On Decidability of MSO Theories of Representable Matroids

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1 Motivation

Graph Tree-Width and Logic

- [Seese, 1975] Undecidability of an *MSO theory of large grids*.
- [Robertson and Seymour, 80's – 90's] The *Graph Minor Project*: WQO of (finite) graphs by minors, notion of tree-width, algorithmic consequences.
- [Courcelle, 1988] Decidability of an *MSO theory* of graphs: The class of all (finite) graphs of bounded tree-width has decidable MS_2 theory. (Independently by [Arnborg, Lagergren, and Seese, 1991].)
- [Seese, 1991] Decidability of the MS_2 theory *implies bounded tree-width*.

Results closely related to *linear-time algorithms* on bounded tree-width graphs.

Extending to Matroids

- [Geelen, Gerards, Robertson, Whittle, and . . . , late 90's – future] Extending the *ideas of graph minors* to matroids (over finite fields).
- [PH, 2002] *Decidability for matroids*: The class of all $GF(q)$-representable matroids of bounded branch-width has a decidable MSO theory.
- [PH, Seese] Decidability of matr. MSO *implies bounded branch-width*.

2 Basics of Matroids

A matroid is a pair $M = (E, \mathcal{B})$ where

- $E = E(M)$ is the *ground set* of M (elements of M),
- $\mathcal{B} \subseteq 2^E$ is a collection of *bases* of M,
- the bases satisfy the "exchange axiom"
  $\forall B_1, B_2 \in \mathcal{B}$ and $\forall x \in B_1 - B_2$,
  $\exists y \in B_2 - B_1$ s.t. $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}$.

Otherwise, a *matroid* is a pair $M = (E, \mathcal{I})$ where

- $\mathcal{I} \subseteq 2^E$ is the collection of *independent sets* (subsets of bases) of M.

The definition was inspired by an abstract view of *independence* in linear algebra and in combinatorics [Whitney, Birkhoff, Tutte,...].

Notice exponential amount of information carried by a matroid.

Literature: J. Oxley, Matroid Theory, Oxford University Press 1992,1997.

Some **elementary matroid terms** are

- *independent set* = a subset of some basis,
  dependent set = not independent,
- *circuit* = minimal dependent set of elements,
  triangle = circuit on 3 elements,
- *hyperplane* = maximal set containing no basis,
- *rank function* $\mathrm{r}_M(X)$ = maximal size of an M-independent subset $I_X \subseteq X$ ("*dimension*" of X).

(Notation is taken from linear algebra and from graph theory. . .)

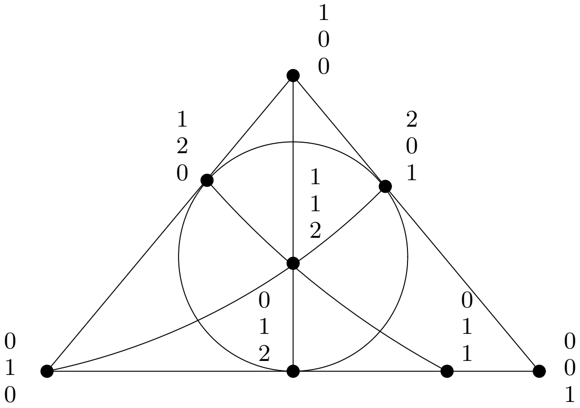
Axiomatic descriptions of matroids via independent sets, circuits, hyperplanes, or rank function are possible, and often used.

Vector matroid — a straightforward motivation:

- Elements are vectors over $\mathbb{F}$,
- independence is usual linear independence,
- the vectors are considered as columns of a matrix $\boldsymbol{A} \in \mathbb{F}^{r \times n}$.
  ($\boldsymbol{A}$ is called a *representation* of the matroid $M(\boldsymbol{A})$ over $\mathbb{F}$.)

Not all matroids are vector matroids.

An example of a rank-3 vector matroid with 8 elements over $GF(3)$:

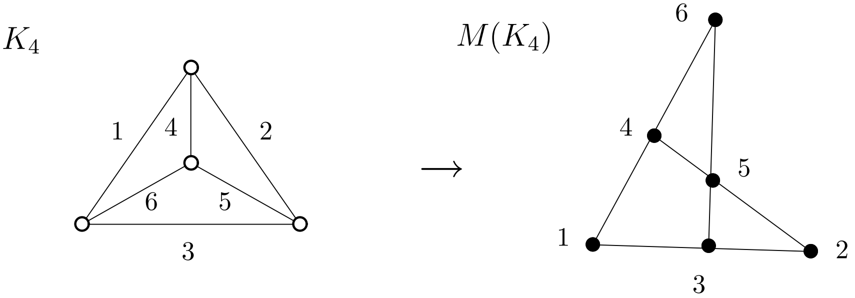


Graphic matroid $M(G)$ — the combinatorial link:

- Elements are the edges of a graph,
- independence $\sim$ acyclic edge subsets,
- bases $\sim$ spanning (maximal) forests,
- circuits $\sim$ graph cycles,
- the rank function $\mathrm{r}_M(X) =$ the number of vertices minus the number of components induced by X.

