Tableau Proofs. Preliminaries

A partial order ... a set $S$ with a binary relation ("less than", written $<$, on $S$ which is transitive and irreflexive.

A tree

König’s lemma: If a finitely branching tree $T$ is infinite, it has an infinite path.

Proof: Logic for Applications, p. 9
Tableau Proofs

- start with a signed formula, $F_{\alpha}$, as the root of a tree
- analyze it into its components to see that any analysis leads to a contradiction
Tableau Proofs in Propositional Calculus I

- signed formula $F_\alpha, T_\alpha$
- atomic tableau, $\alpha$-rules, $\beta$-rules
Tableaux I

A finite tableau is a binary tree, labeled with signed formulas called entries, that satisfies the following inductive definition:

1. All atomic tableaux are finite tableaux.

2. If $\tau$ is a finite tableau, $P$ a path on $\tau$, $E$ an entry of $\tau$ occurring on $P$, and $\tau'$ is obtained from $\tau$ by adjoining the unique atomic tableau with root entry $E$ to $\tau$ at the end of the path $P$, then $\tau'$ is also a finite tableau.
Tableaux II

Let $\mathcal{T}$ be a tableau, $P$ a path on $\mathcal{T}$ and $E$ an entry occurring on $P$.

1. $E$ has been reduced on $P$ if all the entries on one path through the atomic tableau with root $E$ occur on $P$.

2. $P$ is contradictory if, for some proposition $\alpha$, $T\alpha$ and $F\alpha$ are both entries on $P$. $P$ is finished if it is contradictory or every entry on $P$ is reduced on $P$.

3. $\mathcal{T}$ is finished if every path through $\mathcal{T}$ is finished.

4. $\mathcal{T}$ is contradictory if every path through $\mathcal{T}$ is contradictory.
Tableau proof

A *tableau proof* of a proposition $\alpha$ is a contradictory tableau with root entry $F\alpha$.

A *tableau refutation* for a proposition $\alpha$ is a contradictory tableau with root entry $T\alpha$.

tableau provable/refutable proposition
Complete systematic tableaux

Let $R$ be a signed proposition. We define the complete systematic tableau (CST) with root entry $R$ by induction.

1. Let $\tau_0$ be the unique tableau with $R$ at its root.

2. Assume that $\tau_m$ has been defined.

3. Let $n$ be the smallest level of $\tau_m$ containing an entry that is unreduced on some noncontradictory path in $\tau_m$ and let $E$ be the leftmost such entry of level $n$.

4. Let $\tau_{m+1}$ be gotten by adjoining the unique atomic tableau with root $E$ to the end of every noncontradictory path of $\tau_m$ on which $E$ is unreduced.
5. The union of the sequence $\tau_m$ is the desired systematic tableau.
Complete systematic tableaux II

1. Every CST is finished.

2. Every CST is finited.

3. Soundness: $\vdash \Rightarrow \models$
   
   If $\alpha$ is tableau provable, it is valid.

4. Completeness: $\models \Rightarrow \vdash$
   
   If $\alpha$ is valid, then $\alpha$ is tableau provable.

\models \text{ validity } \vdash \text{ provability}
Tableaux from premises

\( \Sigma \) - a possibly infinite set of propositions

1. Every atomic tableau is a finite tableau from \( \Sigma \).

2. If \( \tau \) is a finite tableau from \( \Sigma \) and \( \alpha \in \Sigma \), then the tableau formed by putting \( T\alpha \) at the end of every noncontradictory path not containing it is also a finite tableau from \( \Sigma \).

3. If \( \tau \) is a finite tableau from \( \Sigma \), \( P \) a path on \( \tau \), \( E \) an entry of \( \tau \) occurring on \( P \), and \( \tau' \) is obtained from \( \tau \) by adjoining the unique atomic tableau with root entry \( E \) to \( \tau \) at the end of the path \( P \), then \( \tau' \) is also a finite tableau from \( \Sigma \).
Tableaux from premises II

Every CST is finished.

Both soundness and completeness of deduction from premises hold.

Every CST is finite ... ?

If a CST from $\Sigma$ is a proof, it is finite

Compactness: $\alpha$ is a consequence of $\Sigma$ iff $\alpha$ is a consequence of some finite subset of $\Sigma$. 
Tableaux in predicate calculus

\[ T(\exists x)\phi(x) \]
\[ \mathcal{L}_C \] - adding on a set of constants \( c_0, c_1, \ldots \)
\[ \gamma \] -rules, \( \delta \) -rules

Tableaux in predicate calculus

1. All atomic tableaux are tableaux. The requirement that \( c \) be new in (7b) and (8a) means that \( c \) is one of the constants \( c_i \) added on to \( \mathcal{L} \) to get \( \mathcal{L}_C \) (which therefore does not appear in \( \phi \)).

2. ... adjoining an atomic tableau with root entry \( E \) to \( \tau \) at the end of the path \( P \): \( c \) did not appear in any entries on \( P \).
Tableau is finished:

\[ T(\exists x)\phi(x), F(\forall x)\phi(x) \rightarrow \text{a witness } c \]

\[ T(\forall x)\phi(x), F(\exists x)\phi(x) \]

\[ \rightarrow \]

\[ \text{add } T\phi(t) (F \phi(t)) \text{ for any ground term } t \ldots ? \]
Tableau is finished II

$P$ a path in $\tau$, $E$ an entry on $P$ and $W$ the $i^{th}$ occurrence of $E$ on $P$.

$w$ is reduced

1. ...

2. $E$ is of the form $T(\forall x)\phi(x)$ or $F(\exists x)\phi(x)$, $T\phi(t_i)$ or $F\phi(t_i)$, respectively, is an entry on $P$ and there is an $(i+1)^{st}$ occurrence of $E$ on $P$.

Note: Signed sentences like $T(\forall x)\phi(x)$ must be instantiated for each term $t_i$ in our language before we can say that we have finished with them.
3. finished ...