ATCAGC 2009, Finse, Feb 19-21

21 Years of Negami's Planar Cover Conjecture

(with some very recent development)

Petr Hliněný

Faculty of Informatics
Masaryk University

Botanická 68a, 602 00 Brno, Czech Rep.

e-mail: `hlineny@fi.muni.cz`
`http://www.fi.muni.cz/~hlineny`

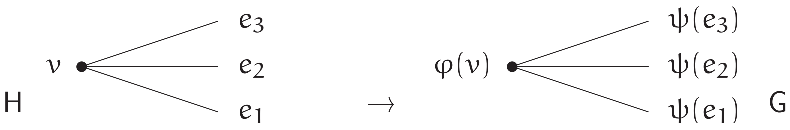
1 Definition

Motivation: Exploring the two graphs locally, we cannot see any difference...

A graph H is a *cover* of a graph G if there exists a pair of onto mappings

(a *projection*) $\varphi : V(H) \rightarrow V(G), \quad \psi : E(H) \rightarrow E(G)$

such that ψ maps the edges incident with each vertex v in H
bijectively onto the edges incident with $\varphi(v)$ in G.



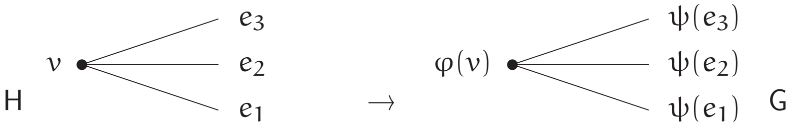
1 Definition

Motivation: Exploring the two graphs locally, we cannot see any difference...

A graph H is a *cover* of a graph G if there exists a pair of onto mappings

(a *projection*) $\varphi : V(H) \to V(G)$, $\psi : E(H) \to E(G)$

such that ψ maps the edges incident with each vertex v in H
bijectively onto the edges incident with $\varphi(v)$ in G.



Remark. The edge $\psi(uv)$ has always ends $\varphi(u), \varphi(v)$, and hence only

$\varphi : V(H) \to V(G)$, the *vertex projection*,

is enough to be specified for simple graphs.

2 Useful basic properties

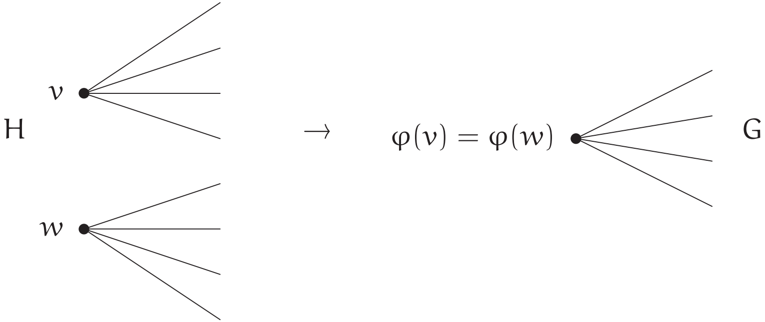
A graph H *is a* cover *of a graph* G *if there exists a pair of onto mappings*

(a projection*)* $\varphi : V(H) \to V(G), \qquad \psi : E(H) \to E(G)$

s.t. ψ *maps the edges inc. with* v *in* H *bijectively onto the edges inc. with* $\varphi(v)$ *in* G.

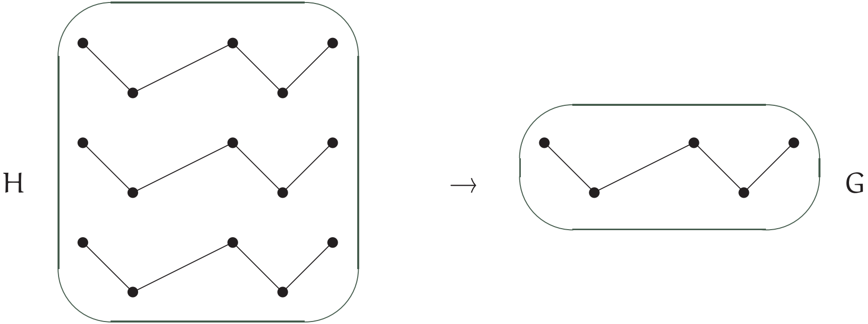
Degree preservation

- $d_H(v) = d_G(\varphi(v))$ for each vertex $v \in V(H)$.



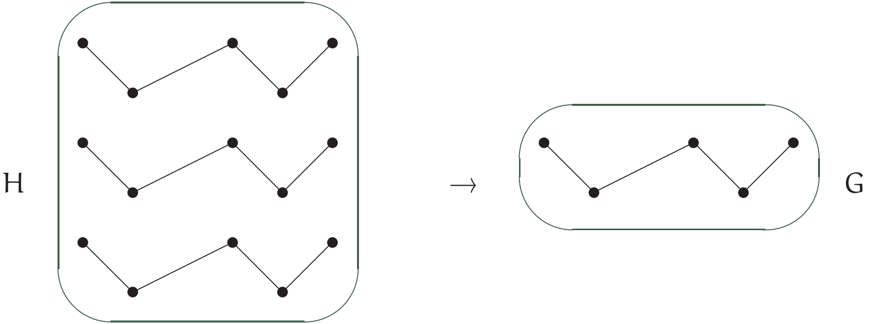
Lifting a path

If G' is a subgraph of G, then the subgraph H' with the vertex set $\varphi^{-1}(V(G'))$ and the edge set $\psi^{-1}(E(G'))$ is called a *lifting of G' into H*.



