Chapter 2

Random variables

Exercise 2.1 (Uniform distribution)

Let X be uniformly distributed on $0, 1, \ldots, 99$. Calculate $\mathcal{P}(X \geq 25)$.

Solution of Exercise 2.1: We have

$$\mathcal{P}(X \geq 25) = 1 - \mathcal{P}(X \leq 24) = 1 - F(24) = 1 - \frac{25}{100} = \frac{3}{4}.$$

Alternative solution:

$$P(X \geq 25) = \sum_{x=25}^{99} P(X = x) = 75\frac{1}{100} = \frac{3}{4}.$$

Exercise 2.2 (Binomial distribution)

Assume the probability of winning a game is $\frac{1}{10}$. If you play the game 10 times, what is the probability that you win at most once?

Solution of Exercise 2.2: Let X be the number of wins. This game represents a binomial situation with $n = 10$ and $p = \frac{1}{10}$. We interpret *win at most once* as meaning $X \leq 1$. Then

$$P(X \leq 1) = P(X = 0) + P(X = 1) = \binom{10}{0}\left(\frac{1}{10}\right)^0\left(\frac{9}{10}\right)^{10} + \binom{10}{1}\left(\frac{1}{10}\right)^1\left(\frac{9}{10}\right)^9$$

$$= \left(\frac{9}{10}\right)^9\left(\frac{19}{10}\right) \equiv 0.736099.$$

Exercise 2.3 (Binomial distribution)

If X is binomial with parameters n and p, find an expression for $P[X \leq 1]$.

Solution of Exercise 2.3:

$$\begin{aligned} P[X \le 1] &= P[X=0] + P[X=1] \\ &= \binom{n}{0} p^0 (1-p)^n + \binom{n}{1} p^1 (1-p)^{n-1} \\ &= (1-p)^n + np(1-p)^{n-1} = (1-p)^{n-1}((1-p)+np) \\ &= (1-p)^{n-1}(1+(n-1)p). \end{aligned}$$

Exercise 2.4 (Geometric distribution)

Consider a biased coin with the probability of "head" equal to $\frac{3}{5}$. We are throwing until the "head" is reached. What is the probability $p_Z(5)$ that the head is reach in the 5th throw?

Solution of Exercise 2.4: The corresponding sequence of outcomes is $TTTTH$. Probability of this sequence is $p_Z(5) = \left(\frac{2}{5}\right)^4 \frac{3}{5}$.

Exercise 2.5 (Geometric distribution)

1. Calculate the aforementioned probability $p_Z(i)$ for a general success probability p and a general number of throws i.
2. Verify that $\sum_{i=1}^{\infty} p_Z(i) = 1$.
3. Determine the distribution function $F_Z(t)$.

Solution of Exercise 2.5:

1. The geometric probability distribution origins from Bernoulli trials as well, but this time we count the number of trials until the first 'success' occurs. The sample space consists of binary strings of the form $S = \{0^{i-1}1 | i \in \mathbb{N}\}$.

   We define the random variable $Z : \{0^i 1 | i \in \mathbb{N}_0\} \to \mathbb{R}$ as $Z(0^{i-1}1) \stackrel{def}{=} i$. Z is the number of trials up to and including the first success. The outcome $0^{i-1}1$ arises from a sequence of independent Bernoulli trials, thus we have

   $$p_Z(i) = (1-p)^{i-1} p. \tag{2.1}$$

2. We use the formula for the sum of geometric series to obtain (verify property (p2))

   $$\sum_{i=1}^{\infty} p_Z(i) = \sum_{i=1}^{\infty} p(1-p)^{i-1} = \frac{p}{1-(1-p)} = \frac{p}{p} = 1.$$

   We require that $p \neq 0$, since otherwise the probabilities do not sum to 1.

3. The corresponding probability distribution function is defined by (for $t \geq 0$)

   $$F_Z(t) = \sum_{i=1}^{\lfloor t \rfloor} p(1-p)^{i-1} = 1 - (1-p)^{\lfloor t \rfloor}.$$

Exercise 2.6 (Geometric distribution)

Suppose that X has a geometric probability distribution with $p = 4/5$. Compute the probability that $4 \le X \le 7$ or $X > 9$.

Solution of Exercise 2.6: We need the following

$$\begin{aligned}
&\big(F(7) - F(3)\big) + [1 - F(9)] = \\
&= \Big(\big(1-(1-p)^7\big) - \big(1-(1-p)^3\big)\Big) + 1 - \big(1-(1-p)^9\big) = \\
&= (1-p)^9 + (1-p)^3 - (1-p)^7.
\end{aligned}$$

Exercise 2.7 (Hypergeometric distribution)

Professor R. A. Bertlmann (*http://homepage.univie.ac.at/reinhold.bertlmann/*) is going to a attend a conference in Erice (Italy) and wants to pack 10 socks. He draws them randomly from a box with 20 socks. However, prof. Bertlmann likes to wear a sock of different color (and pattern) on each leg.

1. What is the probability that he draws out exactly 5 red socks given that there are 7 red socks in the box?
2. Calculate the same probability of obtaining exactly 5 red socks, when drawing 10 socks randomly and the probability to get the red one $p = \frac{7}{20}$ is the same in each trial and trials are independent.

