20 Years of Negami's Planar Cover Conjecture

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Abstract. In 1988, Seiya Negami published a conjecture stating that a graph G has a finite planar cover (i.e. a homomorphism from some planar graph onto G which maps the vertex neighbourhoods bijectively) if and only if G embeds in the projective plane. Though the "if" direction is easy, and over ten related research papers have been published during the past 20 years of investigation, this beautiful conjecture is still open in 2008. We give a short accessible survey on Negami's conjecture and all the (so far) published partial results, and outline some further ideas to stimulate future research towards solving the conjecture.

Key words. graph cover; graph homomorphism, projective graphs

1. Introduction

The concept of covering maps between topological spaces is very well known and useful, e.g. for modeling spaces which are otherwise hard to visualize. Since it is not our intention to give an introductory course on topology here, we just very informally sketch this concept: A covering map f from S onto T maps a small open neighbourhood of every point x in S bijectively to some open neighbourhood of $f(x)$ in T. In this situation, imagine we are locally exploring the space in such a way that we leave no traces and are not allowed to mark any point. Then, basically, the concept of covering map expresses the fact that we are not able to distinguish the spaces S from T.

Examples of simple covering maps include a double-cover of the projective plane by the sphere, or a double-cover of the Klein bottle by itself or by the torus. The covering-map concept easily extends to graphs, both on topological and combinatorial sides, see Section 2. Then, in analogy to mentioned sphere–projective plane cover map, one can get natural Theorems 2, 3 (by Negami) about special planar covers of projective-planar graphs which seem to lie more on the topological side of the story. On the combinatorial side, on the other hand, one expects the closely related statement of Negami's Conjecture 4 to hold true, but that appears a much more involved task than the former.

The goal of this paper is to give a short survey of Negami's planar cover conjecture and all the related partial results published so far, and thus to stimulate future research toward solving this beautiful 20-years open problem in topological graph theory. For more

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details we refer interested readers also to author's dissertation [9] (available online) which is still quite current.

2. Planar covers

We deal only with finite undirected graphs, and assume that the reader is familiar with basic terms of topological graph theory, e.g. with [14].

We first present a precise formal definition of a cover which we then relax to a simpler variant dealing with only simple graphs.

Definition. A graph H is a *cover* of a graph G if there exists a pair of onto mappings (φ, ψ), $\varphi : V(H) \to V(G)$, $\psi : E(H) \to E(G)$, called a (cover) *projection*, such that ψ maps the edges incident with each vertex v in H bijectively onto the edges incident with $\varphi(v)$ in G.

In particular, for $e = uv$ in H, the edge $\psi(e)$ in G has ends $\varphi(u), \varphi(v)$. Thus, for simple graphs, it is enough to specify the vertex projection φ that maps the neighbors of each vertex v in H bijectively onto the neighbors of $\varphi(v)$ in G (a traditional approach). If G' is a subgraph of G, then the graph H' with the vertex set $\varphi^{-1}(V(G'))$ and the edge set $\psi^{-1}(E(G'))$ is called a *lifting of G' into H*.

To illustrate the concept of a cover, we now give several basic properties.

Proposition 1. *In the following claims, H is a cover of a graph G.*

- *It holds $d_H(v) = d_G(\varphi(v))$ for each vertex $v \in V(H)$.*
- *Lifting of a tree T of G into H consists of a collection of disjoint trees isomorphic to T. Hence, if G is connected, then $|\varphi^{-1}(v)| = k$ is the same number for all $v \in V(G)$. We then speak about a k-*fold *cover.*
- *Lifting of a cycle C_n of G into H consists of a collection of disjoint cycles whose lengths are divisible by n.*
- *Any graph embedded in the projective plane has a* double *(2-fold) cover which is planar, via the universal covering map from the sphere onto the projective plane. (See Fig. 1)*
- *If G has a cover which is planar, then so does every* minor *of G.*
- *Let e be an edge of G between two neighbours of some cubic vertex. If $G - e$ has a cover which is planar, then so does G. Therefore, if G has a planar cover and G' is obtained from G by $Y\Delta$-*transformations *(replacing a cubic vertex with a triangle on the neighbours), then G' has a planar cover.*

