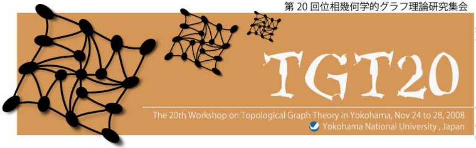

20 Years of Negami's Planar Cover Conjecture

Petr Hliněný

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`http://www.fi.muni.cz/~hlineny`

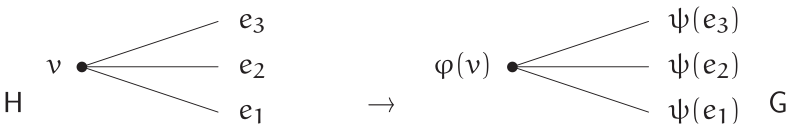
1 Definition

Motivation: Exploring the two graphs locally, we cannot see any difference. . .

A graph H is a *cover* of a graph G if there exists a pair of onto mappings

(a *projection*) $\varphi : V(H) \rightarrow V(G), \quad \psi : E(H) \rightarrow E(G)$

such that ψ maps the edges incident with each vertex v in H
bijectively onto the edges incident with $\varphi(v)$ in G.



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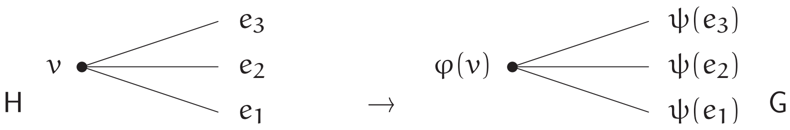


Remark. The edge $\psi(uv)$ has always ends $\varphi(u), \varphi(v)$, and hence only

$\varphi : V(H) \to V(G)$, $\qquad$ the *vertex projection*,

is enough to be specified for simple graphs.

2 Useful basic properties

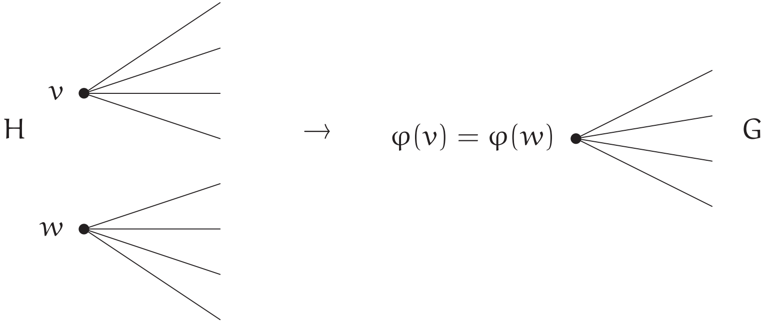
A graph H is a cover of a graph G if there exists a pair of onto mappings

$$\text{(a projection)} \qquad \varphi : V(H) \to V(G), \qquad \psi : E(H) \to E(G)$$

s.t. ψ maps the edges inc. with v in H bijectively onto the edges inc. with $\varphi(v)$ in G.

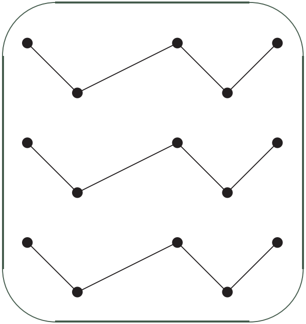
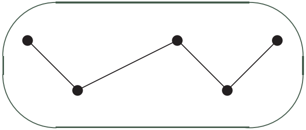
Degree preservation

- $d_H(v) = d_G(\varphi(v))$ for each vertex $v \in V(H)$.



Lifting a path

If G' is a subgraph of G, then the subgraph H' with the vertex set $\varphi^{-1}(V(G'))$ and the edge set $\psi^{-1}(E(G'))$ is called a *lifting of G' into H*.