Solution of Exercise 2.7:

1. This situation is close in its interpretation to the binomial probability distribution except that we consider sampling without replacement. Let us suppose we have two kinds of objects - e.g. r red and $n-r$ black socks in a basket. We have the probability r/n to select a red sock in the first trial. However, the probability of selecting red sock in the second trial is $(r-1)/(n-1)$ if red sock was selected in the first trial, or $r/(n-1)$ if black sock was selected in the first trial. It follows that the assumption of constant probability of every outcome in all trials, as required by the binomial distribution, does not hold. Also, the trials are not independent. In this case we are facing the **hypergeometric distribution** $h(k; m, r, n)$ defined as the probability that there are k red objects in a set of m objects chosen randomly without replacement from n objects containing r red objects.
   There are $\binom{n}{m}$ sample points. The k red socks can be selected from r red socks in $\binom{r}{k}$ ways and $m-k$ black socks can be selected from $n-r$ in $\binom{n-r}{m-k}$ ways. The sample of m socks with k red ones can be selected in

   $$\binom{r}{k}\binom{n-r}{m-k}$$

   ways. Assuming uniform probability distribution on the sample space, the required probability is

$$h(k; m, r, n) = \frac{\binom{r}{k}\binom{n-r}{m-k}}{\binom{n}{m}}, \quad k = 1, 2, \dots \min\{r, m\}$$

In our concrete case we get

$$h(5; 10, 7, 20) = \frac{\binom{7}{5}\binom{13}{5}}{\binom{20}{10}} \approx 0.14628$$

2. Good approximation of the hypergeometric distribution for large n (relatively to m) is the binomial distribution $h(k; m, r, n) \simeq b(k; m, r/n)$. In our concrete case

$$b\left(5; 10, \frac{7}{20}\right) = \left(\frac{7}{20}\right)^5 \left(\frac{13}{20}\right)^5 \binom{10}{5} \approx 0.15357.$$

Exercise 2.8 (Banach's matchbox problem, negative binomial distribution)

Suppose a mathematician carries two matchboxes in his pocket. He chooses either of them with the probability 0.5 when taking a match. Consider the moment when he reaches an empty box in his pocket. Assume there were R matches initially in each matchbox. What is the probability that there are exactly N matches in the nonempty matchbox?

Solution of Exercise 2.8: Let start with the case when the empty matchbox is in the left pocket. Denote choosing the left pocket as a "success" and choosing the right pocket as a "failure". Then we want to know the probability that there were exactly $R - N$ failures until the $(R + 1)^{\text{st}}$ success.

Let us consider the **negative binomial distribution**. It is close in its interpretation to the geometric distribution, we calculate the number of trials until the rth success occurs (in contrast to the 1st success in geometric distribution).

Let T_r be the random variable representing this number. Let us define the following events

- $A =$'$T_r = n$'.
- $B =$'Exactly $(r - 1)$ successes occur in $n - 1$ trials.'
- $C =$'the nth trial results in a success.'

We have that $A = B \cap C$, and B and C are independent giving $P(A) = P(B)P(C)$. Consider a particular sequence of $n - 1$ trials with $r - 1$ successes and $n - 1 - (r - 1) = n - r$ failures. The probability associated with each such sequence is $p^{r-1}(1 - p)^{n-r}$ and there are $\binom{n-1}{r-1}$ such sequences. Therefore

$$P(B) = \binom{n-1}{r-1} p^{r-1}(1 - p)^{n-r}.$$

Since $P(C) = p$ we have

$$\begin{aligned} p_{T_r}(n) =& P(T_r = n) = P(A) \\ =& \binom{n-1}{r-1} p^r (1-p)^{n-r}, \quad n = r, r+1, r+2, \dots \end{aligned}$$

In our case we want to calculate how many matches were removed from the other pocket. We want to calculate the number of failures until the rth success occurs. This is the **modified negative binomial distribution** describing the number of failures until the rth success occurs. The probability distribution is

$$p_Z(n) = \binom{n+r-1}{r-1} p^r (1-p)^n, \quad n \geq 0.$$

(For $r = 1$ we obtain the modified geometric distribution.)

We apply the modified negative binomial distribution to get the probability $p_{\text{left}} = \binom{R-N+R}{R} \left(\frac{1}{2}\right)^{(R+1)} \left(\frac{1}{2}\right)^{R-N}$. The symmetric event (when the matchbox in the right pocket becomes empty) is disjoint, thus the probability of finishing one matchbox when having exactly $N, 0 < N \leq R$ matches in the other one is

$$p = 2p_{\text{left}} = \binom{R-N+R}{R} 2^{N-2R}.$$

Exercise 2.9

Random variables $X_1, X_2, \dots, X_r$ with probability distributions $p_{X_1}, p_{X_2}, \dots, p_{X_r}$ are mutually independent if for all $x_1 \in Im(X_1)$, $x_2 \in Im(X_2)$, ..., $x_r \in Im(X_r)$

$$p_{X_1, X_2, \dots, X_r}(x_1, x_2, \dots, x_r) = p_{X_1}(x_1) p_{X_2}(x_2) \cdots, p_{X_r}(x_r).$$

