How NOT to Characterize Planar-emulable Graphs

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* Joint work with Markus Chimani, Univ. Jena,
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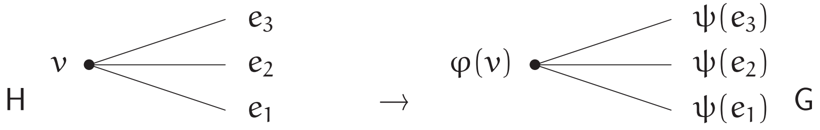
1 Definitions

Motivation: Exploring the two graphs locally, we cannot see any difference...

A graph H is a *cover* of a graph G if there exists a pair of onto mappings

(a *projection*) $\varphi : V(H) \to V(G), \quad \psi : E(H) \to E(G)$

such that ψ maps the edges incident with each vertex v in H bijectively onto the edges incident with $\varphi(v)$ in G.



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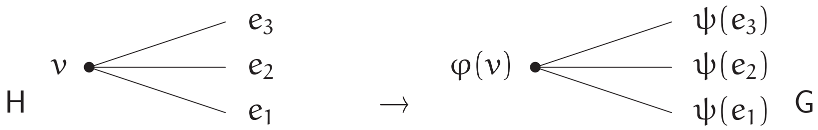


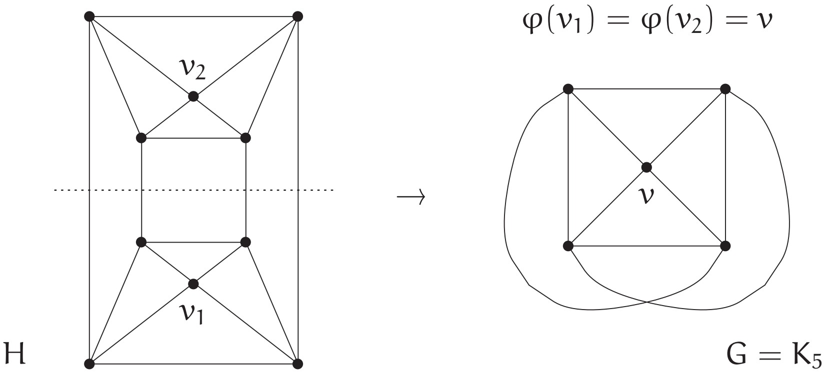
Remark. The edge $\psi(uv)$ has always ends $\varphi(u), \varphi(v)$, and hence only

$\varphi : V(H) \to V(G)$, the *vertex projection*,

is enough to be specified for simple graphs.

Planar covers

- We speak about a *planar cover* if H is a finite planar graph.



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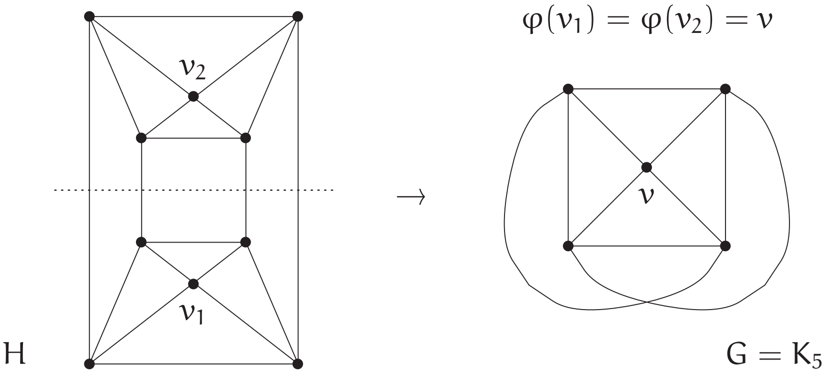
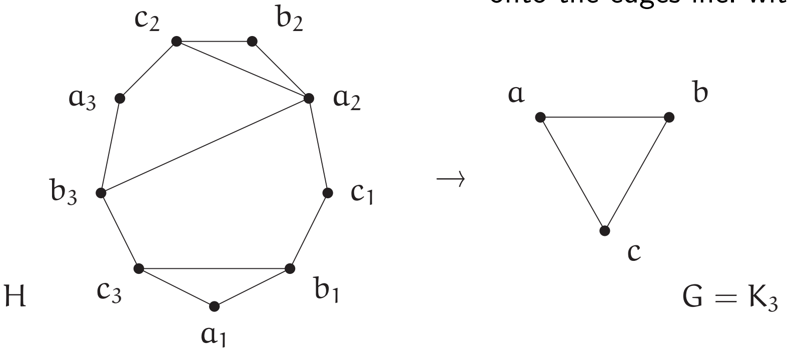


- Graph embedded in the *projective plane* has a double planar cover, *via the universal covering map from the sphere onto the proj. plane.*

Planar emulators

- $\varphi : V(H) \to V(G)$, an *emulator* vs. a cover:

  ... map the edges inc. with v in H surjectively onto the edges inc. with $\varphi(v)$ in G.