Lifting a path

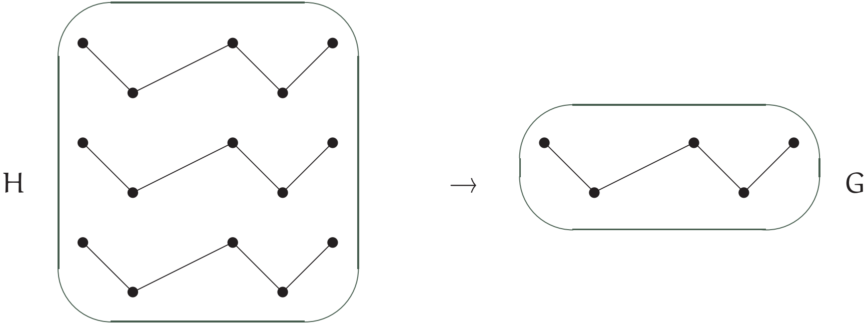
If G' is a subgraph of G, then the subgraph H' with the vertex set $\varphi^{-1}(V(G'))$ and the edge set $\psi^{-1}(E(G'))$ is called a *lifting of G' into H*.



- Lifting of a path from G into H consists of a collection of disjoint isomorphic paths.

Lifting a path

If G' is a subgraph of G, then the subgraph H' with the vertex set $\varphi^{-1}(V(G'))$ and the edge set $\psi^{-1}(E(G'))$ is called a *lifting of G' into H*.



- Lifting of a path from G into H consists of a collection of disjoint isomorphic paths.

- Consequently, if G is connected, then $|\varphi^{-1}(v)| = k$ is a constant.

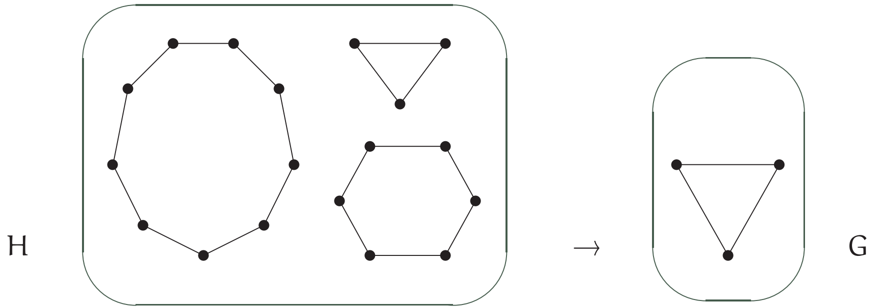
  We then speak about a *k-fold cover*.

- Lifting of a tree into H consists of a collection of disjoint isom. trees.

- Lifting of a tree into H consists of a collection of disjoint isom. trees.

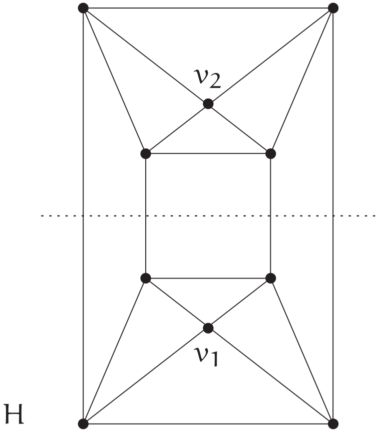
Lifting a cycle

- Lifting of a cycle C_ℓ of G into H consists of a collection of disjoint cycles whose lengths are divisible by ℓ.



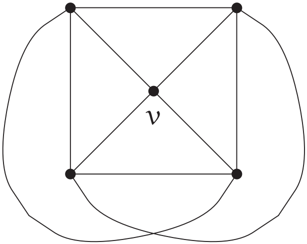
Planar cover

We speak about a *planar cover* if H is a finite planar graph.



$\rightarrow$

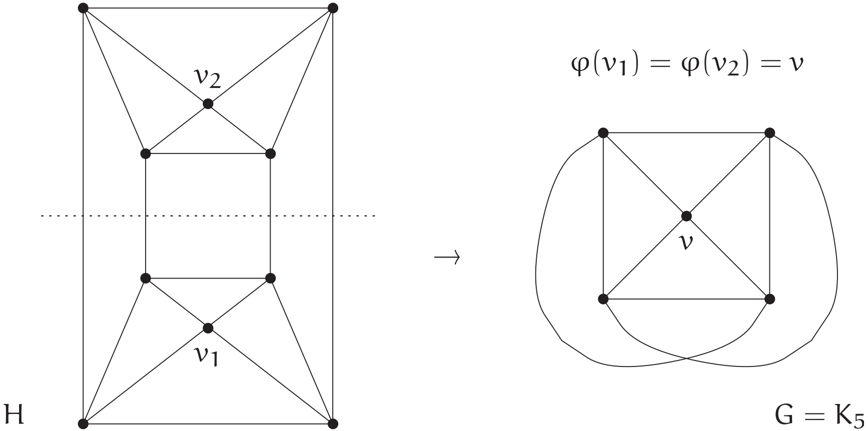
$\varphi(v_1) = \varphi(v_2) = v$



$G = K_5$

Planar cover

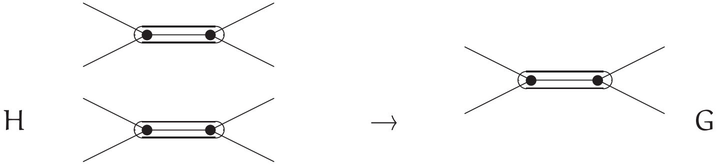
We speak about a *planar cover* if H is a finite planar graph.



- Graph embedded in the *projective plane* has a double planar cover, via the universal covering map from the sphere onto the projective plane.

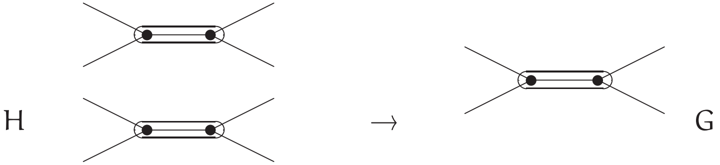
Cover preservation

- If G has a planar cover, then so does every minor of G.

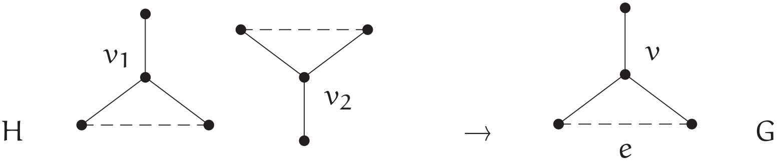


Cover preservation

- If G has a planar cover, then so does every minor of G.