H  $\rightarrow$  G

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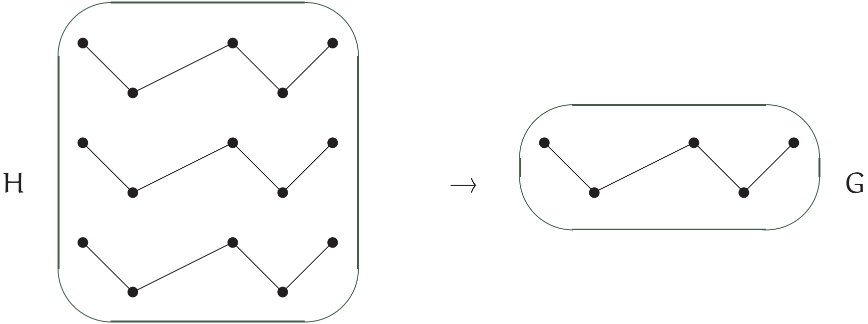

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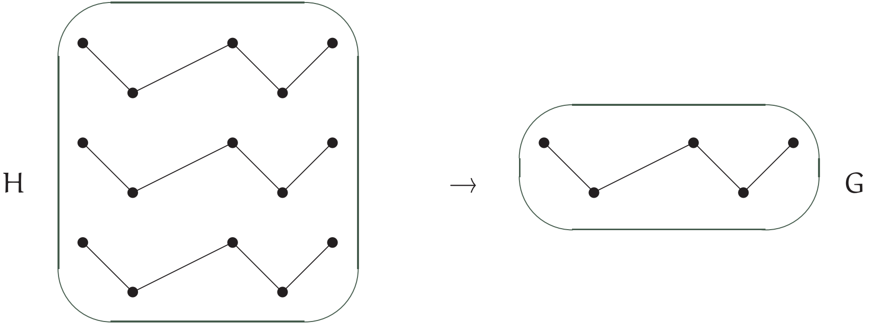

- Lifting of a path from G into H consists of a collection of disjoint isomorphic paths.

- Consequently, if G is connected, then $|\varphi^{-1}(v)| = k$ is a constant.

  We then speak about a *k-fold cover*.

- Lifting of a tree into H consists of a collection of disjoint isom. trees.

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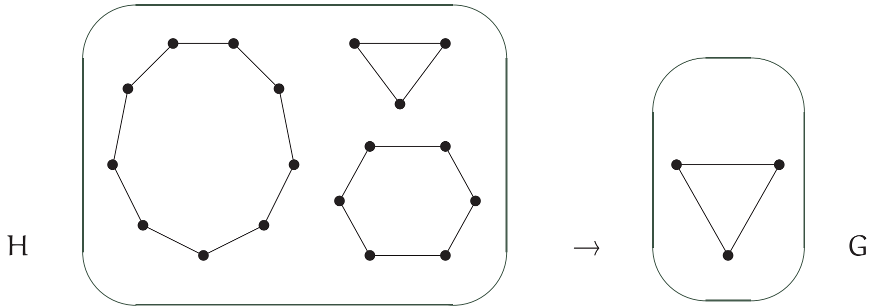
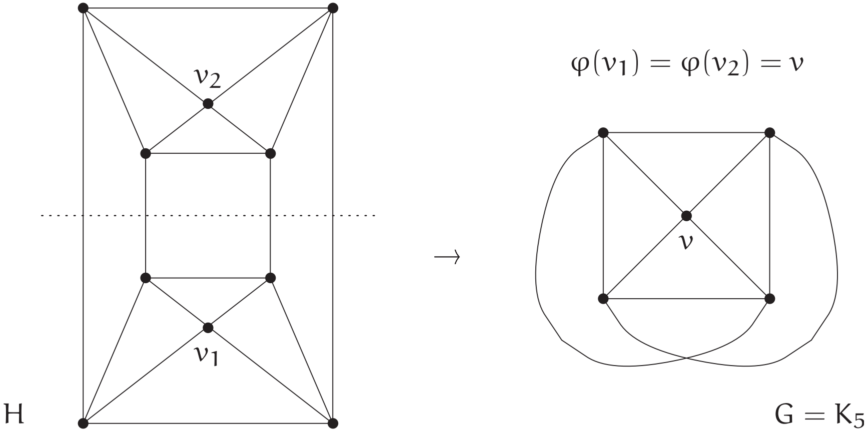
Lifting a cycle

- Lifting of a cycle C_ℓ of G into H consists of a collection of disjoint cycles whose lengths are divisible by ℓ.



Planar cover

We speak about a *planar cover* if H is a finite planar graph.