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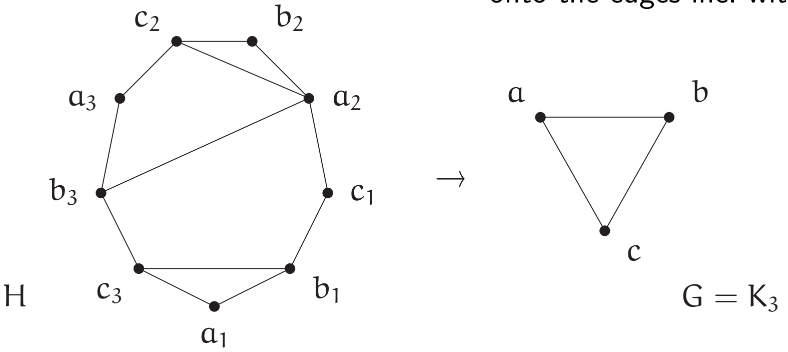

- Can a planar emulator be "more than" a planar cover?
- Not many remarkable results until 2008… Interesting at all?

2 Fellows' planar emulator conjecture

Conjecture 1 (Negami, 1988)

A connected graph has a finite planar cover

$\iff$ *it embeds in the projective plane.*

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Fact. A planar cover is also a planar emulator.

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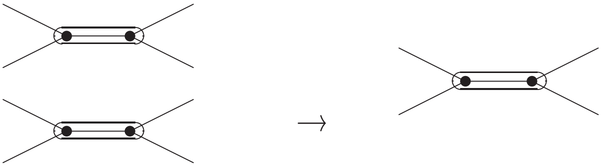
3 Some useful properties

- If G has a planar cover, then so does every minor of G.

H  G

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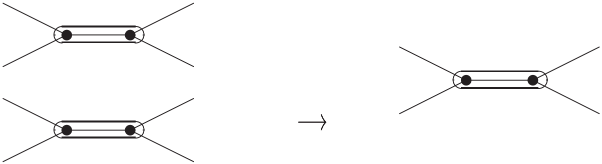
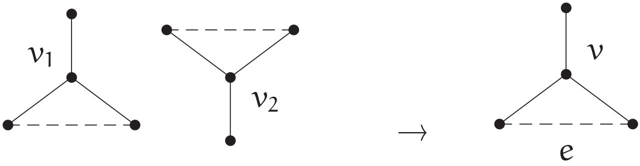
H  G

Consider e between two neighbours of a cubic vertex.
If $G - e$ has a planar cover, then so does G.

H  G

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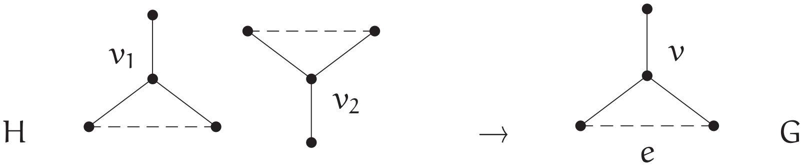

- Therefore, if G has a planar cover, and G' is obtained from G by *YΔ-transformations*, then G' has a planar cover, too.

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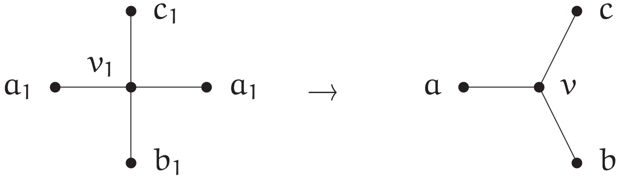

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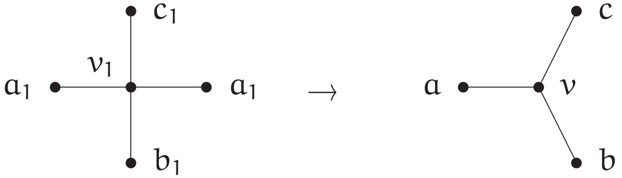