Interest in graphs having a cover which is planar was raised in 1986 by Negami [15] in relation to projective embeddings of 3-connected graphs. Interestingly, a very similar concept of *planar emulators* was introduced and studied independently at the same time by Fellows [4], see Section 5. The main result of [15] relates distinct projective embeddings of 3-connected graphs to their double planar covers, and hence immediately:

Theorem 2. (Negami, 1986) *A connected graph has a double planar cover if and only if it embeds in the projective plane. (Fig. 1)*

A natural extension of this result was brought with the concept of regular covers in [16]. A cover $\varphi : V(H) \to V(G)$ is *regular* if there is a subgroup A of the automorphism group of H such that $\varphi(u) = \varphi(v)$ for $u, v \in V(H)$ if, and only if $\tau(u) = v$ for some automorphism $\tau \in A$.

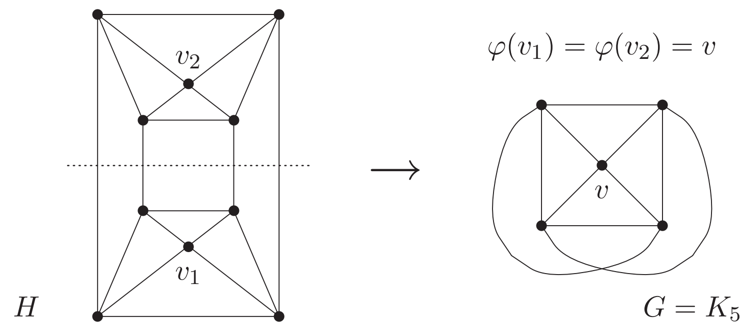

Fig. 1. A double planar cover H of the complete graph K_5.

Theorem 3. (Negami, 1988) *A connected graph has a finite regular planar cover if and only if it embeds in the projective plane.*

It is worth to note that, though these two results are mainly of topological flavor, and a topological argument (the fact that the universal covering space of the projective plane is the sphere) is in the core of Negami's proofs, we can provide our alternative clean combinatorial arguments here.

Sketch of proof. An alternative combinatorial approach to Theorems 2, 3.

Assume that T is any spanning tree of G, and H is a 3-connected double (or finite regular) planar cover of G. Then T lifts into isomorphic copies T' and T'' in H, and there is an automorphism π of H which maps T' onto T''. Let $F(T')$ denote those edges of H having exactly one end in $V(T')$. Notice that our π maps $F(T')$ onto $F(T'')$. See Fig. 2.

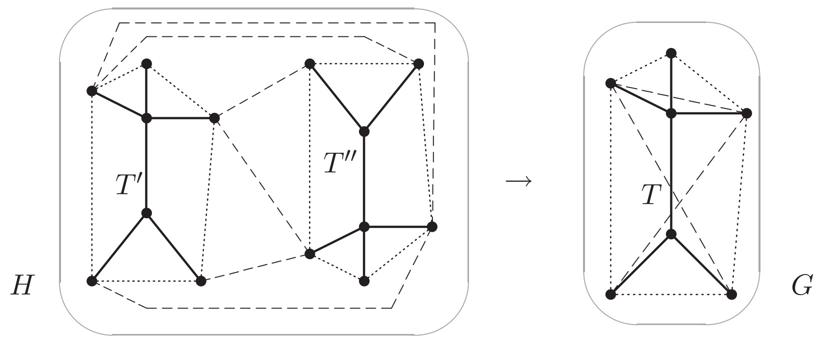

Fig. 2. An illustration of the proof of Theorem 2—lifting a spanning tree T of G into H.

We claim that the cyclic ordering of the edges $F(T')$ around T' in the plane embedding of H is the same (up to mirror image) as that of $F(T'')$ around T''. Indeed, if this was not true, then the action of π onto this particular embedding of H would result in a distinct plane embedding of H, which would contradict 3-connectivity of H.