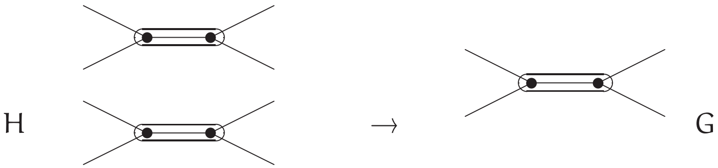


Consider e between two neighbours of a cubic vertex.
If $G - e$ has a planar cover, then so does G.

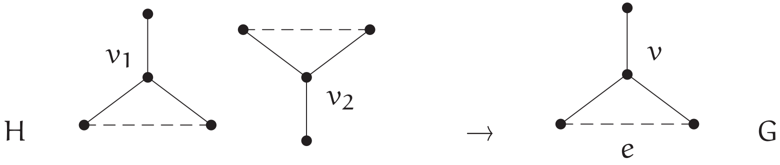


Cover preservation

- If G has a planar cover, then so does every minor of G.



Consider e between two neighbours of a cubic vertex.
If $G - e$ has a planar cover, then so does G.



- Therefore, if G has a planar cover, and G' is obtained from G by $Y\Delta$-transformations, then G' has a planar cover, too.

3 Interest in planar covers

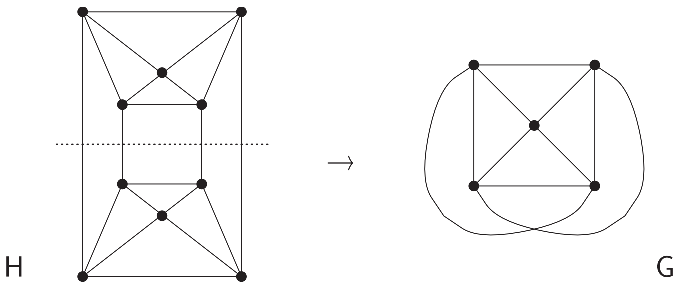
- Raised by Negami [1986] in relation to enumeration of projective embeddings of 3-connected graphs.

3 Interest in planar covers

- Raised by Negami [1986] in relation to enumeration of projective embeddings of 3-connected graphs.
- Independently, *planar emulators* considered by Fellows in his CS-oriented thesis [1985] ("embedding graphs in graphs").

3 Interest in planar covers

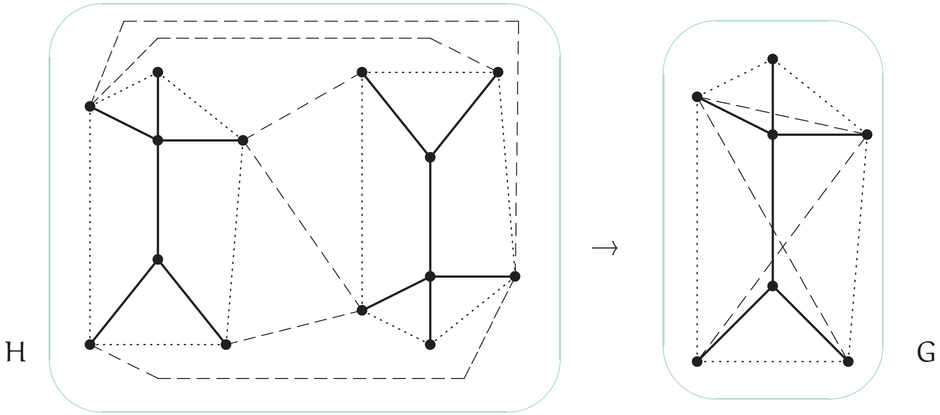
- Raised by Negami [1986] in relation to enumeration of projective embeddings of 3-connected graphs.
- Independently, *planar emulators* considered by Fellows in his CS-oriented thesis [1985] ("embedding graphs in graphs").



Theorem 1 (Negami, 1986**)** *A connected graph has a double planar cover if and only if it embeds in the projective plane.*

Theorem 1 (Negami, 1986) *A connected graph has a double planar cover if and only if it embeds in the projective plane.*

Proof sketch. If a *3-connected planar* graph H is a double cover of a graph G, then G embeds with at most one crosscap:



– this is a purely combinatorial argument. . .

Negami's planar cover conjecture

A cover $\varphi : V(H) \to V(G)$ is *regular*

if there is a subgroup $A \subseteq Aut(H)$ such that $\varphi(u) = \varphi(v)$
for $u, v \in V(H)$ if, and only if $\tau(u) = v$ for some $\tau \in A$.

Theorem 2 (Negami, 1988) *A connected graph has a finite regular planar cover if and only if it embeds in the projective plane.*

Negami's planar cover conjecture

A cover $\varphi : V(H) \to V(G)$ is *regular*

if there is a subgroup $A \subseteq Aut(H)$ such that $\varphi(u) = \varphi(v)$
for $u, v \in V(H)$ if, and only if $\tau(u) = v$ for some $\tau \in A$.

Theorem 2 (Negami, 1988) *A connected graph has a finite regular planar cover if and only if it embeds in the projective plane.*

And now an immediate generalization reads...

Conjecture 3 (Negami, 1988)

A connected graph has a finite ~~regular~~ planar cover
if and only if
it embeds in the projective plane.

4 Approaching Negami's conjecture

A connected graph has a finite planar cover if and only if it embeds in the projective plane.

We recall the above basic properties of covers. . .

- Assume a projective graph G. Then G has a double planar cover.

4 Approaching Negami's conjecture

A connected graph has a finite planar cover if and only if it embeds in the projective plane.

We recall the above basic properties of covers...

- Assume a projective graph G. Then G has a double planar cover.
- Conversely, assume connected G is not projective.
  Then G contains some F of the *forbidden minors* for the projective plane.
  We just have to show that this F has no finite planar cover.