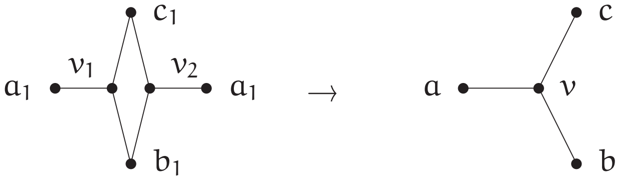

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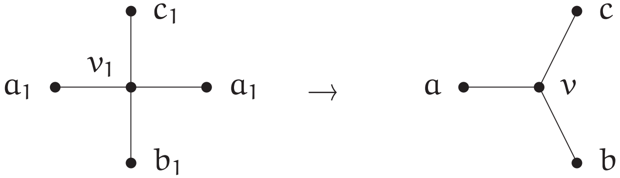

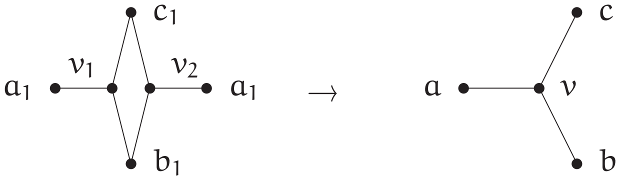

Therefore, if G has a planar emulator, and G' is obtained from G by *YΔ-transformations*, then G' has a planar emulator, too.

4 Approaching the conjectures

A connected graph has a finite planar cover / emulator if, and only if, it embeds in the projective plane.

We recall the above basic properties. . .

- Assume a *projective graph* G. Then G has a double planar cover / emulator.

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- Conversely, assume connected G is not projective.
  Then G contains some F of the *forbidden minors* for the projective plane. We just have to show that this connected F has no finite planar cover / emulator.

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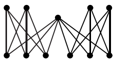
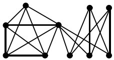
- Conversely, assume connected G is not projective.
  Then G contains some F of the *forbidden minors* for the projective plane. We just have to show that this connected F has no finite planar cover / emulator.
- Furthermore, it is enough to consider only those F which are YΔ-transforms of some forbidden minor in G.

  [Archdeacon]



$K_{3,3} \cdot K_{3,3}$



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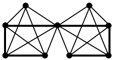

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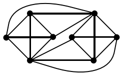
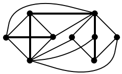
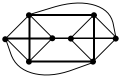


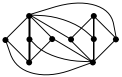
$\mathcal{B}_3$



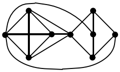
$\mathcal{C}_2$



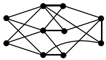
$\mathcal{C}_7$



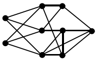
$\mathcal{D}_1$



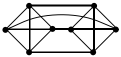
$\mathcal{D}_4$



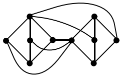
$\mathcal{D}_9$



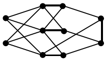
$\mathcal{D}_{12}$



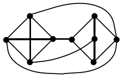
$\mathcal{D}_{17}$



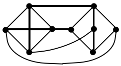
$\mathcal{E}_6$



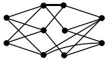
$\mathcal{E}_{11}$



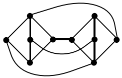
$\mathcal{E}_{19}$



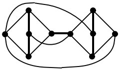
$\mathcal{E}_{20}$



$\mathcal{E}_{27}$



$\mathcal{F}_4$



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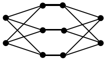
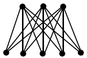
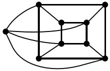


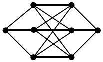
$\mathcal{G}_1$



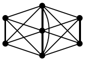
$K_{3,5}$



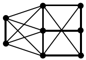
$K_{4,5} - 4K_2$



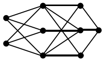
$K_{4,4} - e$



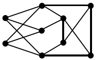
$K_7 - C_4$



$\mathcal{D}_3$



$\mathcal{E}_5$



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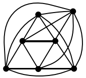
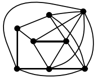


$K_{1,2,2,2}$



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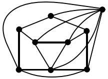

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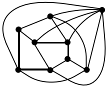
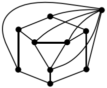
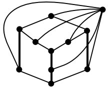


$\mathcal{C}_4$



$\mathcal{D}_2$



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Known results (and a big surprise)

Long-term development around Negami's conjecture led to...