Furthermore, we can resolve the technical details concerning non-3-connected cover graphs H analogously to [15, 16]. Theorem 2 thus follows easily now: If $T'' \cup F(T'')$ has the same orientation as $T' \cup F(T')$ in the plane embedding of H, then G has a plane embedding, too. If $T'' \cup F(T'')$ is a mirror image of $T' \cup F(T')$ (Fig. 2), then G embeds in the projective plane such that all the edges lifting into $F(T') \cup F(T'')$ pass though the crosscap.

Regarding Theorem 3, we need one more technical argument (actually similar to the proof of Theorem 5) in addition to our above claim, which we skip in this short survey. □

3. Negami's conjecture

Theorem 3 suggests the following very natural generalization [16] which, informally saying, takes the whole subject from topological land to pure combinatorics.

Conjecture 4. (Negami, 1988) A connected graph has a finite planar cover if and only if it embeds in the projective plane.

Although Conjecture 4 only relaxes the regularity condition, the jump in difficulty seems enormous. No proof ideas of Theorem 3 reasonably extend towards solving Conjecture 4; the main reason being that lack of "regularity", or symmetry, in the cover graph. Consequently, despite a chain of promising partial results over the years, Conjecture 4 is still open in 2008.

All the known partial results of this conjecture follow a simple scheme developed at the beginning by Archdeacon and Negami:

– If a graph G embeds in the projective plane, then it has a double planar cover (Proposition 1).
– Conversely, there is a known list [6, 1] of all the 35 forbidden minors for graphs embeddable in the projective plane (see them in the Appendix).
– So, if a connected graph G does not embed in the projective plane, then G has F, one of the 32 connected graphs of that list, as a minor. If we manage to prove that F has no finite planar cover, then neither has G by Proposition 1.
– Furthermore, as observed by Archdeacon, the list to consider can be shortened using $Y\Delta$-transformations (Proposition 1 again).

Though the problem now reduces to a case-by-case check of (at most) 32 graphs, we remind the reader that even looking for a planar cover of a particular graph does not seem to be a finite task at all.

Disjoint k-graphs

Actually, more than half of the 32 cases can be covered with a simple general argument discovered by Negami [17] and Archdeacon: It is enough to know that a graph contains "two disjoint k-graphs" to argue that it has no finite planar cover. The rather complicated notion of k-graphs was introduced already in [6] and we refer the reader to e.g. [9, Section 2.3] for a precise formulation. We also remind the reader of the graphical list of all the 32 graphs in the Appendix.

Theorem 5. (Negami / Archdeacon, 1988)
Neither of the graphs $K_{3,3} \cdot K_{3,3}$, $K_5 \cdot K_{3,3}$, $K_5 \cdot K_5$, $\mathcal{B}_3$, $\mathcal{C}_2$, $\mathcal{C}_7$, $\mathcal{D}_1$, $\mathcal{D}_4$, $\mathcal{D}_9$, $\mathcal{D}_{12}$, $\mathcal{D}_{17}$, $\mathcal{E}_6$, $\mathcal{E}_{11}$, $\mathcal{E}_{19}$, $\mathcal{E}_{20}$, $\mathcal{E}_{27}$, $\mathcal{F}_4$, $\mathcal{F}_6$, $\mathcal{G}_1$ have a finite planar cover.

Sketch of proof. We briefly describe the proof [17] on a particular case of $K_5 \cdot K_5$, but a full generalization is quite straightforward.

Let x be the degree-8 vertex of $K_5 \cdot K_5$, and A_4 and B_4 be the two components of $K_5 \cdot K_5 - x$, as in Fig. 3. Consider a finite plane-embedded cover H of $K_5 \cdot K_5$, and