4 Approaching Negami's conjecture

A connected graph has a finite planar cover if and only if it embeds in the projective plane.

We recall the above basic properties of covers...

- Assume a projective graph G. Then G has a double planar cover.
- Conversely, assume connected G is not projective.
  Then G contains some F of the *forbidden minors* for the projective plane.
  We just have to show that this F has no finite planar cover.
- Furthermore, it is enough to consider only those F which are $Y\Delta$-transforms of some forbidden minor in G.

4 Approaching Negami's conjecture

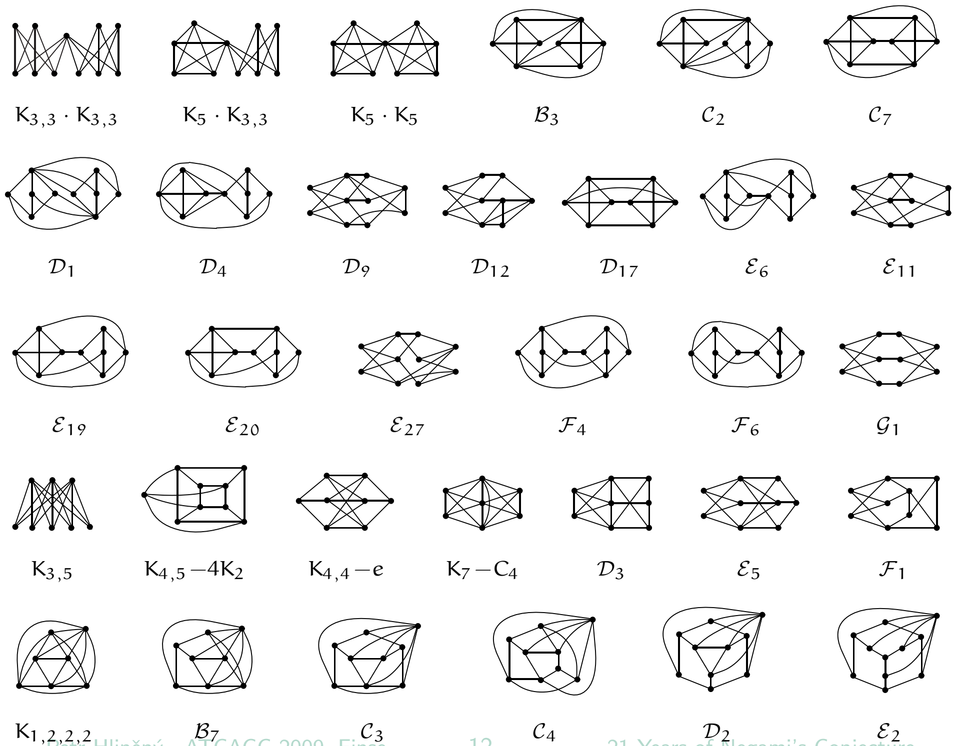
A connected graph has a finite planar cover if and only if it embeds in the projective plane.

We recall the above basic properties of covers...

- Assume a projective graph G. Then G has a double planar cover.
- Conversely, assume connected G is not projective.
  Then G contains some F of the *forbidden minors* for the projective plane.
  We just have to show that this F has no finite planar cover.
- Furthermore, it is enough to consider only those F which are $Y\Delta$-transforms of some forbidden minor in G.

Does this sound like a piece of cake now?

Unfortunately, the difficulties are just coming...


$K_{3,3} \cdot K_{3,3}$
$K_5 \cdot K_{3,3}$
$K_5 \cdot K_5$
$\mathcal{B}_3$
$\mathcal{C}_2$
$\mathcal{C}_7$
$\mathcal{D}_1$
$\mathcal{D}_4$
$\mathcal{D}_9$
$\mathcal{D}_{12}$
$\mathcal{D}_{17}$
$\mathcal{E}_6$
$\mathcal{E}_{11}$
$\mathcal{E}_{19}$
$\mathcal{E}_{20}$
$\mathcal{E}_{27}$
$\mathcal{F}_4$
$\mathcal{F}_6$
$\mathcal{G}_1$
$K_{3,5}$
$K_{4,5} - 4K_2$
$K_{4,4} - e$
$K_7 - C_4$
$\mathcal{D}_3$
$\mathcal{E}_5$
$\mathcal{F}_1$
$K_{1,2,2,2}$
$\mathcal{B}_7$
$\mathcal{C}_3$
$\mathcal{C}_4$
$\mathcal{D}_2$
$\mathcal{E}_2$

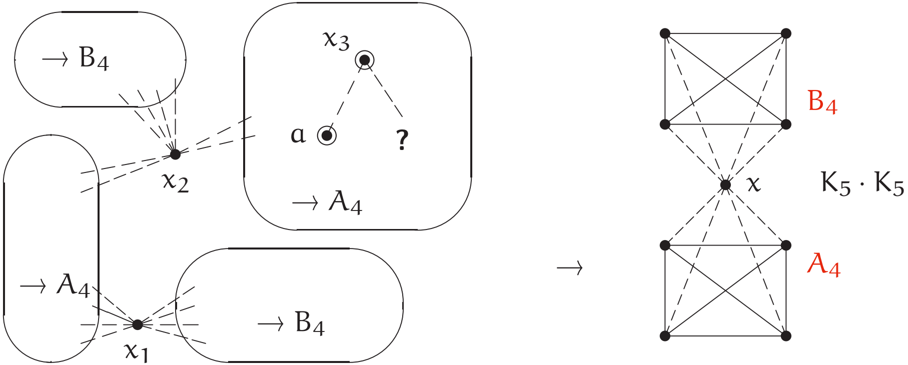
Disjoint k-graphs

Theorem 4 (Negami / Archdeacon, 1988)
Neither of the graphs $K_{3,3} \cdot K_{3,3}$, $K_5 \cdot K_{3,3}$, $K_5 \cdot K_5$, $\mathcal{B}_3$, $\mathcal{C}_2$, $\mathcal{C}_7$, $\mathcal{D}_1$, $\mathcal{D}_4$, $\mathcal{D}_9$, $\mathcal{D}_{12}$, $\mathcal{D}_{17}$, $\mathcal{E}_6$, $\mathcal{E}_{11}$, $\mathcal{E}_{19}$, $\mathcal{E}_{20}$, $\mathcal{E}_{27}$, $\mathcal{F}_4$, $\mathcal{F}_6$, $\mathcal{G}_1$ *have a finite planar cover.*