Theorem 3 (A+N+F+H, since 1998)
If $K_{1,2,2,2}$ *had no finite planar cover, then Negami's c. would be proved.*

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Fact. The graph $K_{4,5} - 4K_2$ has no finite planar cover.

Theorem 4 (Rieck and Yamashita 2008)
The graphs $K_{1,2,2,2}$ and $K_{4,5} - 4K_2$ do have finite planar emulators!!!

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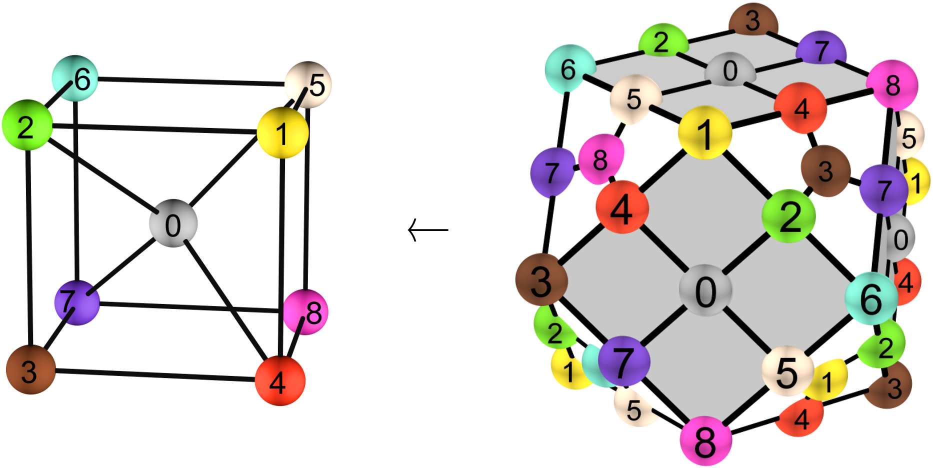
Theorem 4 (Rieck and Yamashita 2008)
The graphs $K_{1,2,2,2}$ *and* $K_{4,5} - 4K_2$ *do have finite planar emulators!!!*

- Now we know that the class of graphs having finite *planar emulators*
  - is different from the class of graphs having finite *planar covers*,
  - and different from the class of *projective planar* graphs, too.
- So, let us study this class...!

$K_{4,5} - 4K_2$



(A picture by Yamashita.)

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  (based on the notion of *internal 4-connectivity*)

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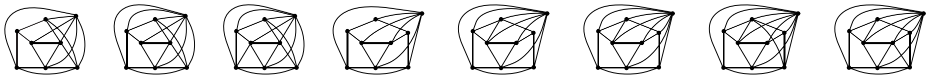
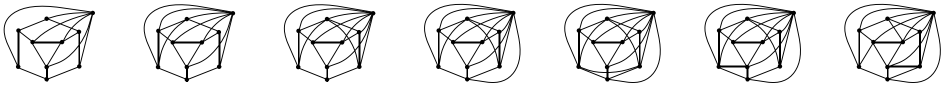
  (based on the notion of *internal 4-connectivity*)

Theorem 5 (Thomas and PH 1999, 2004)

If a connected graph G *had a finite planar cover but no projective embedding, then* G *would be a planar expansion of* $K_{1,2,2,2}$ *or one of:*



$\mathcal{B}_7$ $\mathcal{B}_7'$ $\mathcal{B}_7''$ $\mathcal{C}_3$ $\mathcal{C}_3'$ $\mathcal{C}_3''$ $\mathcal{C}_3^\bullet$ $\mathcal{C}_3^\circ$



$\mathcal{D}_2$ $\mathcal{D}_2'$ $\mathcal{D}_2''$ $\mathcal{D}_2'''$ $\mathcal{D}_2^\bullet$ $\mathcal{D}_2^\circ$ $\mathcal{D}_2^\star$

Computer search of internally 4-connected exceptions

- Starting from the whole $K_{1,2,2,2}$ family, or from $K_{4,5} - 4K_2$,

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  The graph K_7-C_4 also has a planar emulator! [Klusáček, 2011]