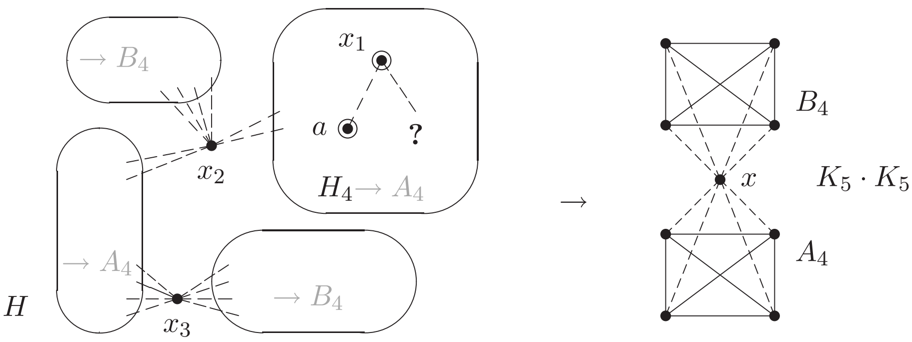

Fig. 3. An illustration of the proof of Theorem 5.

assume, up to symmetry, that it is a component H_4 of the lifting of A_4 into H that is "minimal", i.e. H_4 contains no part of the lifting of B_4 inside. However, H_4 as a cubic graph cannot be outerplanar, and hence some internal vertex a of H_4 is adjacent to some x_1 in the lifting of x into H, and this x_1 must be adjacent to some vertices in the lifting of B_4, a contradiction. Hence $K_5 \cdot K_5$ has no finite planar cover. □

Two discharging arguments

Discharging is a proof method developed mainly to study the Four colour problem. The method simply applies Euler's formula in a clever way.

A very easy discharging argument shows that the graph $K_{3,5}$ cannot have a finite planar cover. Though this claim is first attributed to Fellows, it does not occur in [4]. A short published proof can be found, e.g., in [12].

Theorem 6. (1988, 1993) *The graph $K_{3,5}$ has no finite planar cover.*

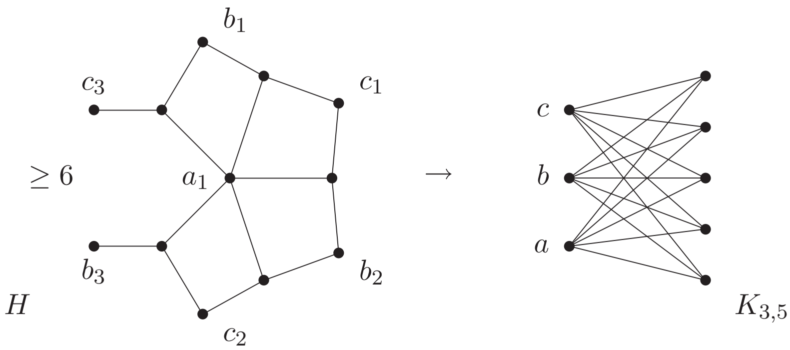

Fig. 4. An illustration of the proof of Theorem 6.

Proof. Suppose that a (bipartite) graph H was a finite cover of $K_{3,5}$ embedded in the plane. We assign charge of $3(4 - d_H(v))$ to every vertex v, and of $3(4 - len(\phi))$ to every face ϕ of H. By Euler's formula, the total charge of H is positive $12{\cdot}2$. Then every 3-vertex of H sends its charge equally 1 to each neighbour. So every 5-vertex, say a_1, of H now has charge of $-3 + 5 = 2$. That charge is subsequently sent from a_1 to any incident ≥ 6-face of H. If a_1 covers a of $K_{3,5}$, then the second neighbourhood of a_1 in H contains vertices

$b_1, c_1, b_2, c_2, \ldots$ alternating between the liftings of b and c of $K_{3,5}$. See Fig. 4. That clearly cannot happen with all the incident faces of a_1 quadrilaterals. Therefore, all vertices of H end up with nonpositive charge, and so do all the faces. This contradiction concludes the proof. □

Another, significantly more involved discharging argument was found several years later by the author [7] for the case of $K_{4,4}-e$.

Theorem 7. (PH, 1998) *The graph $K_{4,4}-e$ has no finite planar cover.*

A noticeable feature of the proof is that discharging is applied not to the supposed planar cover itself, but to a special simplification of it. That seems the right way to go, as also a successful case of $\mathcal{E}_2$ in Theorem 10 shows.

Structural approach

Yet another approach to prove nonexistence of a planar cover was discovered by Archdeacon already in 1988, but the proof had not been published until much later in [2].

