Notes on satisfiability

Skolemization

Resolution
Skolemization. An example

Example 1: Prove that $\forall x \phi(x, f(x)) \Rightarrow \forall x \exists y \phi(x, y)$ holds.
Skolemization. An example

Example 1: Prove that $\forall x \phi(x, f(x)) \Rightarrow \forall x \exists y \phi(x, y)$ holds.

Example 2: Prove that $\forall x \exists y \phi(x, y) \Rightarrow \forall x \phi(x, f(x))$ not.
Skolemization I

(Nerode, Shore, Logic for Applications)

**Theorem 9.4** For every sentence $\phi$ in a given language $\mathcal{L}$ there is a universal formula $\phi'$ in an expanded language $\mathcal{L}'$ gotten by the addition of new function symbols such that $\phi$ and $\phi'$ are equisatisfiable.
Skolemization II

**Theorem 9.4** For every sentence $\phi$ in a given language $\mathcal{L}$ there is a universal formula $\phi'$ in an expanded language $\mathcal{L}'$ gotten by the addition of new function symbols such that $\phi$ and $\phi'$ are equisatisfiable.

**Lemma 9.5** For any sentence $\phi = \forall x_1 \ldots \forall x_n \exists y \psi$ of a language $\mathcal{L}$ $\phi$ and $\phi' = \forall x_1 \ldots \forall x_n \psi(y/f(x_1, \ldots, x_n))$ are equisatisfiable when $f$ is a function symbol not in $\mathcal{L}$.
Lemma 9.5 For any sentence $\phi = \forall x_1 \ldots \forall x_n \exists y \psi$ of a language $\mathcal{L}$ and $\phi' = \forall x_1 \ldots \forall x_n \psi(y/f(x_1, \ldots, x_n))$ are equisatisfiable when $f$ is a function symbol not in $\mathcal{L}$.

Proof: $\mathcal{L}'$ ... $\mathcal{L}$ extended with the function symbol $f$

If $A'$ is a structure for $\mathcal{L}'$ and $A$ is a structure obtained from $A'$ by omitting the function interpreting $f$, and $A' \models \phi'$ then $A \models \phi$.

On the other hand, if $A$ is a structure for $\mathcal{L}$ and $A \models \phi$, we can extend $A$ to a structure $A'$ by defining $f^{A'}$ so that for every $a_1, \ldots, a_n \in A = A'$, $A \models \psi(y/f(a_1, \ldots, a_n))$. Then $A' \models \phi'$. ($n$ may be 0, $f$ be a constant).
Resolution. An example

\{\{p, q\}, \{r, \neg q\}\}
Resolution. An example

\{ \{p, q\}, \{r, \neg q\} \} \text{ satisfiable } \Rightarrow \text{ the resolvent } \{p, r\} \text{ satisfiable}

\{p, r\} \text{ unsatisfiable } \Rightarrow \{\{p, q\}, \{r, \neg q\}\} \text{ unsatisfiable}
Resolution

Lemma 8.12 If the formula (i.e., set of clauses) $S = \{C_1, C_2\}$ is satisfiable and $C$ is a resolvent of $C_1$ and $C_2$, then $C$ is satisfiable. Any assignment $\mathcal{A}$ satisfying $S$ satisfies $C$.

Proof: $C_1 = \{l\} \cup C'_1$, $C_2 = \{\neg l\} \cup C'_2$, the resolvent is $C = C'_1 \cup C'_2$. As $\mathcal{A}$ is an assignment that satisfies $S = \{C_1, C_2\}$ it cannot be that both $l \in \mathcal{A}$ and $\neg l \in \mathcal{A}$. Say $\neg l \notin \mathcal{A}$. As $\mathcal{A} \models C_2$ and $\neg l \notin \mathcal{A}$, $\mathcal{A} \models C'_2$ and so $\mathcal{A} \models C$. Similarly for $l \notin \mathcal{A}$. 