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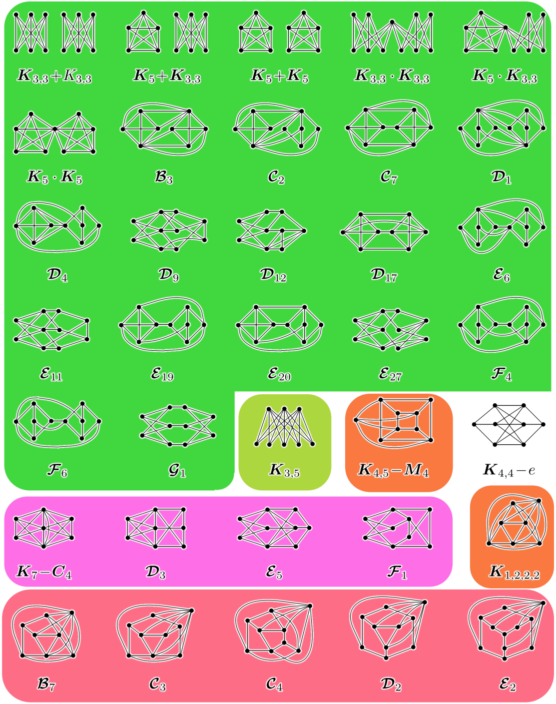
- and new $\mathbf{K_7-C_4}$ and its whole family! by [Klusáček, 2011].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$\mathcal{F}_6$
$\mathcal{G}_1$
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$K_{4,5}-M_4$
$K_{4,4}-e$
K_7-C_4
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$\mathcal{E}_5$
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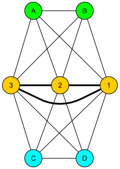
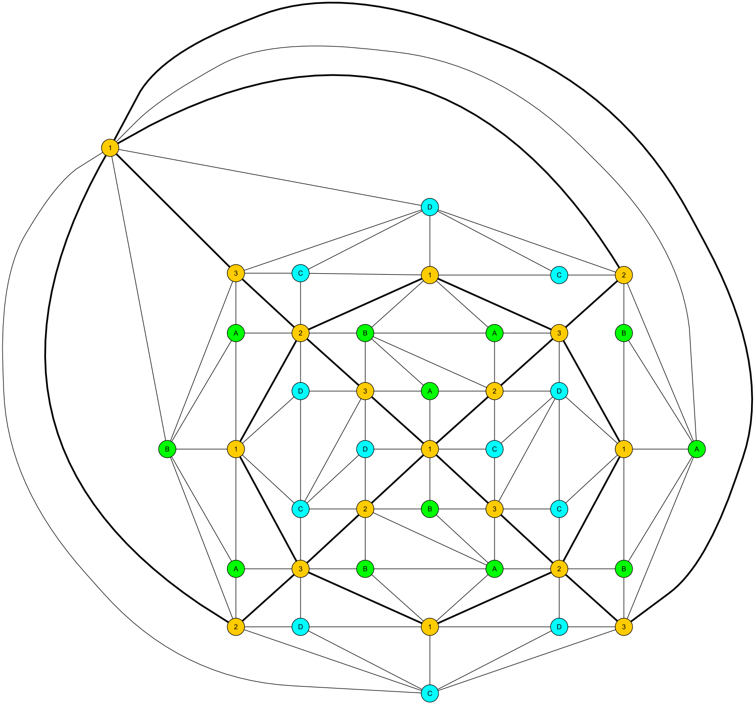
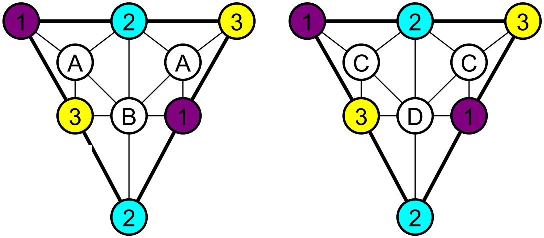
K7-C4







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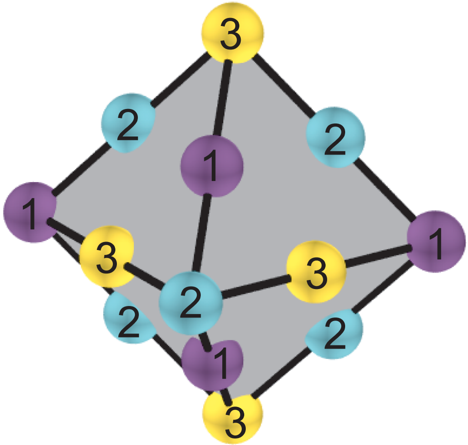