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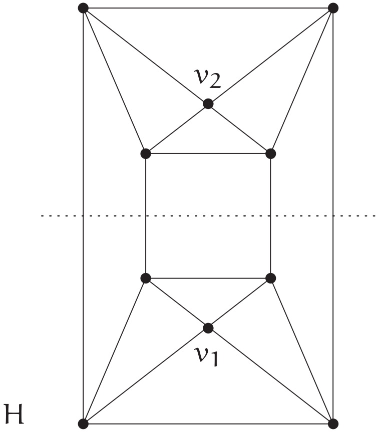
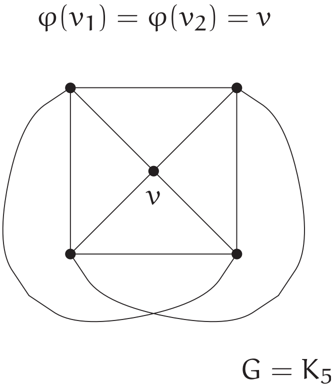


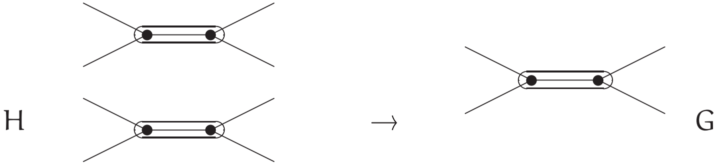
$\rightarrow$



- Graph embedded in the *projective plane* has a double planar cover, via the universal covering map from the sphere onto the projective plane.

Cover preservation

- If G has a planar cover, then so does every minor of G.



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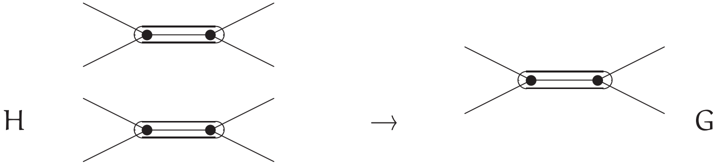
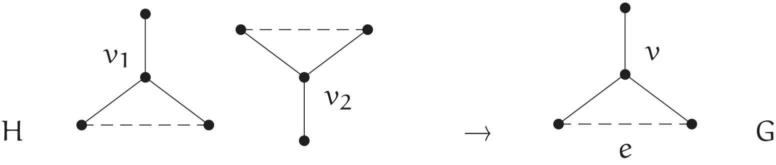


Consider e between two neighbours of a cubic vertex.
If $G - e$ has a planar cover, then so does G.



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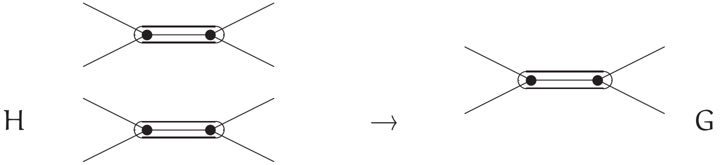

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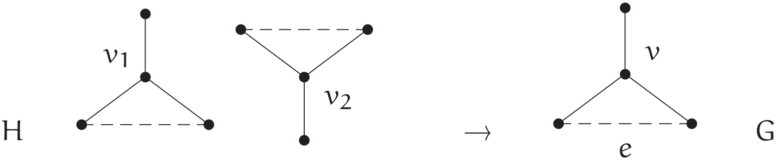

- Therefore, if G has a planar cover, and G' is obtained from G by $Y\Delta$-transformations, then G' has a planar cover, too.

3 Interest in planar covers

- Raised by Negami [1986] in relation to enumeration of projective embeddings of 3-connected graphs.

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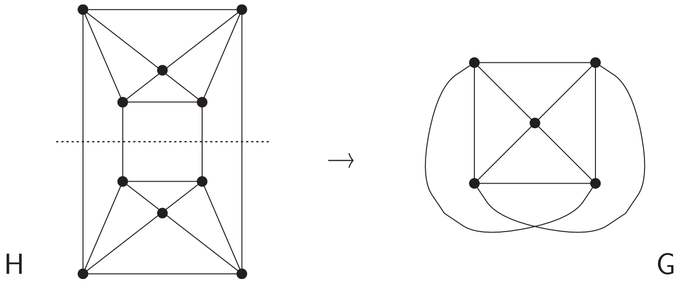

Theorem 1 (Negami, 1986) *A connected graph has a double planar cover if and only if it embeds in the projective plane.*