Does this imply that for any $q \leq r$ and any set $i_1, i_2 \dots i_q \in \{1, 2 \dots r\}$ of distinct indices we have

$$p_{X_{i_1}, X_{i_2}, \dots, X_{i_q}}(x_{i_1}, x_{i_2}, \dots, x_{i_q}) = p_{X_{i_1}}(x_{i_1}) p_{X_{i_2}}(x_{i_2}) \cdots, p_{X_{i_q}}(x_{i_q})?$$

Solution of Exercise 2.9: Yes. We only prove the particular case for $q = r - 1$, the rest follows (I hope :-)). Let j be the index such that $j \neq i_k$ for any k. We have

$$\begin{aligned} & p_{X_1, \dots, X_{j-1}, X_{j+1}, \dots, X_r}(x_1, \dots x_{j-1}, x_{j+1}, \dots x_r) = \\ & \quad = \mathcal{P}(X_1 = x_1, \dots, X_{j-1} = x_{j-1}, X_{j+1} = x_{j+1}, \dots X_r = x_r) \\ & \quad = \sum_y \mathcal{P}(X_1 = x_1, \dots X_{j-1} = x_{j-1}, X_j = y, X_{j+1} = x_{j+1}, \dots X_r = x_r) \\ & \quad = \sum_y p_{X_1}(x_1) \cdots p_{X_{j-1}}(x_{j-1}) p_{X_j}(y) p_{X_{j+1}}(x_{j+1}) \cdots p_{X_r}(x_r) \\ & \quad = \left(\sum_y p_{X_j}(y)\right) p_{X_1}(x_1) \cdots p_{X_{j-1}}(x_{j-1}) p_{X_{j+1}}(x_{j+1}) \cdots p_{X_r}(x_r) \\ & \quad = p_{X_1}(x_1) \cdots p_{X_{j-1}}(x_{j-1}) p_{X_{j+1}}(x_{j+1}) \cdots p_{X_r}(x_r) \end{aligned}$$

Exercise 2.10

Let $n \in \mathbb{N}$ and let

$$f(x) = \begin{cases} c2^x, & x = 0, 1, 2, \dots, n \\ 0, & \text{otherwise} \end{cases}$$

Find the value of c such that f is a probability distribution.

Solution of Exercise 2.10: We need to find $c > 0$ such that $\sum_{x=0}^{n} c2^x = 1$. Recall that the geometric series sums as $\sum_{x=0}^{n} y^x = \frac{y^{n+1}-1}{y-1}$ for $y \neq 1$. We proceed as follows:

$$1 = \sum_{x=0}^{n} c2^x = c\sum_{x=0}^{n} 2^x = c(2^{n+1} - 1),$$

what gives $c = \frac{1}{2^{n+1}-1}$.

Exercise 2.11

Prove that $\binom{n-1}{r-1} p^r (1-p)^{n-r} = \binom{-r}{n-r}(-1)^{n-r} p^r (1-p)^{n-r}$.

Solution of Exercise 2.11: We need to prove $\binom{n-1}{r-1} = \binom{-r}{n-r}(-1)^{n-r}$. For all a,b we have $\binom{a}{b} = \binom{a}{a-b}$ and if $a \le 0$ and $b \ge 0$, then $\binom{-a}{b} = (-1)^b \binom{a+b-1}{b}$ from definition.

Using the latter, we have $\binom{-r}{n-r}(-1)^{n-r} = \binom{r+n-r-1}{n-r} = \binom{n-1}{n-r}$ and using the first, we obtain $\binom{n-1}{n-r} = \binom{n-1}{n-1-(n-r)} = \binom{n-1}{r-1}$.

Exercise 2.12

Use the previous Exercise to show that probabilities of the negative binomial distribution sum to 1.

Solution of Exercise 2.12: We want to show that for every r

$$\sum_{n=r}^{\infty} p_{T_r}(n) = \sum_{n=r}^{\infty} \binom{n-1}{r-1} p^r (1-p)^{n-r} = 1. \tag{2.2}$$

Previous exercise shows that

$$p_{T_r}(n) = p^r \binom{-r}{n-r} (-1)^{n-r} (1-p)^{n-r}. \tag{2.3}$$

We use the Taylor expansion of $(1-t)^{-r}$ for $-1 < t < 1$:

$$(1-t)^{-r} = \sum_{n=r}^{\infty} \binom{-r}{n-r} (-t)^{n-r}.$$

and the substitution $t = (1-p)$ to get

$$p^{-r} = \sum_{n=r}^{\infty} \binom{-r}{n-r} (-1)^{n-r} (1-p)^{n-r}. \tag{2.4}$$

Substituting Eq. (2.4) and Eq. (2.3) into Eq. (2.2) gives the desired result

$$1 = \sum_{n=r}^{\infty} p^r \binom{-r}{n-r} (-1)^{n-r} (1-p)^{n-r}.$$

Note that the summation from r is correct since clearly $p_{T_r}(n) = 0$ for $n < r$.