Petr Hliněný

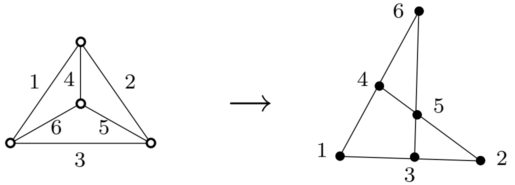
On the Computational Complexity of Matroid Minors

Department of Computer Science
FEI, VŠB – Technical University of Ostrava

and

Institute of Theoretical Computer Science
MFF, Charles University, Czech Republic

e-mail: petr.hlineny@vsb.cz
http://www.cs.vsb.cz/hlineny



2000 Math Subjects Classification: 05B35, 05C83, 68R05

1 Motivation

The Graph Minor Project 80's – 90's
[Robertson and Seymour, others later...]

- Proved *Wagner's conjecture* – WQO property of graph minors.
  (Among the partial steps: WQO of graphs of bounded tree-width, excluded grid theorem, description of graphs excluding a complete minor.)

- Testing for an arbitrary fixed graph *minor in cubic time*.

Extending to Matroids late 90's – future
[Geelen, Gerards, Robertson, Whittle, ...]

- WQO property of minors for matroids of bounded branch-width over a fixed finite field.

- "Excluded grid" theorem for matroid branch-width (without long lines).

- Geelen: Conjectured structure of finite-field representable matroids excluding a projective geometry minor.

- So, what is the *complexity* of testing for a fixed *matroid minor*?

2 Basics of Matroids

A matroid is a pair $M = (E, \mathcal{B})$ where

- $E = E(M)$ is the *ground set* of M (elements of M),
- $\mathcal{B} \subseteq 2^E$ is a collection of *bases* of M,
- the bases satisfy the "exchange axiom"
  $\forall B_1, B_2 \in \mathcal{B}$ and $\forall x \in B_1 - B_2$,
  $\exists y \in B_2 - B_1$ s.t. $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}$.

Otherwise, a *matroid* is a pair $M = (E, \mathcal{I})$ where

- $\mathcal{I} \subseteq 2^E$ is the collection of *independent sets* (subsets of bases) of M.

The definition was inspired by an abstract view of *independence* in linear algebra and in combinatorics [Whitney, Birkhoff, Tutte,. . .].

Notice exponential amount of information carried by a matroid.

Literature: J. Oxley, Matroid Theory, Oxford University Press 1992,1997.

Some **elementary matroid terms** are

- *independent set* = a subset of some basis,
  dependent set = not independent,
- *circuit* = minimal dependent set of elements,
  triangle = circuit on 3 elements,
- *hyperplane* = maximal set containing no basis,
- *rank function* $\mathrm{r}_M(X)$ = maximal size of an M-independent subset $I_X \subseteq X$ ("*dimension*" of X).

(Notation is taken from linear algebra and from graph theory...)

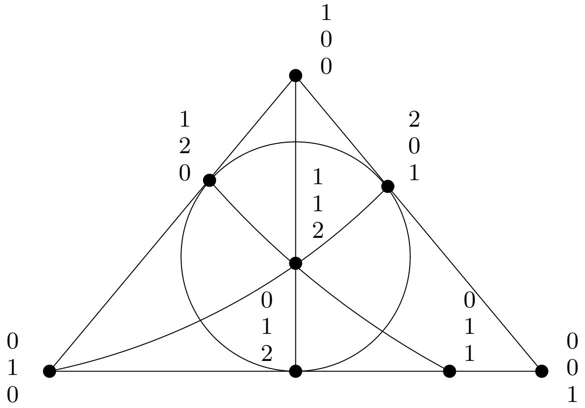
Axiomatic descriptions of matroids via independent sets, circuits, hyperplanes, or rank function are possible, and often used.

Vector matroid — a straightforward motivation:

- Elements are vectors over $\mathbb{F}$,
- independence is usual linear independence,
- the vectors are considered as columns of a matrix $\boldsymbol{A} \in \mathbb{F}^{r \times n}$.
  ($\boldsymbol{A}$ is called a *representation* of the matroid $M(\boldsymbol{A})$ over $\mathbb{F}$.)

Not all matroids are vector matroids.