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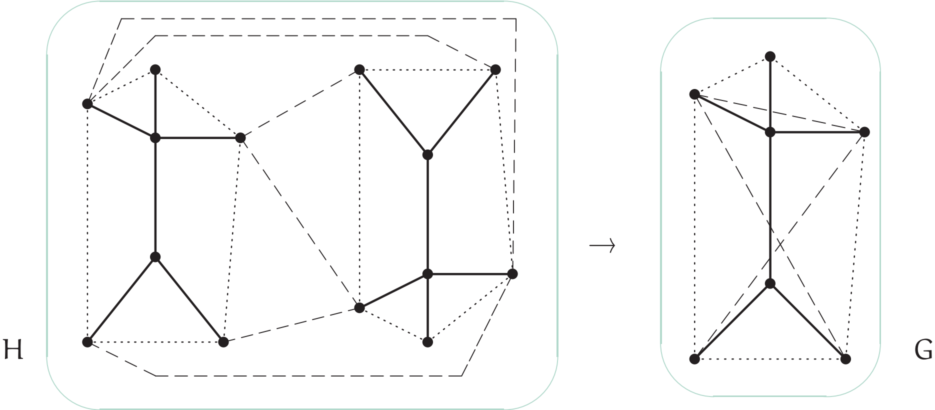
Proof sketch. If a *3-connected planar* graph H is a double cover of a graph G, then G embeds with at most one crosscap:



– this is a purely combinatorial argument...

Negami's planar cover conjecture

A cover $\varphi : V(H) \to V(G)$ is *regular*

if there is a subgroup $A \subseteq Aut(H)$ such that $\varphi(u) = \varphi(v)$ for $u, v \in V(H)$ if, and only if $\tau(u) = v$ for some $\tau \in A$.

Theorem 2 (Negami, 1988) *A connected graph has a finite regular planar cover if and only if it embeds in the projective plane.*

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And now an immediate generalization reads...

Conjecture 3 (Negami, 1988)

A connected graph has a finite ~~regular~~ planar cover
if and only if
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4 Approaching Negami's conjecture

A connected graph has a finite planar cover if and only if it embeds in the projective plane.

We recall the above basic properties of covers...

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- Furthermore, it is enough to consider only those F which are $Y\Delta$-transforms of some forbidden minor in G.

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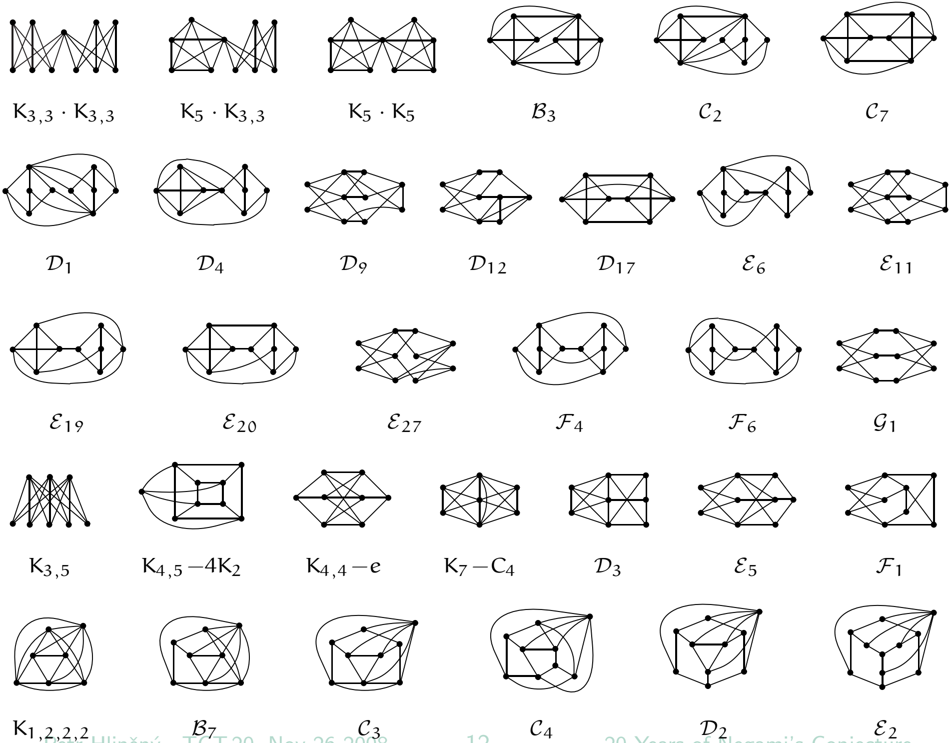
- Furthermore, it is enough to consider only those F which are $Y\Delta$-transforms of some forbidden minor in G.