Disjoint k-graphs

Theorem 4 (Negami / Archdeacon, 1988)
Neither of the graphs $K_{3,3} \cdot K_{3,3}$, $K_5 \cdot K_{3,3}$, $K_5 \cdot K_5$, $\mathcal{B}_3$, $\mathcal{C}_2$, $\mathcal{C}_7$, $\mathcal{D}_1$, $\mathcal{D}_4$, $\mathcal{D}_9$, $\mathcal{D}_{12}$, $\mathcal{D}_{17}$, $\mathcal{E}_6$, $\mathcal{E}_{11}$, $\mathcal{E}_{19}$, $\mathcal{E}_{20}$, $\mathcal{E}_{27}$, $\mathcal{F}_4$, $\mathcal{F}_6$, $\mathcal{G}_1$ *have a finite planar cover.*

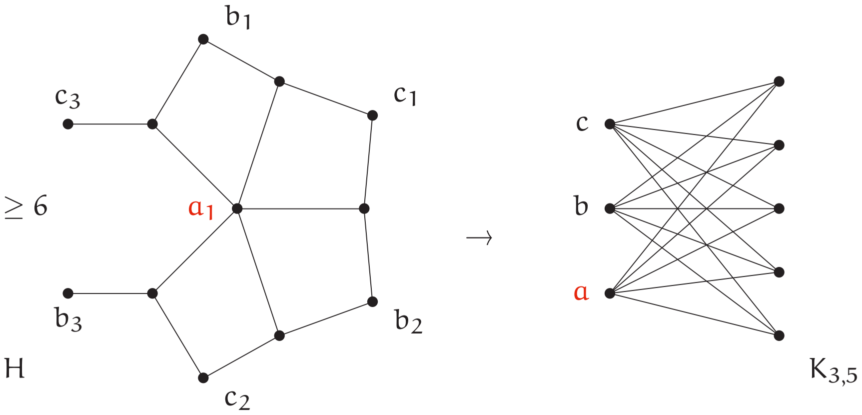
Proof sketch. We choose the $K_5 \cdot K_5$ case for an illustration. . .



Discharging technique

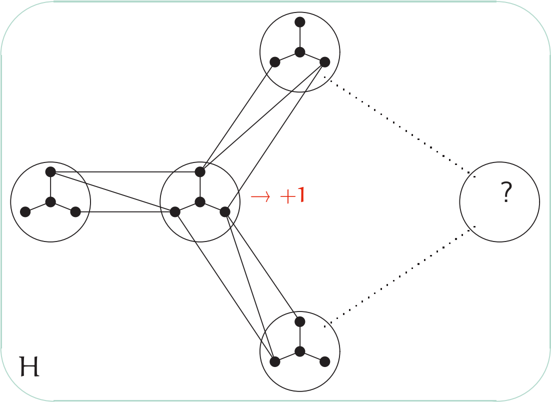
Theorem 5 (?? 1988, 1993) *The graph* $K_{3,5}$ *has no finite planar cover.*

Proof sketch. Assuming H is a finite planar cover of $K_{3,5}$, we shall derive a contradiction to Euler's formula (or, easy *discharging*)...

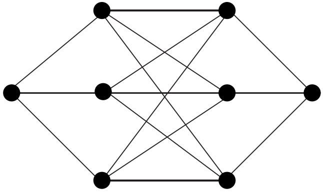


Theorem 6 (PH, 1998) *The graph* $K_{4,4}-e$ *has no finite planar cover.*

Brief idea. Form "thick" metavertices from the 3-stars in a supposed cover H.



→

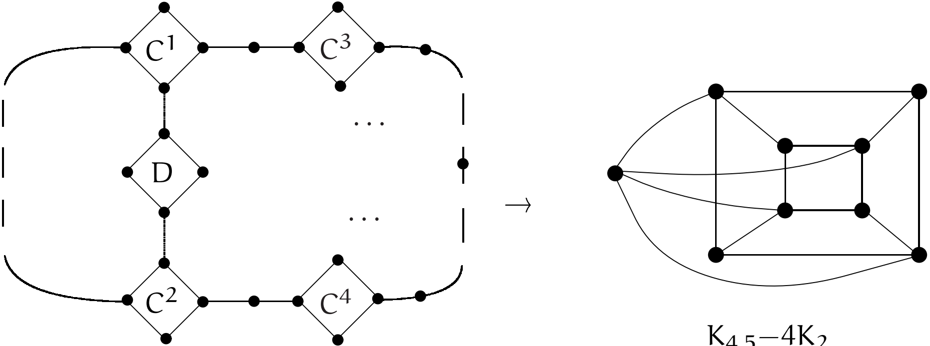


This metagraph is planar bipartite, and so it has a degree-3 vertex. Here we apply *discharging* to get a contradiction again...

"Necklace" argument

Theorem 7 (Archdeacon 1988, 2002) (indep. Thomas and PH 1999)
The graphs $K_7 - C_4$ *and* $K_{4,5} - 4K_2$ *have no finite planar covers.*

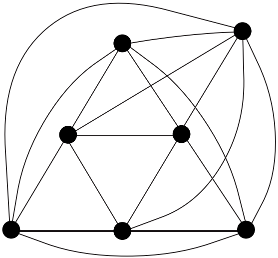
Brief idea. Find the shortest "necklace" (a *reduced semi-cover*), and shorten it further...