Only few matroids are graphic, but all *graphic ones are vector matroids* over any field.

Example:



Matroid operations

Rank of X $\sim$ matrix rank, or the number of vertices minus the number of components induced by X in graphs.

Matroid duality M^* (exchanging bases with their complements) $\sim$ topological duality in planar graphs, or transposition of standard-form matrices (i.e. without some basis).

Matroid element deletion $\sim$ usual deletion of a graph edge or a vector.

Matroid element contraction (corresponds to deletion in the dual matroid) $\sim$ edge contraction in a graph, or projection of the matroid from a vector (i.e. a linear transformation having a kernel formed by this vector).

Matroid minor — obtained by a sequence of element deletions and contractions, order of which does not matter.

3 MSO Theories

MSOL – monadic second-order logic:
propositional + quantification over elements and sets.

MSOL + class of structures $\Longrightarrow$ **MSO theory** (of the structures).

For graphs

- *Adjacency graphs* – formed by vertices and an adjacency relation.
  $\rightarrow$ MS_1 theory.
- *Incidence graphs* – formed by vertices and edges (two-sorted structure), with an incidence relation.
  $\rightarrow$ MS_2 theory.

Decidability of theories

A theory $\mathrm{Th}_L(\mathcal{S})$ – language L (=MSO) applied to a class $\mathcal{S}$ of structures.

Input: A sentence $\phi \in L$ (closed formula).

Question: Is $\mathcal{S} \models \phi$? In other words, is ϕ true for all structures $S \in \mathcal{S}$?

This problem algorithmically solvable $\leftrightarrow$ $\mathrm{Th}_L(\mathcal{S})$ *decidable*.

MSO Theory of Matroids

A **matroid** *in logic* – the ground set $E = E(M)$ with all subsets 2^E,
– and a predicate *indep* on 2^E, s.t. *indep*(F) iff $F \subseteq E$ is independent.

The *MSO theory of matroids* – language of MSOL applied to such matroids.

Basic expressions:

- *basis*$(B) \equiv$ *indep*$(B) \wedge \forall D(B \not\subseteq D \vee B = D \vee \neg$ *indep*$(D))$
  A basis is a maximal independent set.

- *circuit*$(C) \equiv \neg$ *indep*$(C) \wedge \forall D(D \not\subseteq C \vee D = C \vee$ *indep*$(D))$
  A circuit C is dependent, but all proper subsets of C are independent.

- *cocircuit*$(C) \equiv \forall B[$ *basis*$(B) \rightarrow \exists x(x \in B \wedge x \in C)] \wedge$
  $\wedge \forall X[X \not\subseteq C \vee X = C \vee \exists B($ *basis*$(B) \wedge \forall x(x \notin B \vee x \notin X))]$
  A cocircuit C (a dual circuit) intersects every basis, but each proper subset of C is disjoint from some basis.

Theorem 3.1. (PH) *Any MS_2 sentence about a graph G can be formulated as a (matroidal) MSO sentence about the cycle matroid $M(G \uplus K_3)$.*

($G \uplus K_3$, adding three vertices adjacent to everything, makes a 3-*connected graph*.)

Computability and decidability on matroids

Graph or matroid *branch-width* – like tree-width, being "close to a tree".

A matroid M of bounded branch-width $\rightarrow$ a *parse tree* $\bar{T}$ for $M = P(\bar{T})$.

Theorem 3.2. (PH) *Let $\mathbb{F}$ be a finite field, $t \geq 1$, and ϕ be a sentence in MSOL of matroids. Then there exists a (constructible) finite tree automaton $\mathcal{A}_t^\phi$ accepting those parse trees for matroids which posses ϕ, i.e. those $\bar{T}$ such that $P(\bar{T}) \models \phi$.*

This result, together with an algorithm constructing the parse tree, provides an efficient way to verify MSO-definable properties over matroids of bounded branch-width.

Corollary 3.3. *If $\mathcal{B}_t$ is the class of all matroids representable over $\mathbb{F}$ of branch-width at most t, then the theory $\mathrm{Th}_{MSO}(\mathcal{B}_t)$ is decidable.*

Sketch: It is enough to verify emptiness of the complementary automaton $\neg\mathcal{A}_t^\phi$ over all valid parse trees.