Does this sound like a piece of cake now?

Unfortunately, the difficulties are just coming...


$K_{3,3} \cdot K_{3,3}$
$K_5 \cdot K_{3,3}$
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$\mathcal{B}_3$
$\mathcal{C}_2$
$\mathcal{C}_7$
$\mathcal{D}_1$
$\mathcal{D}_4$
$\mathcal{D}_9$
$\mathcal{D}_{12}$
$\mathcal{D}_{17}$
$\mathcal{E}_6$
$\mathcal{E}_{11}$
$\mathcal{E}_{19}$
$\mathcal{E}_{20}$
$\mathcal{E}_{27}$
$\mathcal{F}_4$
$\mathcal{F}_6$
$\mathcal{G}_1$
$K_{3,5}$
$K_{4,5}-4K_2$
$K_{4,4}-e$
K_7-C_4
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Disjoint k-graphs

Theorem 4 (Negami / Archdeacon, 1988)
Neither of the graphs $K_{3,3} \cdot K_{3,3}$, $K_5 \cdot K_{3,3}$, $K_5 \cdot K_5$, $\mathcal{B}_3$, $\mathcal{C}_2$, $\mathcal{C}_7$, $\mathcal{D}_1$, $\mathcal{D}_4$, $\mathcal{D}_9$, $\mathcal{D}_{12}$, $\mathcal{D}_{17}$, $\mathcal{E}_6$, $\mathcal{E}_{11}$, $\mathcal{E}_{19}$, $\mathcal{E}_{20}$, $\mathcal{E}_{27}$, $\mathcal{F}_4$, $\mathcal{F}_6$, $\mathcal{G}_1$ *have a finite planar cover.*

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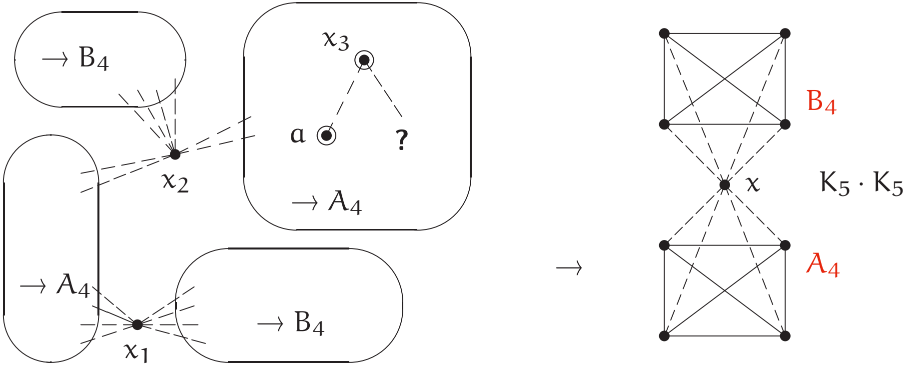
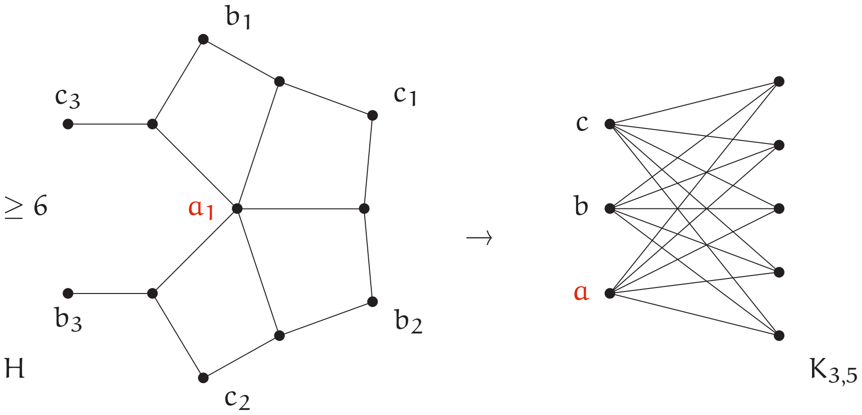
Proof sketch. We choose the $K_5 \cdot K_5$ case for an illustration. . .