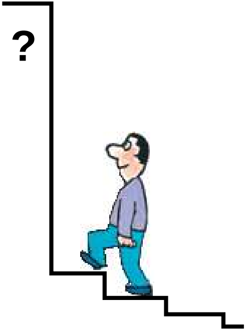
$K_{4,5} - 4K_2$

At the end, we reduce the (length-2) necklace to a projective embedding!

5 The bad guys: $K_{1,2,2,2}$ and its relatives



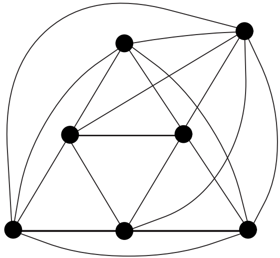
$K_{1,2,2,2}$



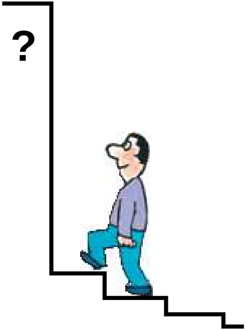
Fact. (since 1995/8)
If $K_{1,2,2,2}$ had no finite planar cover, then Negami's conjecture would be proved.

However;

5 The bad guys: $K_{1,2,2,2}$ and its relatives



$K_{1,2,2,2}$

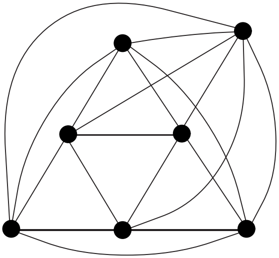


Fact. (since 1995/8)
If $K_{1,2,2,2}$ had no finite planar cover, then Negami's conjecture would be proved.

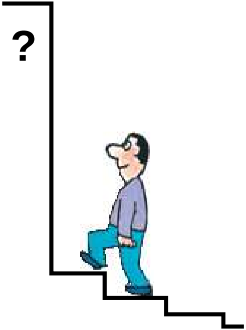
However;

- We are being stuck at this last step for more than 10 years now!

5 The bad guys: $K_{1,2,2,2}$ and its relatives



$K_{1,2,2,2}$

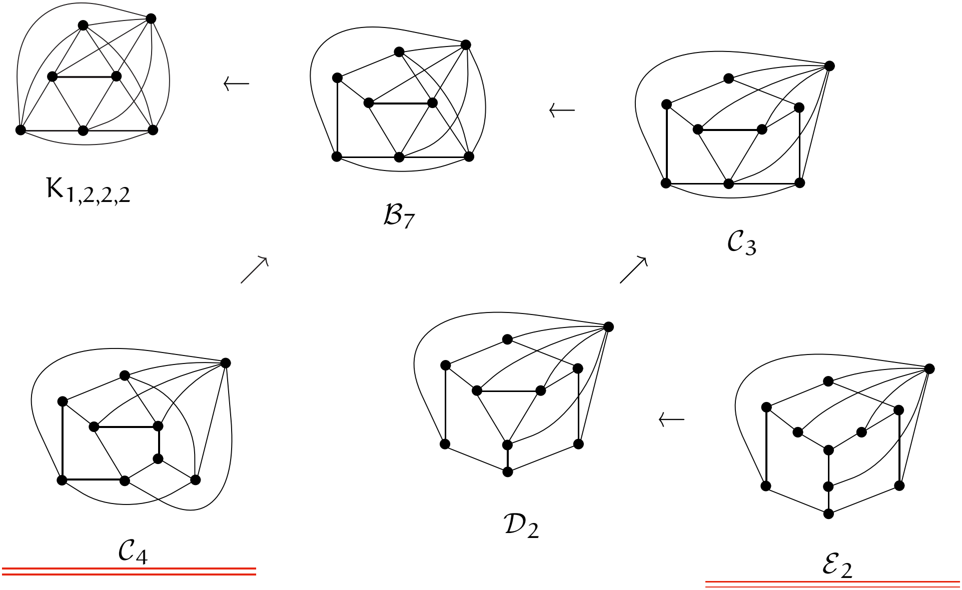


Fact. (since 1995/8)
If $K_{1,2,2,2}$ had no finite planar cover, then Negami's conjecture would be proved.

However;

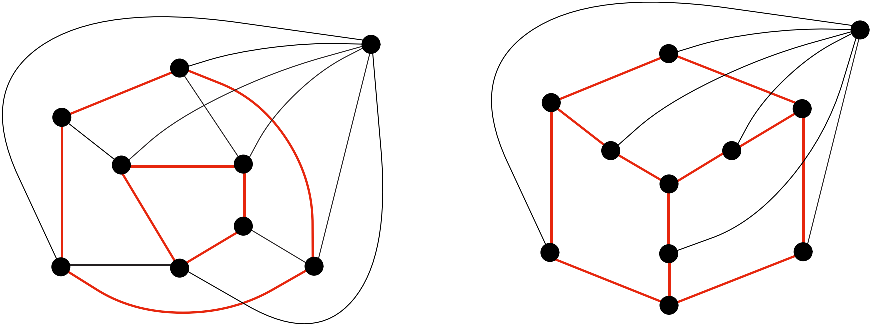
- We are being stuck at this last step for more than 10 years now!
- Fortunately, some finer development "under the surface" is possible and has actually happened...

Fact. Among all the forbidden minors for the projective plane, five more $Y\Delta$-transform to $K_{1,2,2,2}$:



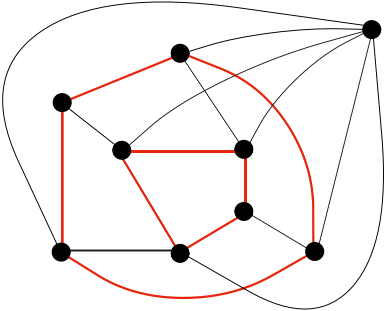
Theorem 8 (PH 1999, 2001)

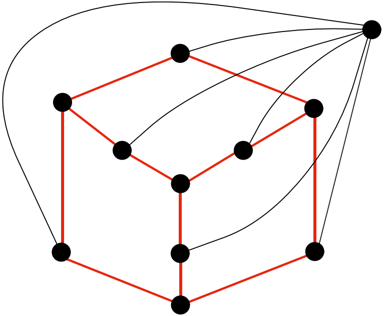
Both the "bottom" graphs $\mathcal{C}_4$ and $\mathcal{E}_2$ have no finite planar covers.



