) = m(1-2^{-k}).$$

Let $p = Pr(N_c \geq m(1-2^{-k}))$, and observe that $N_c \leq m$. It then follows that

$$\begin{aligned} m(1-2^{-k}) &= E[N_c] \\ &= \sum_{i \leq m(1-2^{-k})-1} iPr(N_c = i) + \sum_{i \geq m(1-2^{-k})} iPr(N_c = i) \\ &\leq (1-p)(m(1-2^{-k})-1) + pm, \end{aligned}$$

which implies that

$$p \geq \frac{1}{1+m2^{-k}}.$$

Therefore, the expected number of samples before finding an assignment satisfying at least $m(1-2^{-k})$ clauses is $1/p$, which is at most $1+m2^{-k}$. Testing to see if $(N_c \geq m(1-2^{-k}))$ can be done in $O(km)$ time. As such the algorithm can be done in polynomial time.

(b) Assign values to the variables deterministically – one at a time – in any order $x_1, x_2, \ldots, x_n$. Suppose that we have assigned the first k variables. Let $y_1, y_2, \ldots, y_k$be the corresponding assigned values. We compute the the quantities;

(i) $E[N_c | x_1 = y_1, x_2 = y_2, \ldots, x_k = y_k, x_{k+1} = \mathrm{T}]$

(ii) $E[N_c | x_1 = y_1, x_2 = y_2, \ldots, x_k = y_k, x_{k+1} = \mathrm{F}]$.

and then choose the setting with the larger expectation.

Exercise 2

(a) Prove that, for every integer n, there exists a coloring of the edges of the complete graph K_n by two colours so that the total number of monochromatics copies of K_4 is at most $\binom{n}{4}2^{-5}$.

(b) Give a randomized algorithm for finding a colouring with at most $\binom{n}{4}2^{-5}$ monochromatic copies of K_4 that runs in expected time polynomial in n.

(c) Show how to construct such a colouring deterministically in polynomial time using the method of conditional expectations.

Answer of exercise 2

(a) X is the random variable denoting the number of monochromatics copies of K_4. The probability that a certain 4-subset forms a monochromatic K_4 is 2.2^{-6} – where 2 is for the two different colours. Then

$$E[X] = \underbrace{\binom{n}{4}}_{\text{choose 4 vertices from n}} .2..2^{-6} = \binom{n}{4}2^{-5}.$$

(b) Colour edges independently and uniformly. Let $p = Pr(X \leq \binom{n}{4}2^{-5})$. Then, we have

$$\begin{aligned}
\binom{n}{4}2^{-5} &= E[X] \\
&= \sum_{i \leq \binom{n}{4}2^{-5}} iPr(X=i) + \sum_{i \geq \binom{n}{4}2^{-5}} iPr(X=i) \\
&\geq p + (1-p)(\binom{n}{4}2^{-5} + 1),
\end{aligned}$$

which implies that

$$\frac{1}{p} \le \binom{n}{4} 2^{-5}.$$

Thus, the expected number of samples is at most $\binom{n}{4}2^{-5}$. Testing this to see if $X \le \binom{n}{4}2^{-5}$ can be done in $O(n^4)$ time. As such the algorithm can be done in polynomial time.

(c) Follow the solution method in 1(b).

Exercise 3

Given an n-vertex undirected graph $G = (V, E)$, consider the following method of generating an independent set. Given a permutation σ of the vertices, define a subset $S(\sigma)$ of the vertices as follows: for each vertex i, $i \in S(\sigma)$ iff no neighbour j of i precedes i in the permutation σ.

(a) Show that each $S(\sigma)$ is an independent set in G.

(b) Suggest a natural randomized algorithm to produce σ for which you can show that the expected cardinality of $S(\sigma)$ is

$$\sum_{i=1}^{n} \frac{1}{d_i + 1},$$

where d_i denotes the degree of vertex i.

(c) Prove the G has an independent set of size at least $\sum_{i=1}^{n} \frac{1}{d_i+1}$.

Answer of exercise 3

(a) For any edge (i, j), if $i \in S(\sigma)$ then it implies that $(\sigma(i) < \sigma(j))$. If $j \in S(\sigma)$, then it implies that $(\sigma(j) < \sigma(i))$. But it is impossible that these two cases occur at the same time. Therefore $S(\sigma)$ is an independent set in G.

(b) Choose the permutation σ randomly – with respect to the uniform distribution. For any vertex i, let U_i be the union of i and its neighbours. As the degree of i is d_i, U_i has $d_i + 1$ elements. By definition – of the question – $i \in S(\sigma)$ iff $\sigma(i)$ is 'smallest' among $\sigma(x)$, $x \in U_i$. By symmetry, the prob of $i \in S(\sigma)$ is $1/(d_i + 1)$. Therefore, by linearity of expectation, the prob of $i \in S(\sigma)$ is

$$E[|S(\sigma)|] = \sum_{i=1}^{n} Pr(i \in S(\sigma)) = \sum_{i=1}^{n} \frac{1}{d_i + 1}.$$

(c) By an expectation argument, there must be at least one $S(\sigma)$ whose value is at least $E[|S(\sigma)|]$. And then, $S(\sigma)$ is an independent set in G. Therefore, G has an independent set of size at least $\sum_{i=1}^{n} \frac{1}{d_i+1}$.