An example of a rank-3 vector matroid with 8 elements over $GF(3)$:

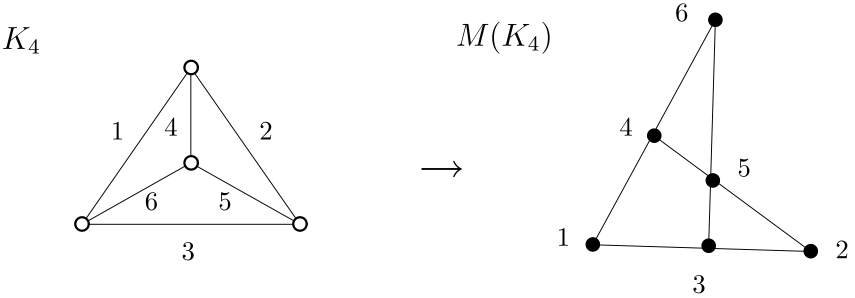


Graphic matroid $M(G)$ — the combinatorial link:

- Elements are the edges of a graph,
- independence $\sim$ acyclic edge subsets,
- bases $\sim$ spanning (maximal) forests,
- circuits $\sim$ graph cycles,
- the rank function $\mathrm{r}_M(X) =$ the number of vertices minus the number of components induced by X.

Only few matroids are graphic, but all *graphic ones are vector matroids* over any field.

Example:



Matroid operations

Rank of X $\sim$ matrix rank, or the number of vertices minus the number of components induced by X in graphs.

Matroid duality M^* (exchanging bases with their complements) $\sim$ topological duality in planar graphs, or transposition of standard-form matrices (i.e. without some basis).

Matroid element deletion $\sim$ usual deletion of a graph edge or a vector.

Matroid element contraction (corresponds to deletion in the dual matroid) $\sim$ edge contraction in a graph, or projection of the matroid from a vector (i.e. a linear transformation having a kernel formed by this vector).

Matroid minor — obtained by a sequence of element deletions and contractions, order of which does not matter.

3 Importance of Matroid Minors

Usual way to describe (characterize) matroid properties. . .

- [Tutte] Matroid is representable over $GF(2)$ iff it contains no $U_{2,4}$ (4-element line) as a minor.
- [Tutte] Matroid is graphic iff it contains no $U_{2,4}$, no F_7 (Fano plane), no F_7^*, $M(K_5)^*$, $M(K_{3,3})^*$ as a minor.
- [Tutte] Matroid is representable over any field iff it contains no $U_{2,4}$, F_7, F_7^* as a minor.
- [Bixby, Seymour] Matroid is representable over $GF(3)$ iff it contains no $U_{2,5}$ (5-element line), no $U_{3,5} = U_{2,5}^*$, F_7, F_7^* as a minor.
- [Geelen, Gerards, Kapoor] The excluded minors for matroid representability over $GF(4)$.
- [Geelen, Gerards, Whittle] Matroid representable over finite field has small branch-width iff it contains no matroid of a large grid graph as a minor. ("*Excluded grid*" theorem)

4 Complexity of Matroid Minors

We consider the following **matroid N-minor problem**:

Input. An ($\mathbb{F}$-represented) matroid M on n-elements.

Parameter. An arbitrary matroid N.

Question. Is N isomorphic to some minor of M?

(N arbitrary, but fixed, not part of the input!)

Remark. About matroids on an input:

To describe an n-element matroid, one has to specify properties of all 2^n subsets. So giving a complete description on the input would *ruin any complexity measures*.

Solutions:

Give a special matroid with a particular *small representation*. (Likewise a matrix for a vector matroid.)

Give a matroid via a *rank oracle* – answering queries about the rank.

Known Results

	N a planar matroid	N an arbitrary matroid
M is a graphic matroid	$O(n)$	$O(n^3)$
M an "abstract" matroid	$NPH\ (EXP)$	$NPH\ (EXP)$
M of bounded branch-width represented over finite field	$O(n^3)$	$O(n^3)$
M represented over finite field	$O(n^3)$	??
M of branch-width 3 represented over $\mathbb{Q}$	NPC	NPC

Finite field — $GF(q)$.

$\mathbb{Q}$ — rational numbers (holds also for other infinite fields).

Planar matroid — of a planar graph.

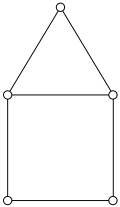
Small branch-width $\sim$ structured "almost" like a tree.