Theorem 8 (PH 1999, 2001)

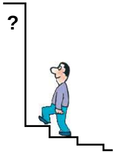
Both the "bottom" graphs $\mathcal{C}_4$ and $\mathcal{E}_2$ have no finite planar covers.





- While the left-hand case appears "structural" — we manage to apply a necklace argument, generalizing Theorem 7,
- the right-hand case is a "counting" one — we get a specialized discharging contradiction.

6 Deeper look – possible counterexamples?

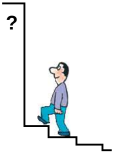


While we are not able to climb the last step $K_{1,2,2,2}$ *directly . . .*

. . . we should perhaps try some detour?!

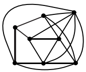
6 Deeper look – possible counterexamples?

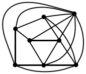
While we are not able to climb the last step $K_{1,2,2,2}$ *directly . . .*

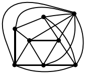


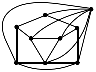
. . . we should perhaps try some detour?!

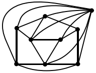
Theorem 9 (Thomas and PH 1999, 2004)
If a connected graph G *has a finite planar cover but no projective embedding, then* G *is a planar expansion of* $K_{1,2,2,2}$ *or some graph from:*

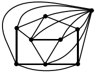

$\mathcal{B}_7$

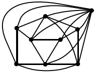

$\mathcal{B}_7'$

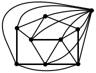

$\mathcal{B}_7''$

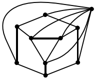

$\mathcal{C}_3$

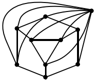

$\mathcal{C}_3'$

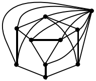

$\mathcal{C}_3''$

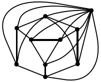

$\mathcal{C}_3^\bullet$

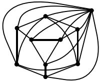

$\mathcal{C}_3^\circ$

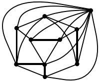

$\mathcal{D}_2$

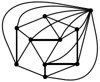

$\mathcal{D}_2'$


$\mathcal{D}_2''$


$\mathcal{D}_2'''$


$\mathcal{D}_2^\bullet$


$\mathcal{D}_2^\circ$


$\mathcal{D}_2^\star$

The following scheme orders the possible counterexamples by their "*difficulty*"

$$K_{1,2,2,2} \leftarrow \mathcal{B}_7 \leftarrow \mathcal{B}_7' \leftarrow \mathcal{B}_7'' \leftarrow \mathcal{C}_3 \leftarrow \mathcal{C}_3' \leftarrow \mathcal{C}_3'' -$$

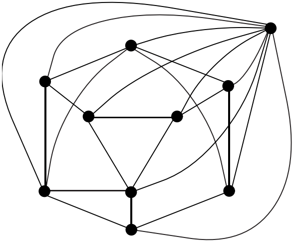
$$- \; \mathcal{C}_3^\bullet \; - \qquad\qquad\qquad\qquad \mathcal{D}_2^\bullet$$

$$\swarrow \qquad\qquad \nwarrow \qquad\qquad\qquad \swarrow$$

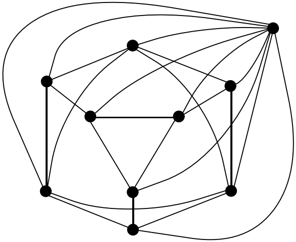
$$\mathcal{C}_3'' \leftarrow \mathcal{C}_3^\circ \leftarrow \mathcal{D}_2 \leftarrow \mathcal{D}_2' \leftarrow \mathcal{D}_2'' \leftarrow \mathcal{D}_2''' \leftarrow \mathcal{D}_2^\circ$$

$$\mathcal{D}_2 \updownarrow \mathcal{D}_2^\star$$

and so one should try one of the two new "*bottom*" cases:



$\mathcal{D}_2^\circ$



$\mathcal{D}_2^\bullet$

7 Additional remarks

Surface extensions

- Is it true that a graph has a finite cover embeddable on a given *nonorientable surface* iff it embeds there?

  cf. [PH 1999], [Negami 2005]

7 Additional remarks

Surface extensions

- Is it true that a graph has a finite cover embeddable on a given *nonorientable surface* iff it embeds there?

  cf. [PH 1999], [Negami 2005]

ACCOTA 2004

- One surprising announcement, but no proof has appeared since. . .

7 Additional remarks

Surface extensions

- Is it true that a graph has a finite cover embeddable on a given *nonorientable surface* iff it embeds there?

  cf. [PH 1999], [Negami 2005]

ACCOTA 2004

- One surprising announcement, but no proof has appeared since. . .

Related recent research

- [Negami and Watanabe, 2002] The conjecture is true for cubic graphs.

7 Additional remarks

Surface extensions

- Is it true that a graph has a finite cover embeddable on a given *nonorientable surface* iff it embeds there?

  cf. [PH 1999], [Negami 2005]

ACCOTA 2004

- One surprising announcement, but no proof has appeared since. . .

Related recent research

- [Negami and Watanabe, 2002] The conjecture is true for cubic graphs.
- [Negami 2003] Composite planar coverings,
  [Negami 2005] Projective planar double coverings.

7 Additional remarks

Surface extensions

- Is it true that a graph has a finite cover embeddable on a given *nonorientable surface* iff it embeds there?

  cf. [PH 1999], [Negami 2005]

ACCOTA 2004

- One surprising announcement, but no proof has appeared since. . .

Related recent research

- [Negami and Watanabe, 2002] The conjecture is true for cubic graphs.
- [Negami 2003] Composite planar coverings,
  [Negami 2005] Projective planar double coverings.
- [Rieck and Yamashita, preprint 2006] Extending the regular-cover thm.

