Finding Branch-decompositions and Rank-decompositions

Petr Hliněný

Faculty of Informatics, Masaryk University
Botanická 68a, 602 00 Brno, Czech Rep.

e-mail: `hlineny@fi.muni.cz`
`http://www.fi.muni.cz/~hlineny`

joint work with **Sang-il Oum**

Department of Combinatorics and Optimization,
University of Waterloo, Canada.

e-mail: `sangil@math.uwaterloo.ca`.

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- and few more notions (path-width, bandwidth, cut-width, * * * * *).

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The **hardness results**
carry over to matroid branch-width / tree-width, and to graph rank-width.

The **FPT results** here
can be extended to matroids over finite fields, and to graph rank-width. . . .

2 Branch-width, Definition

A ground set E, with a **connectivity function** λ (arbitr. symm. submod.)
$\longrightarrow$ a **branch decomposition:**

- E decomposed to a *sub-cubic tree* (degrees ≤ 3), and
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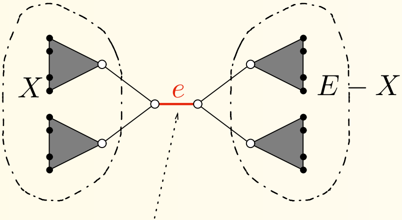

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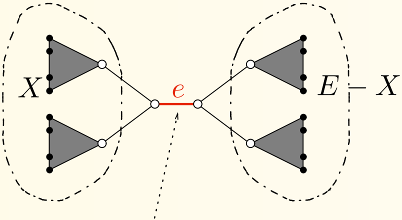

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Branch-width $\mathrm{bw}(\lambda)$ = min. of max. edge widths over all decompositions.

Branch-width variants, rank-width

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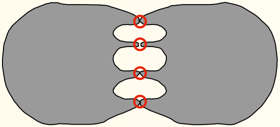

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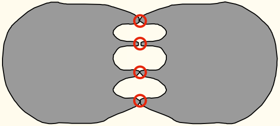

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- *Matroid branch-width*:

$E = E(M)$ and $\lambda(X) = \mathrm{r}_M(X) + \mathrm{r}_M(E \setminus X) - \mathrm{r}(M) + 1$

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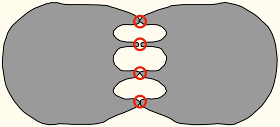

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- *Graph rank-width*:

(as motivated by clique-width)

$$\boldsymbol{A}(G) \;\rightarrow\; \begin{array}{c|ccc} & & V(G)\setminus X & \\ \hline & 0 & 1 & 1 \\ X & 1 & 0 & 1 \\ & 0 & 1 & 0 \\ \hline \end{array}$$

$E = V(G)$ and $\lambda(X) = \mathrm{rank}(\, A(G)[X, E \setminus X]\,)$ over $GF(2)$.

Computing branch-width and rank-width

Theorem 1. (Bodlaender & Thilikos, 1997) *There is an $O(n)$-time FPT algorithm that, for each fixed k, either confirms that the branch-width of a given graph G is $> k$, or it finds a branch-decomposition of G of width $\leq \boldsymbol{k}$.*

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. . . . **new** . . . .

Theorem 4. (PH & Oum, 2007) *There is an $O(n^3)$-time FPT algorithm that, for each fixed k, either confirms that the branch-width of a given matroid M over a finite field (rank-width of a given graph G) is $> k$, or it finds a branch-decomposition of M (a rank-decomposition of G) of width $\leq \boldsymbol{k}$.*

3 Width (a number) → Decomposition

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- Matroids (over finite fields) — there is a computable finite set of forbidden minors for the matroids of branch-width $\leq k$, and we have a decomposition of width $\leq 3k$ from Theorem 2.

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- Same with graph rank-width — there is a computable finite set of forbidden vertex-minors for the graphs of rank-width $\leq k$, and we have a decomposition of width $\leq 3k$ from Theorem 3.

  How can we get an optimal decomposition?

An idea motivated by Geelen [private communication]...

1. Take an arbitrary partition $\mathcal{P}$ of our E, and extend connectivity λ to $\lambda^{\mathcal{P}}$ on $\mathcal{P}$ naturally. Assume the branch-width of $\lambda^{\mathcal{P}}$ is efficiently computable.

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- At deeper levels, process only such pairs that P_1 of it has been processed one level up.

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Titanic gadget – for $P \in \mathcal{P}$ and $\ell = \lambda(P)$,

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Starting with a partitioned matroid $M, \mathcal{P}$, we arrive at *normalized* matroid $M^{\#}$.

($M^{\#}$ may require a slightly larger field to be represented over.)

Theorem 6. *The branch-width of* $\lambda^{\mathcal{P}}$ *on* M *is equal to* $\mathrm{bw}(M^{\#})$*.*

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Altogether, we really get $n^2 \times O(n) + n \times O(n^2) = O(n^3)$ time!

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Fact. The rank-width of a bipartite graph equals the branch-width of the binary matroid represented by its *bipartite adjacency matrix*.

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- What to do for non-bipartite graphs?

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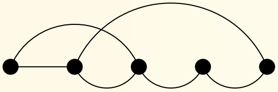
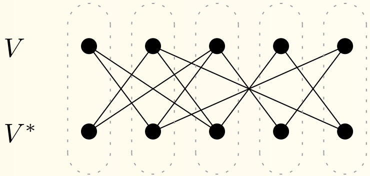
\hline
 & 0 & 1 & 1 & \\
U & 1 & 0 & 1 & \boldsymbol{I}_U \\
 & 0 & 1 & 0 & \\
\hline
\end{array}
$$

Fact. The rank-width of a bipartite graph equals the branch-width of the binary matroid represented by its *bipartite adjacency matrix*.

- What to do for non-bipartite graphs?





Fact. We change every graph G to the associated bipartite graph with its canonical vertex partition. The value of rank-width exactly doubles.