Discharging technique

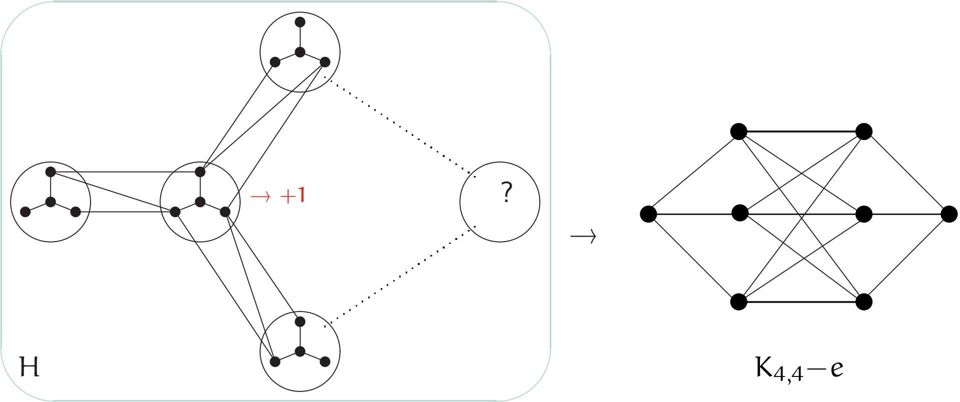
Theorem 5 (?? 1988, 1993) *The graph $K_{3,5}$ has no finite planar cover.*

Proof sketch. Assuming H is a finite planar cover of $K_{3,5}$, we shall derive a contradiction to Euler's formula (or, easy *discharging*)...



Theorem 6 (PH, 1998) *The graph* $K_{4,4}-e$ *has no finite planar cover.*

Brief idea. Form "thick" metavertices from the 3-stars in a supposed cover H.

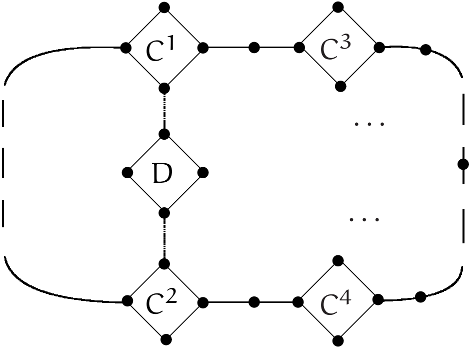


This metagraph is planar bipartite, and so it has a degree-3 vertex. Here we apply *discharging* to get a contradiction again...

"Necklace" argument

Theorem 7 (Archdeacon 1988, 2002**)** (indep. Thomas and PH 1999)
The graphs $K_7 - C_4$ *and* $K_{4,5} - 4K_2$ *have no finite planar covers.*

Brief idea. Find the shortest "necklace" (a *reduced semi-cover*), and shorten it further...



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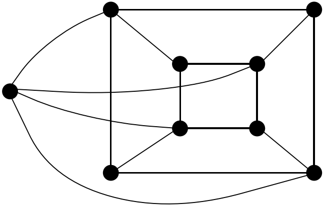
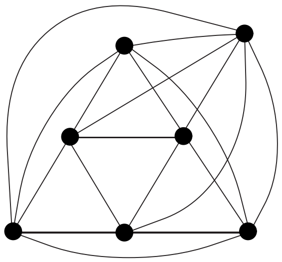


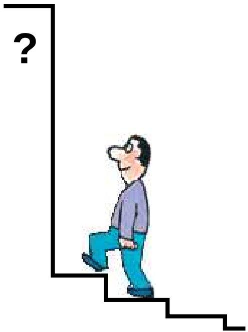
$K_{4,5} - 4K_2$

At the end, we reduce the (length-2) necklace to a projective embedding!

5 The bad guys: $K_{1,2,2,2}$ and its relatives



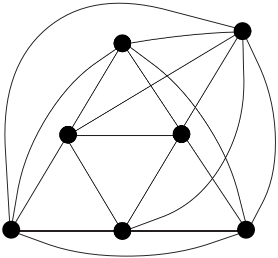
$K_{1,2,2,2}$



Fact. (since 1995/8)
If $K_{1,2,2,2}$ had no finite planar cover, then Negami's conjecture would be proved.