7 Additional remarks

Surface extensions

- Is it true that a graph has a finite cover embeddable on a given *nonorientable surface* iff it embeds there?

  cf. [PH 1999], [Negami 2005]

ACCOTA 2004

- One surprising announcement, but no proof has appeared since. . .

Related recent research

- [Negami and Watanabe, 2002] The conjecture is true for cubic graphs.
- [Negami 2003] Composite planar coverings,
  [Negami 2005] Projective planar double coverings.
- [Rieck and Yamashita, preprint 2006] Extending the regular-cover thm.

  and a very recent NEW: [**Rieck and Yamashita**, preprint 2008]. . .

Planar emulators

- Introduced by [Fellows 1985], independently of Negami, and considered also by [Kitakubo 1991] as *branched planar covers*.
- Not many remarkable results until 2008... Interesting at all?

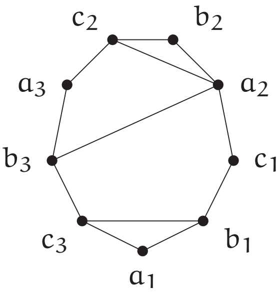
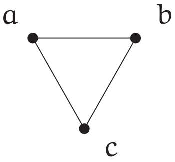
Planar emulators

- Introduced by [Fellows 1985], independently of Negami, and considered also by [Kitakubo 1991] as *branched planar covers*.

- Not many remarkable results until 2008. . . Interesting at all?

- $\varphi : \mathbf{V}(\mathbf{H}) \to \mathbf{V}(\mathbf{G})$, an *emulator* vs. a cover:

. . . map the edges inc. with v in H surjectively onto the edges inc. with $\varphi(v)$ in G.

H  $\rightarrow$  G

Why a planar emulator should be "more than" a planar cover???

Why a planar emulator should be "more than" a planar cover???

- We only "use more edges" – this takes us farther away from planarity!

Why a planar emulator should be "more than" a planar cover???

- We only "use more edges" – this takes us farther away from planarity!
- Until the end of 2008, most people considered planar emulators just as a strange redefinition of covers. . . [Fellows 1985], [Kitakubo 1991]

. . . and then. . .

Why a planar emulator should be "more than" a planar cover???

- We only "use more edges" – this takes us farther away from planarity!
- Until the end of 2008, most people considered planar emulators just as a strange redefinition of covers... [Fellows 1985], [Kitakubo 1991]

... and then...

Theorem 10 (Rieck and Yamashita 2008)
The graphs $K_{1,2,2,2}$ *and* $K_{4,5}-4K_2$ *do have finite planar emulators.*

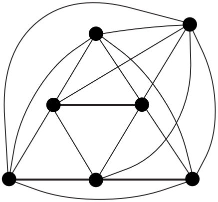
Why a planar emulator should be "more than" a planar cover???

- We only "use more edges" – this takes us farther away from planarity!
- Until the end of 2008, most people considered planar emulators just as a strange redefinition of covers... [Fellows 1985], [Kitakubo 1991]

... and then...

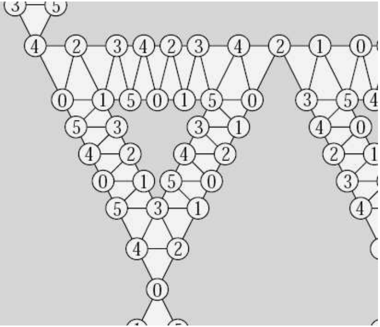
Theorem 10 (Rieck and Yamashita 2008)
The graphs $K_{1,2,2,2}$ *and* $K_{4,5}-4K_2$ *do have finite planar emulators.*

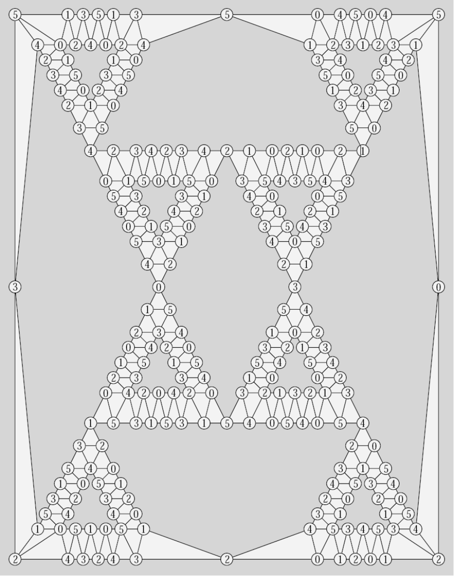
- Now we know that the class of graphs having finite *planar emulators*
  - is different from the class of graphs having finite *planar covers*,
  - and from the class of *projective planar* graphs.
- So, let us study this class...!

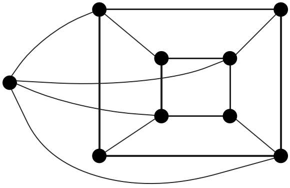


$K_{1,2,2,2}$

$\leftarrow$

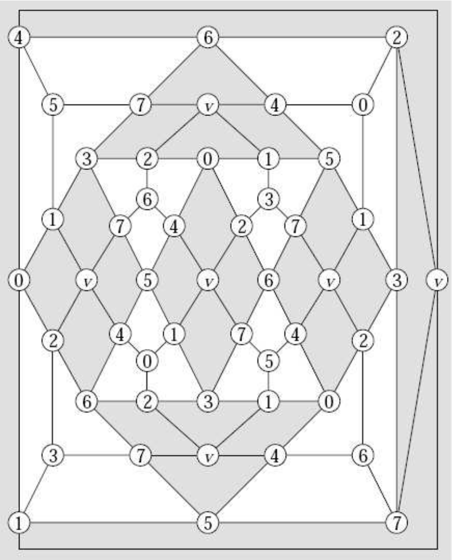






$K_{4,5} - 4K_2$

←


4 6 2
5 7 v 4 0
3 2 0 1 5
6 3
1 7 4 2 7 1
0 v 5 v 6 v 3 v
4 1 7 4
2 2
0 5
6 2 3 1 0
3 7 v 4 6
1 5 7