Theorem 8. (Archdeacon, 1988, 2002)
The graphs K_7-C_4 and $K_{4,5}-4K_2$ have no finite planar covers.

Here the proof cannot be easily sketched, and so we only mention that one looks for a short "necklace" of interconnected 4-cycles in the supposed cover, and then finds a way the necklace can be made even shorter, arriving at a contradiction. This idea can be regarded as a wide generalization of the "disjoint k-graph" argument of Theorem 5.

Interestingly, the exactly same proof was rediscovered (independently) by Thomas and the author 10 years later, see [9], and subsequently generalized by the author to cover also the case of $\mathcal{C}_4$ in Theorem 10 in the next section. The particularly nice feature of this generalization is that its proof directly constructs from the shortest necklace a projective embedding of the covered graph, instead of deriving an artificial contradiction, cf. also Theorems 2,3.

4. The bad: $K_{1,2,2,2}$ and relatives

After all, putting together Theorems 5, 6, 7, and 8, and applying $Y\Delta$-transformations to the graphs $\mathcal{D}_3, \mathcal{E}_5, \mathcal{F}_1, \mathcal{B}_7, \mathcal{C}_3, \mathcal{C}_4, \mathcal{D}_2, \mathcal{E}_2$, leaves only one following case to be resolved (Fig. 5).

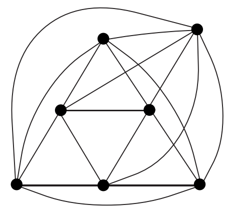

Fig. 5. The graph $K_{1,2,2,2}$.

Corollary 9. (1998) *If the graph $K_{1,2,2,2}$ (the octahedron with an extra vertex) had no finite planar cover, then Conjecture 4 would hold true.*

It might have appeared that Corollary 9 "almost solved" Negami's conjecture, but the opposite is showing true—even nowadays, more than 10 years later, Conjecture 4 is still wide open. In fact, it seems that this last step towards resolving the conjecture captures all difficulties of the problem!

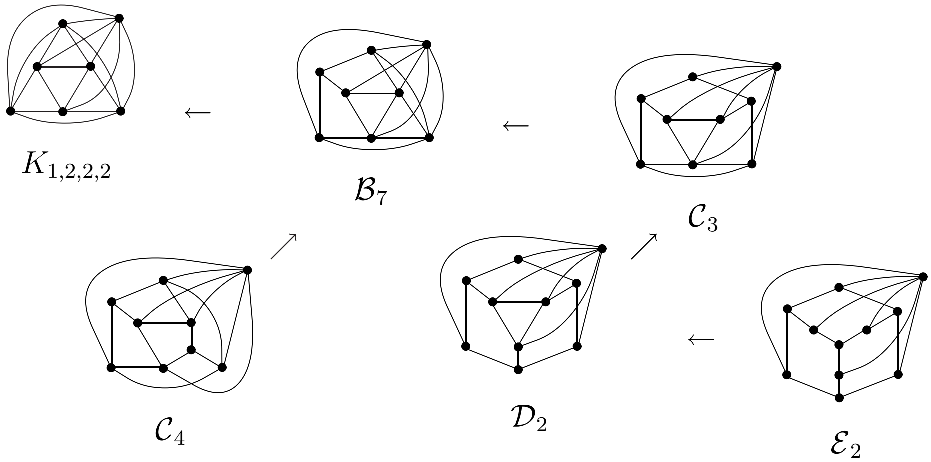

Fig. 6. The "family" of $K_{1,2,2,2}$ via $Y\Delta$-transformations.

Furthermore, notice that there are other graphs on the list of the 32 projective obstructions which are unsolved yet. Namely, the graph $\mathcal{C}_4$ reduces via $\mathcal{B}_7$ to $K_{1,2,2,2}$, and so does the graph $\mathcal{E}_2$ via $\mathcal{D}_2$, $\mathcal{C}_3$ and $\mathcal{B}_7$ (Fig. 6). Hence in the situation when we are not able to attack the final case of $K_{1,2,2,2}$ directly, it might perhaps help to "train our muscles" on some of the supposedly easier cases. This strategy led us to the following new results [9, 10]:

Theorem 10. (PH, 1999 and 2001)
The graphs $\mathcal{C}_4$ and $\mathcal{E}_2$ (cf. Fig. 6) have no finite planar covers.