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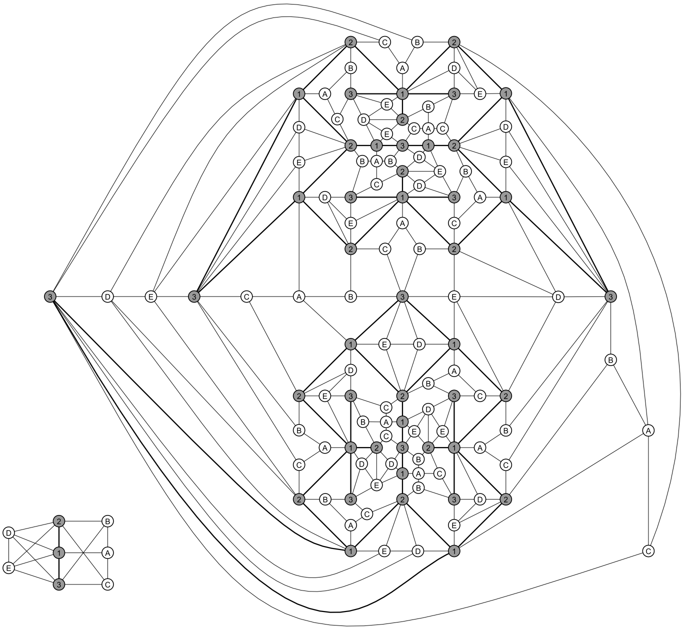

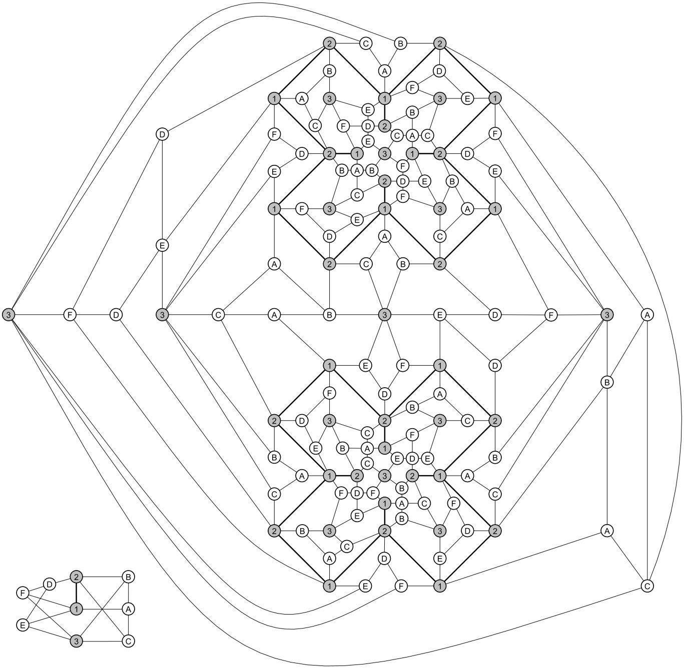

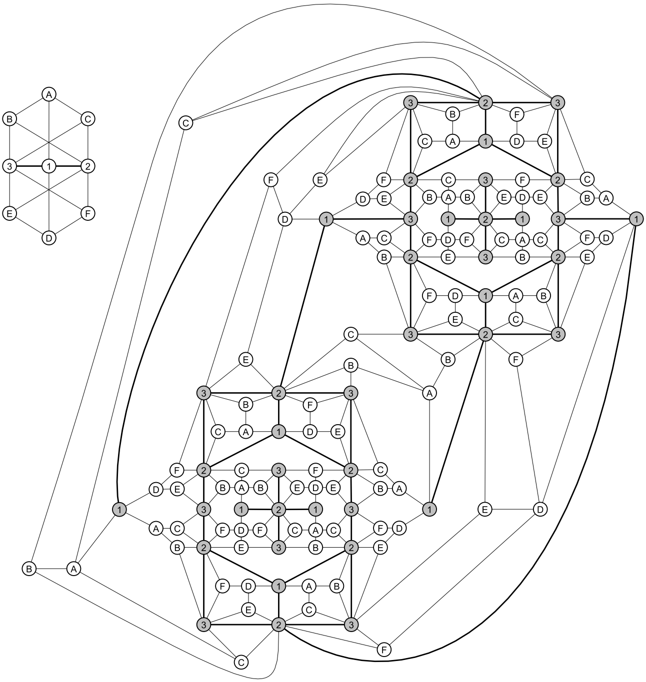

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- And, of course, do not forget about Negami's conjecture!