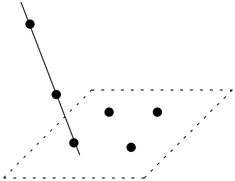
References

- About minors in graphic matroids:
  [Robertson and Seymour: Graph Minors],
  and [Bodlaender: A linear time algorithm for tree-width].
- About minors of small branch-width matroids:
  [PH: Recognizability of MSO-definable properties of representable matroids].
- About planar minors in matroids:
  [Geelen, Gerards, Whittle: An "Excluded grid" theorem for matroid branch-width].
- The new NP-hardness results:

Theorem 4.1. *Given an n-element $\mathbb{Q}$-represented matroid M of branch-width 3, it is NP-complete to decide whether M has a minor isomorphic to the (planar) cycle matroid $M(G_6)$.*

G_6

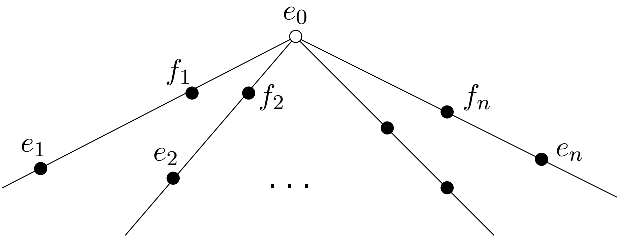




$M(G_6)$

5 Recognizing the Free Spikes

Definition. Let S_0 be a matroid circuit on $e_0, e_1, \ldots, e_n$, and S_1 an arbitrary simple matroid obtained from S_0 by adding n new elements f_i such that e_0, e_i, f_i are triangles. Then the matroid $S = S_1 \setminus e_0$ is a **rank-n spike**.



A typical matrix representation of a spike ($x_i \neq 1$):

$$
\begin{array}{ccccc c ccccc}
e_1 & e_2 & \ldots & e_{n-1} & e_n & & f_1 & f_2 & \ldots & f_{n-1} & f_n
\end{array}
$$

$$
\left[\begin{array}{ccccc c ccccc}
1 & 0 & \cdots & 0 & 0 & & x_1 & 1 & \cdots & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & & 1 & x_2 & 1 & 1 & 1 \\
\vdots & 0 & \ddots & 0 & \vdots & & \vdots & 1 & \ddots & 1 & \vdots \\
0 & 0 & 0 & 1 & 0 & & 1 & 1 & 1 & x_{n-1} & 1 \\
0 & 0 & \cdots & 0 & 1 & & 1 & 1 & \cdots & 1 & x_n
\end{array}\right]
$$

Sketch of proof of Theorem 4.1:

Lemma 5.1. (folklore) *Let S be a rank-n spike where $n \geq 3$. Then*

(a). the union of any two legs forms a 4-element circuit in S,

(b). every other circuit intersects all legs of S, and

(c). branch-width of S is 3.

Definition. The *free spike* is a spike having no unforced dependencies.

Theorem 5.2. *Let $n \geq 5$, and let S be a $\mathbb{Q}$-represented rank-n spike. Then it is NP-hard to recognize that S is not the free spike.*

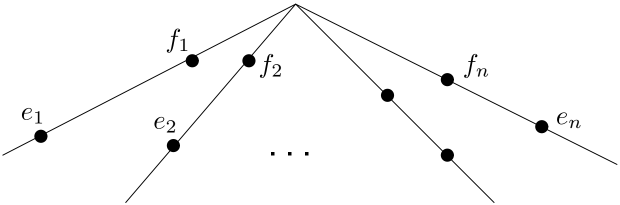
Matroid structure is determined by the subdeterminants of the reduced representation, in this case by subdeterminants of the following kind:

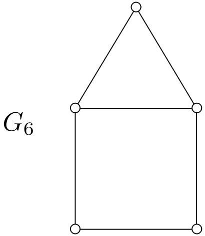
$$\{y_1, y_2, \ldots, y_k\} \subseteq \{x_1, x_2, \ldots, x_k\}\ , \qquad x_i \neq 1$$

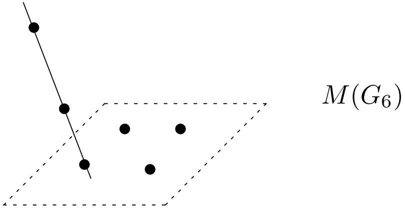
$$\begin{vmatrix} y_1 & 1 & \cdots & 1 \\ 1 & y_2 & \cdots & 1 \\ \vdots & & \ddots & \vdots \\ 1 & 1 & \cdots & y_k \end{vmatrix} = \begin{vmatrix} y_1 & 1 & \cdots & 1 \\ 1-y_1 & y_2-1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 1-y1 & 0 & \cdots & y_k-1 \end{vmatrix} = \left(\prod_{i=1}^{k}(y_i-1)\right)\cdot\left(1-\sum_{i=1}^{k}\frac{1}{1-y_i}\right)$$

Lemma 5.3. *Let S be a rank-n spike for $n \geq 5$ that. Then S is not the free spike iff S has an $M(G_6)$-minor.*

Use one of the leg cycles to get the triangle, and one of the extra dependencies to get the quadrangle. . .







. . . . . .