Despite the graphs $\mathcal{C}_4$ and $\mathcal{E}_2$ are "relatives", the proofs for each one of them in Theorem 10 are completely different and incomparable. While an involved discharging argument is applied in the case of $\mathcal{E}_2$, the other case of $\mathcal{C}_4$ is covered by a generalization of the necklace argument from Theorem 8. Unfortunately, neither of these arguments can be directly generalized to any other of the missing cases. Hence, we suggest that the right way to attack the last case of $K_{1,2,2,2}$ is to find a suitable *common generalization of the structural and discharging approaches* of Section 3.

Possible counterexamples?

Trying to understand, in view of Theorem 10, the difficulties surrounding the last case of $K_{1,2,2,2}$, one should naturally ask for which of all graphs, to our current knowledge, Conjecture 4 might possibly fail. That direction has been taken by Thomas and the author in [9] and [11]. A *planar expansion* of a graph G is a graph which results from G by adding a planar graph sharing one vertex with G, or by replacing an edge or a cubic vertex with a planar graph with its attachments on the outer face. We refer to [9, Section 6.1] for a formal description.

Theorem 11. (PH and Thomas, 1999 and 2004)
Let Π be the set of $K_{1,2,2,2}$ and the 15 graphs listed in Fig. 7. If a connected graph G has a finite planar cover but no projective embedding, then G is a planar expansion of some graph from Π.

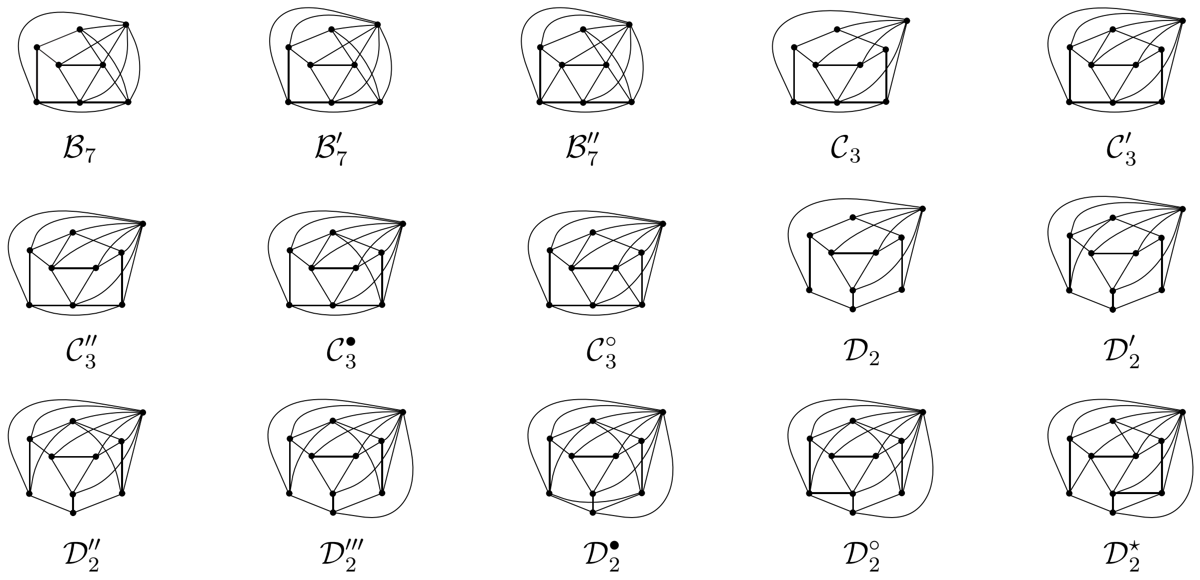

Fig. 7. The family Π of Theorem 11.

Although it might seem that Theorem 11 brings no useful new information in the case Conjecture 4 holds, we suggest that quite the opposite is true—the theorem shows that there are only at most 15 discrete steps left toward solving Conjecture 4. In other words, Theorem 11 acts as a selection filter for ideas—if a new idea should put us forward in solving Negami's conjecture, then it has to be applicable to at least one of the graphs in Π.

