Proof Call a formula independent of $\sigma_{\mathscr{I}_A}$ if its value does not depend on $\sigma_{\mathscr{I}_A}$. Let $A' = \forall x A_1(x)$ be a (not necessarily proper) subformula of A, where A' is *not* contained in the scope of any other quantifier. Then $v_{\sigma_{\mathscr{I}_A}}(A') = T$ iff $v_{\sigma_{\mathscr{I}_A}[x \leftarrow d]}(A_1)$ for all $d \in D$. But x is the only free variable in A_1, so A_1 is independent of $\sigma_{\mathscr{I}_A}$ since what is assigned to x is replaced by the assignment $[x \leftarrow d]$. A similar results holds for an existential formula $\exists x A_1(x)$.

The theorem can now be proved by induction on the depth of the quantifiers and by structural induction, using the fact that a formula constructed using Boolean operators on independent formulas is also independent. ■

By the theorem, if A is a closed formula we can use the notation $v_{\mathscr{I}}(A)$ without mentioning an assignment.

Example 7.21 Let us check the truth values of the formula $A = \forall x p(a, x)$ under the interpretations given in Example 7.17:

- $v_{\mathscr{I}_1}(A) = T$: For *all* $n \in \mathcal{N}$, $0 \le n$.
- $v_{\mathscr{I}_2}(A) = F$: It is not true that for *all* $n \in \mathcal{N}$, $1 \le n$. If $n = 0$ then $1 \not\le 0$.
- $v_{\mathscr{I}_3}(A) = F$: There is no smallest integer.
- $v_{\mathscr{I}_4}(A) = T$: By definition, the null string is a substring of every string.

The proof of the following theorem is left as an exercise.

Theorem 7.22 *Let* $A' = A(x_1, \dots, x_n)$ *be a (non-closed) formula with free variables* $x_1, \dots, x_n$*, and let* $\mathscr{I}$ *be an interpretation. Then*:

- $v_{\sigma_{\mathscr{I}_A}}(A') = T$ *for* some *assignment* $\sigma_{\mathscr{I}_A}$ *iff* $v_{\mathscr{I}}(\exists x_1 \cdots \exists x_n A') = T$.
- $v_{\sigma_{\mathscr{I}_A}}(A') = T$ *for* all *assignments* $\sigma_{\mathscr{I}_A}$ *iff* $v_{\mathscr{I}}(\forall x_1 \cdots \forall x_n A') = T$.

7.3.2 Validity and Satisfiability

Definition 7.23 Let A be a closed formula of first-order logic.

- A is *true* in $\mathscr{I}$ or $\mathscr{I}$ is a *model* for A iff $v_{\mathscr{I}}(A) = T$. Notation: $\mathscr{I} \models A$.
- A is *valid* if for *all* interpretations $\mathscr{I}$, $\mathscr{I} \models A$. Notation: $\models A$.
- A is *satisfiable* if for some interpretation $\mathscr{I}$, $\mathscr{I} \models A$.
- A is *unsatisfiable* if it is not satisfiable.
- A is *falsifiable* if it is not valid. ■

Example 7.24 The closed formula $\forall x p(x) \rightarrow p(a)$ is valid. If it were not, there would be an interpretation $\mathscr{I} = (D, \{R\}, \{d\})$ such that $v_{\mathscr{I}}(\forall x p(x)) = T$ and $v_{\mathscr{I}}(p(a)) = F$. By Theorem 7.22, $v_{\sigma_{\mathscr{I}}}(p(x)) = T$ for all assignments $\sigma_{\mathscr{I}}$, in particular for the assignment $\sigma'_{\mathscr{I}}$ that assigns d to x. But $p(a)$ is closed, so $v_{\sigma'_{\mathscr{I}}}(p(a)) = v_{\mathscr{I}}(p(a)) = F$, a contradiction. ■

Example 7.25

- $\forall x \forall y (p(x, y) \rightarrow p(y, x))$
  The formula is satisfiable in an interpretation where p is assigned a symmetric relation like $=$. It is not valid because the formula is falsified in an interpretation that assigns to p a non-symmetric relation like $<$.
- $\forall x \exists y p(x, y)$
  The formula is satisfiable in an interpretation where p is assigned a relation that is a total function, for example, $(x, y) \in R$ iff $y = x + 1$ for $x, y \in \mathscr{Z}$. The formula is falsified if the domain is changed to the negative numbers because there is no negative number y such that $y = -1 + 1$.
- $\exists x \exists y (p(x) \wedge \neg p(y))$
  This formula is satisfiable only in a domain with at least two elements.
- $\forall x p(a, x)$
  This expresses the existence of an element with special properties. For example, if p is interpreted by the relation $\le$ on the domain $\mathcal{N}$, then the formula is true for $a = 0$. If we change the domain to $\mathscr{Z}$ the formula is false for the same assignment of $\le$ to p.
- $\forall x (p(x) \wedge q(x)) \leftrightarrow (\forall x p(x) \wedge \forall x q(x))$
  The formula is valid. We prove the forward direction and leave the converse as an exercise. Let $\mathscr{I} = (D, \{R_1, R_2\}, \{\})$ be an arbitrary interpretation. By Theorem 7.22, $v_{\sigma_{\mathscr{I}}}(p(x) \wedge q(x)) = T$ for all assignments $\sigma_{\mathscr{I}}$, and by the inductive definition of an interpretation, $v_{\sigma_{\mathscr{I}}}(p(x)) = T$ and $v_{\sigma_{\mathscr{I}}}(q(x)) = T$ for all assignments $\sigma_{\mathscr{I}}$. Again by Theorem 7.22, $v_{\mathscr{I}}(\forall x p(x)) = T$ and $v_{\mathscr{I}}(\forall x q(x)) = T$, and by the definition of an interpretation $v_{\mathscr{I}}(\forall x p(x) \wedge \forall x q(x)) = T$.
  Show that $\forall$ does not distribute over disjunction by constructing a falsifying interpretation for $\forall x (p(x) \vee q(x)) \leftrightarrow (\forall x p(x) \vee \forall x q(x))$.
- $\forall x (p(x) \rightarrow q(x)) \rightarrow (\forall x p(x) \rightarrow \forall x q(x))$
  We leave it as an exercise to show that this is a valid formula, but its converse $(\forall x p(x) \rightarrow \forall x q(x)) \rightarrow \forall x (p(x) \rightarrow q(x))$ is not. ■

7.3.3 An Interpretation for a Set of Formulas

In propositional logic, the concept of interpretation and the definition of properties such as satisfiability can be extended to sets of formulas (Sect. 2.2.4). The same holds for first-order logic.

Definition 7.26 Let $U = \{A_1, \dots\}$ be a set of formulas where $\{p_1, \dots, p_m\}$ are all the predicates appearing in all $A_i \in S$ and $\{a_1, \dots, a_k\}$ are all the constants appearing in all $A_i \in S$. An *interpretation* $\mathscr{I}_U$ for S is a triple:

$$(D, \{R_1, \dots, R_m\}, \{d_1, \dots, d_k\}),$$