Our main result reads:

Theorem 3.4. *Let $\mathbb{F}$ be a finite field, and let $\mathcal{N}$ be a class of matroids that are representable by matrices over $\mathbb{F}$. If the monadic second-order theory $\mathrm{Th}_{MSO}(\mathcal{N})$ is decidable, then the class $\mathcal{N}$ has bounded branch-width.*

(Analogous to a result of [Seese, 1991] on the MS_2 theory of graphs.)

4 Interpretation of Theories

...the way to prove (un)decidability of logic theories.

An interpretation I of a theory $\mathrm{Th}_L(\mathcal{K})$ in a theory $\mathrm{Th}_{L'}(\mathcal{K}')$ is as follows:

$$\begin{array}{ccc} \forall\ \varphi \in L & \xrightarrow{\quad I \quad} & \varphi^I \in L' \\ \forall\ H \in \mathcal{K} & & G \in \mathcal{K}' \\ & & \\ H \simeq G^I & \xleftarrow{\quad I \quad} & \forall\ G \end{array}$$

We are using the results of:

Theorem 4.1. (Rabin) *If* $\mathrm{Th}_L(\mathcal{K})$ *is interpretable in* $\mathrm{Th}_{L'}(\mathcal{K}')$*, then undecidability of* $\mathrm{Th}_L(\mathcal{K})$ *implies undecidability of* $\mathrm{Th}_{L'}(\mathcal{K}')$*.*

Theorem 4.2. (Seese) *Let* $\mathcal{K}$ *be a class of adjacency graphs such that for every integer* $m > 1$ *there is a graph* $G \in \mathcal{K}$ *such that* G *has the* $m \times m$ *grid* Q_m *as an induced subgraph. Then the* MS_1 *theory of* $\mathcal{K}$ *is undecidable.*

Theorem 4.3. (Geelen, Gerards, Whittle) *For every finite field* $\mathbb{F}$*; a class* $\mathcal{N}$ *of* $\mathbb{F}$*-representable matroids has bounded branch-width if and only if there exists a constant* m *such that no matroid* $N \in \mathcal{N}$ *has a minor isomorphic to* $M(Q_m)$*.*

Hence, we interpret the MS_1 theory of grids in the matroidal MSO theory of $\mathcal{N}$...

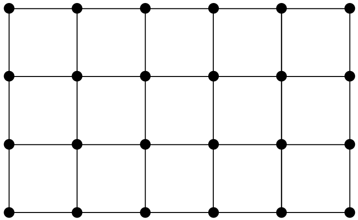
Sketch of the interpretation proof

The seemingly straightforward way – interpreting graphs ($\sim$large grids) in their cycle matroids, gets stuck due to technical details.
(Definability of the underlying graphs, insufficient connectivity, ...)

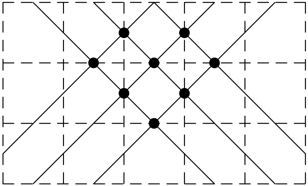
Our approach to *Theorem 3.4* ("decidability $\Rightarrow$ bounded branch-width"):

- If the matroid theory $\mathrm{Th}_{MSO}(\mathcal{N})$ is decidable, then so is the theory $\mathrm{Th}_{MSO}(\mathcal{N}_{\text{minor}})$ of all minors of the class $\mathcal{N}$
(i.e. minors are definable in matroid MSOL).

- By Theorem 4.3 [Geelen, Gerards, Whittle], the class $\mathcal{N}_{\text{minor}}$ contains arbitrary matroid grids $M(Q_m)$ if $\mathcal{N}$ has unbounded branch-width.



- A *4CC-graph of a matroid M* is the graph on the vertex set $E(M)$, with edges joining those pairs of elements which belong to a common 4-element circuit and a 4-element cocircuit (bond) in M.



The 4CC-graph of the grid matroid $M(Q_m)$ ($m \geq 6$ even) has an induced subgraph isomorphic to the grid Q_{m-2}.

- Let $\mathcal{F}_\mathcal{N}$ be the class of all 4CC-graphs of matroids in $\mathcal{N}$. Then the theory $\mathrm{Th}_{MS_1}(\mathcal{F}_\mathcal{N})$ is interpretable in the theory $\mathrm{Th}_{MSO}(\mathcal{N})$.
Hence by Theorem 4.1 [Rabin], undecidability of $\mathrm{Th}_{MS_1}(\mathcal{F}_\mathcal{N})$ implies undecidability of $\mathrm{Th}_{MSO}(\mathcal{N})$.

- Finally, Theorem 4.2 [Seese] states that $\mathrm{Th}_{MS_1}(\mathcal{F}_\mathcal{N})$ is undecidable if $\mathcal{F}_\mathcal{N}$, hence also if $\mathcal{N}$, contain large grids.

  Conversely, if $\mathrm{Th}_{MSO}(\mathcal{N})$ is decidable, then the previous (large grids) cannot be true, and so $\mathcal{N}$ has bounded branch-width. ∎