However;

5 The bad guys: $K_{1,2,2,2}$ and its relatives



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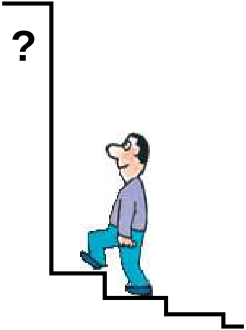

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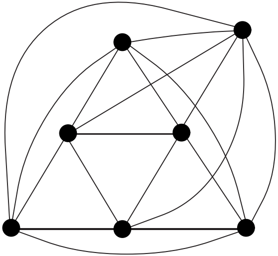

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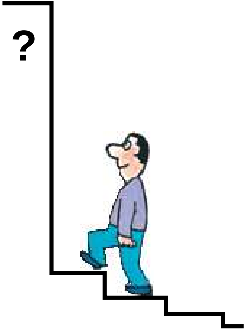

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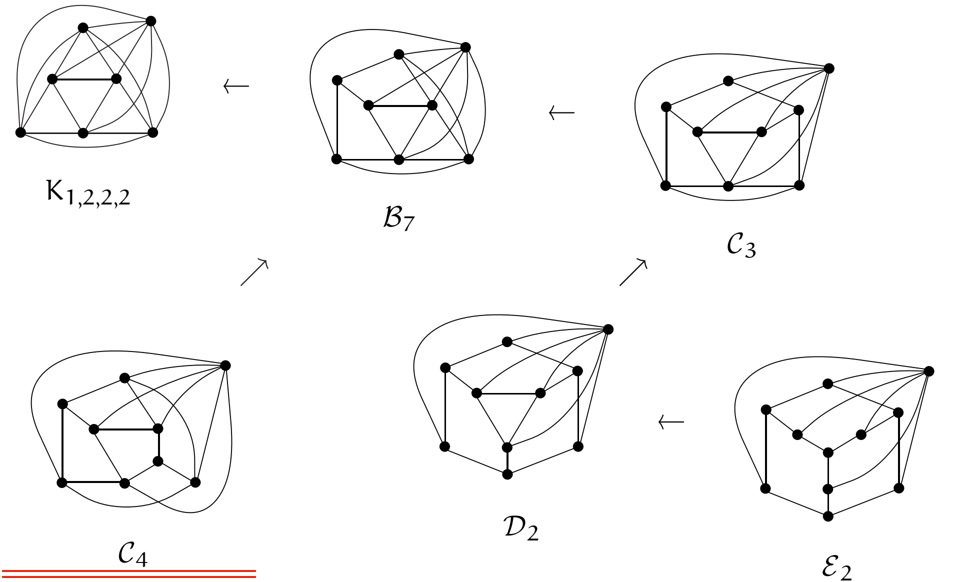
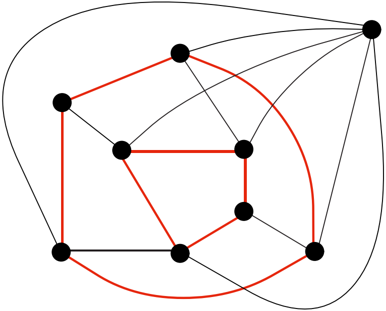
However;

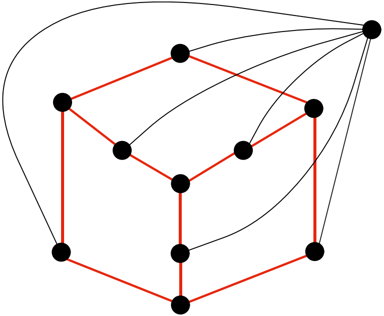
- We are being stuck at this last step for more than 10 years now!
- Fortunately, some finer development "under the surface" is possible and has actually happened. . .

Fact. Among all the forbidden minors for the projective plane, five more $Y\Delta$-transform to $K_{1,2,2,2}$:



Theorem 8 (PH 1999, 2001)
Both the "bottom" graphs $\mathcal{C}_4$ and $\mathcal{E}_2$ have no finite planar covers.