Furthermore, we can order the graphs from Π according to their difficulty as follows:

Proposition 12. *Let us write $G \xrightarrow[NC]{} H$ to mean that "if G has no planar cover, then neither has H". Then:*

$$K_{1,2,2,2} \xrightarrow[NC]{} \mathcal{B}_7 \xrightarrow[NC]{} \mathcal{B}'_7 \xrightarrow[NC]{} \mathcal{B}''_7 \xrightarrow[NC]{} \mathcal{C}_3 \xrightarrow[NC]{} \mathcal{C}'_3 \xrightarrow[NC]{} \mathcal{C}''_3 \rightarrow$$

$$\begin{array}{ccccccccccccc} & \nearrow_{NC} \rightarrow & \mathcal{C}^\bullet_3 & \rightarrow \searrow^{NC} & & & & & & & & \nearrow_{NC} & \mathcal{D}^\bullet_2 \\ \mathcal{C}''_3 & \xrightarrow[NC]{} & \mathcal{C}^\circ_3 & \xrightarrow[NC]{} & \mathcal{D}_2 & \xrightarrow[NC]{} & \mathcal{D}'_2 & \xrightarrow[NC]{} & \mathcal{D}''_2 & \xrightarrow[NC]{} & \mathcal{D}'''_2 & \xrightarrow[NC]{} & \mathcal{D}^\circ_2 \\ & & & & \updownarrow{\scriptstyle NC} & & & & & & & & \\ & & & & \mathcal{D}^\star_2 & & & & & & & & \end{array}$$

The task now is: Can we, at least, prove that the "easiest" two tail cases of Proposition 12 have no finite planar covers? Unfortunately, no progress on either case has been made till now (2008). Still, that may be just because no one (including us) has tried seriously enough, and so we would like to encourage the interested readers to try it. The reason we believe it is important to follow this direction, is that solving each new particular case must bring some new ideas or methods which can later be, perhaps, combined together into a final solution to Negami's conjecture.

5. Additional remarks

Several other research papers studying planar covers of graphs, but not in a direct relation to solving Conjecture 4, have been published over the years. In [3], for instance, it is proved that no nonplanar graph has an odd-fold planar cover. In [18] it is proved that Conjecture 4 holds for all cubic graphs, but that claim is indeed a trivial corollary of Theorem 11.

In [8], a natural way of extending Conjecture 4 is outlined: the conjecture is equivalent to saying that a connected graph has a finite projective cover if and only if it is projective. Such a formulation can be easily extended to any nonorientable surface (while it is trivially false for all orientable surfaces), and little support for the Klein bottle extension has been provided there [8], too. Then, the weaker projective-planar double-covering variant of this reformulation has been proved by Negami in [20], using the idea of so called composite coverings [19].

Planar emulators

At last, we return to related Fellows' problem of *planar emulators* [4, 5], cf. also "branched coverings" by Kitakubo [13].

Definition. A graph H is an *emulator* of a graph G if there exists a pair of onto mappings (φ, ψ), $\varphi : V(H) \to V(G)$, $\psi : E(H) \to E(G)$, such that ψ maps the edges incident with each vertex v in H surjectively onto the edges incident with $\varphi(v)$ in G.

Easily, a planar cover is a planar emulator but not the opposite. It appears very natural to extend questions and results on planar covers to planar emulators. For instance, although emulator projections do not preserve the vertex degrees nor the fold number, the last two important points of Proposition 1 still hold [5] (though the proof is no longer trivial):

Proposition 13. (Fellows) *The property of having a finite planar emulator is preserved under taking minors and $Y\Delta$-transformations.*

The following [5] also appeared to be a very natural extension:

Conjecture 14. (Fellows, 1988) A connected graph has a finite planar emulator if and only if it has a finite planar cover.

The strategy for attacking this conjecture would be the same as for Conjecture 4—to study possible planar emulators of all the 32 connected obstructions for the projective plane. In this manner it is easy to extend [5] the proofs of Theorems 5 and 6 to emulators, but no further extension is known.