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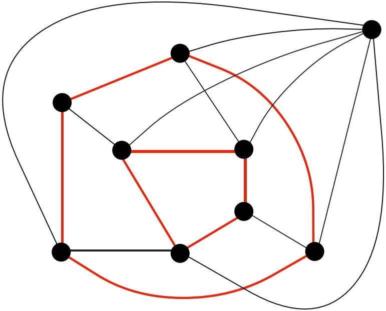
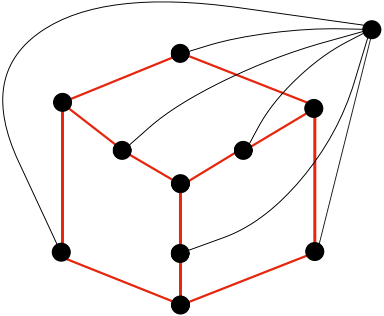
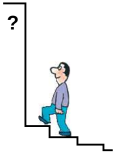




- While the left-hand case appears "structural" — we manage to apply a necklace argument, generalizing Theorem 7,
- the right-hand case is a "counting" one — we get a specialized discharging contradiction.

6 Deeper look – possible counterexamples?

While we are not able to climb the last step $\mathrm{K}_{1,2,2,2}$ *directly . . .*



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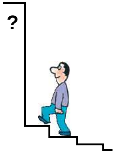

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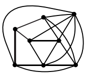
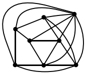
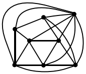
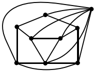
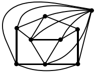
Theorem 9 (Thomas and PH 1999, 2004)
If a connected graph G *has a finite planar cover but no projective embedding, then* G *is a planar expansion of* $K_{1,2,2,2}$ *or some graph from:*

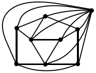

$\mathcal{B}_7$

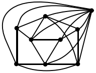

$\mathcal{B}_7'$

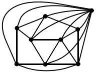

$\mathcal{B}_7''$

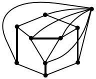

$\mathcal{C}_3$

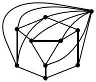

$\mathcal{C}_3'$

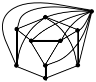

$\mathcal{C}_3''$

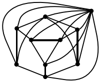

$\mathcal{C}_3^\bullet$

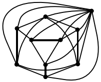

$\mathcal{C}_3^\circ$

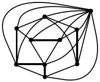

$\mathcal{D}_2$

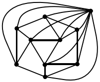

$\mathcal{D}_2'$


$\mathcal{D}_2''$


$\mathcal{D}_2'''$


$\mathcal{D}_2^\bullet$


$\mathcal{D}_2^\circ$

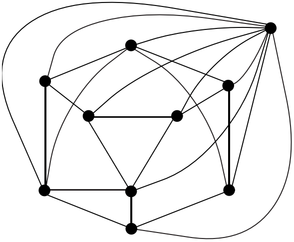

$\mathcal{D}_2^\star$

The following scheme orders the possible counterexamples by their "*difficulty*"

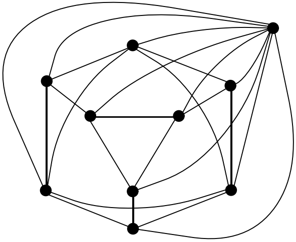
$$\mathsf{K}_{1,2,2,2} \leftarrow \mathcal{B}_7 \leftarrow \mathcal{B}'_7 \leftarrow \mathcal{B}''_7 \leftarrow \mathcal{C}_3 \leftarrow \mathcal{C}'_3 \leftarrow \mathcal{C}''_3 -$$

$$\begin{tikzcd}
& {-\ \mathcal{C}_3^\bullet\ -} \arrow[dl] & & & & & \mathcal{D}_2^\bullet \arrow[dl] \\
\mathcal{C}''_3 & \mathcal{C}^\circ_3 \arrow[l] & \mathcal{D}_2 \arrow[l] \arrow[d, leftrightarrow] & \mathcal{D}'_2 \arrow[l] \arrow[ull] & \mathcal{D}''_2 \arrow[l] & \mathcal{D}'''_2 \arrow[l] & \mathcal{D}^\circ_2 \arrow[l] \\
& & \mathcal{D}_2^\star & & & &
\end{tikzcd}$$

and so one should try one of the two new "*bottom*" cases:



$\mathcal{D}_2^\circ$



$\mathcal{D}_2^\bullet$

7 Additional remarks

Related recent research

- [Negami, Watanabe, 2002] The conjecture is true for cubic graphs.

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Surface extensions

- Is it true that a graph has a finite cover embeddable on a given nonorientable surface iff it embeds there?

  cf. [PH 1999], [Negami 2005]

Planar emulators (*branched planar covers*)

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  iff it has a finite planar cover?

  (Not studied much so far, to our knowledge.)

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So, who wants to be ~~a millionaire~~ the conjecture solver?