However, a surprising new result [21] has appeared just recently:

Theorem 15. (Rieck and Yamashita, 2008) *The graphs $K_{4,5} - 4K_2$ and $K_{1,2,2,2}$ do have finite planar emulators. Hence Conjecture 14 is false.*

Inspired by this breakthrough statement, Chimani and the author [unpublished] have subsequently constructed finite planar emulators for the graphs $\mathcal{C}_4$ and $\mathcal{E}_2$. Using Proposition 13 and arguments of [11, Proposition 6.1], this gives finite planar emulators for all the 16 graphs listed in Theorem 11. Particularly, one can construct in this way an emulator of $K_{1,2,2,2}$ which is smaller than the one in [21].

Although Theorem 15 has no direct relevance to solving Conjecture 4 (which still remains open), it is now desirable to find (or at least conjecture) the full list of forbidden minors for the graphs having a finite planar emulator. Right now, we consider it a very interesting open question whether the graphs $K_7 - C_4$ and $K_{4,4} - e$ have finite planar emulators.

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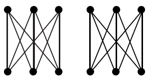
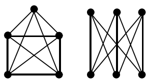
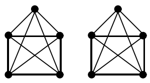
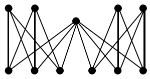
Appendix: The obstructions for projective plane

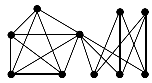
This is a list of all the 35 minor-minimal non-projective graphs [6,1].

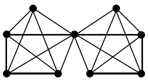

$K_{3,3}+K_{3,3}$

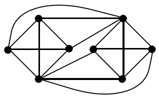

$K_5+K_{3,3}$

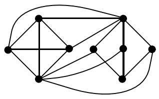

K_5+K_5

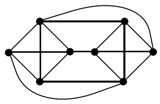

$K_{3,3}\cdot K_{3,3}$

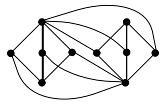

$K_5\cdot K_{3,3}$

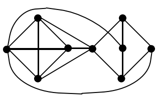

$K_5\cdot K_5$


$\mathcal{B}_3$


$\mathcal{C}_2$


$\mathcal{C}_7$


$\mathcal{D}_1$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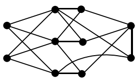
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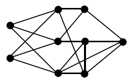
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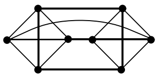
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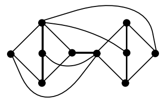
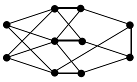
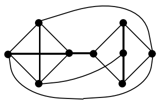
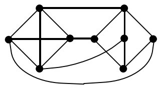

$\mathcal{E}_6$


$\mathcal{E}_{11}$


$\mathcal{E}_{19}$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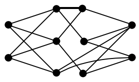
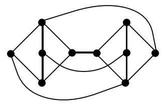

$\mathcal{E}_{27}$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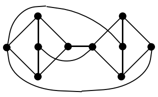
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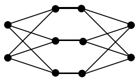
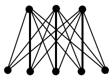
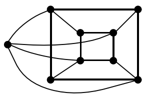
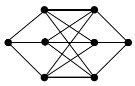
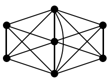
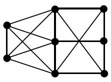

$\mathcal{G}_1$

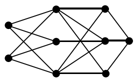

$K_{3,5}$

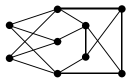

$K_{4,5}-4K_2$


$K_{4,4}-e$


K_7-C_4


$\mathcal{D}_3$


$\mathcal{E}_5$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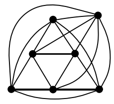
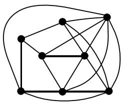
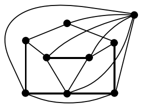

$K_{1,2,2,2}$


$\mathcal{B}_7$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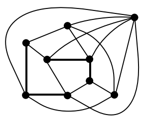
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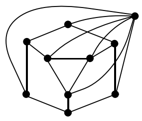
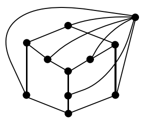

$\mathcal{D}_2$


$\mathcal{E}_2$