Some Recent Additions to Matroid Tree-Width

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1 TREE-WIDTH - an Overview

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- “bags” (subsets) of vertices at the tree nodes,
- each edge of $G$ belongs to some bag, and
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**Tree-width** = $\min_{\text{decompositions of } G} \max \{|B| - 1 : B \text{ bag in decomp.}\}$
Alternative traditional definition

- The tree-width of $G$ equals the smallest possible clique number minus one of a chordal supergraph of $G$. 
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- The tree-width of $G$ equals the smallest possible clique number minus one of a chordal supergraph of $G$.
- This can be much easier understood via $k$-trees, see e.g. a 2-tree:

![Diagram of $k$-trees](image)

[Beineke & Pippert, 68 – 69], [Rose 74], [Arnborg & Proskurowski, 86].

- A graph $G$ has tree-width $\leq k$ iff $G$ is a partial (subgraph of a) $k$-tree.
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- Logic side:
  - Decidability of **MSO theories** of the graphs of bounded tree-width [Courcelle 88]; a converse by [Seese 91].
2 “Vertex-free” Tree-Decompositions

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- **Node with of** $x = |V(G)| + (d - 1) \cdot c(G) - \sum_{i=1}^{d} c(G - F_i)$, where $F_i$ are the edges mapped to the subtrees $T - x$, and $c()$ denotes the number of components.
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where $F_i$ are the edges mapped to the subtrees $T - x$, and $c()$ denotes the number of components.

**VF Tree-width** = min_{decompositions of $G$} max \{ node-width in decomp. \}. 

---

**Diagram:**

- $E \rightarrow F_1 \rightarrow x \rightarrow T_1 \rightarrow T_2 \rightarrow F_2 \rightarrow T_3 \rightarrow F_3$
- Node with of $x = |V(G)| + (d - 1) \cdot c(G) - \sum_{i=1}^{d} c(G - F_i)$,
Are these two parameters really the same?
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Check the following examples for an illustration…

\[
\text{node-with of } x = |V(G)| + (d - 1) \cdot c(G) - \sum_{i=1}^{d} c(G - F_i)
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Where this idea comes from?

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**Definition:** A tree-decomposition of a matroid $M$ is a tree $T$ with

- an arbitrary $\tau : E(M) \to V(T)$, without further restrictions.

\[
\begin{align*}
E &\rightarrow \quad T_1 \quad F_2 \quad T_2 \\
F_1 \quad T_1 \quad x \quad T_3 &\rightarrow \quad F_3
\end{align*}
\]

- **Node with of** $x = \sum_{i=1}^{d} r(M \setminus F_i) - (d - 1) \cdot r(M)$,

where $r()$ denotes the matroid rank ("dimension").
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$T_1 \quad T_2 \quad T_3$\
$F_1 \quad F_2 \quad F_3$

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(M) **Tree-width** = $\min_{\text{decomps. of } M} \max \{ \text{node-width in decomp.} \}$.

- BTW, if a matroid $M$ has tree-width $k$ and branch-width $b$ (which readily extends to matroids), then $b - 1 \leq k \leq \max(2b - 1, 1)$ — that is nice...
Comparing the tree-width parameters

**Theorem** [PH & Whittle, 03]. Let a graph $G$ has an edge, and $M$ be the cycle matroid of $G$. Then the tree-width of $G$ equals the tree-width of $M$. 
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- An equality between the above node-width formulas for graphs and matroids is easy to show.
- For vector matroids, a tree-decomposition has a nice “visualization” with
  - affine *subspaces* modelling the traditional “bags”,
  - with *dimension* in place of bag size, and an *interpolation* property.
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- For vector matroids, a tree-decomposition has a nice “visualization” with

  - affine *subspaces* modelling the traditional “bags”,
  
  - with *dimension* in place of bag size, and an *interpolation* property.

- An ordinary tree-decomposition can be *readily translated* into a VF tree-decomposition; just find a bag hosting each edge of $G$. 
3 From one Decomposition to Another

• Where we stand?
  – The VF tree-width is at most the ordinary tree-width; since an ordinary tree-decomposition naturally translates to a VF tree-decomposition of at most the same width.
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• What happens in the converse direction?
  – Again, any VF tree-decomposition naturally translates into an ordinary decomposition (just apply the interpolation property to the ends of mapped edges).
  – However, the width may increase (dramatically)!

The problem is that edges mapped to a branch in the decomposition may induce a disconnected subgraph, hence further decreasing the node-width in the VF setting.

\[
\text{node-with of } x = |V(G)| + (d - 1) \cdot c(G) - \sum_{i=1}^{d} c(G - F_i)
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An example of a “disconnected” decomposition

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Easy to check that all six nodes in this VF tree-decomposition have width 4.
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\text{node-with formula} = |V(G)| + (d - 1) \cdot c(G) - \sum_{i=1}^{d} c(G - F_i)
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Easy to check that all six nodes in this VF tree-decomposition have width 4. However, the central two nodes induce bags of size 9 in an ordinary tree-decomposition! (tree-width up to 8)
Handling a “disconnected” decomposition

• If we want to get an ordinary tree-decomposition of the same width, we have to alter “disconnected” spots of a VF tree-decomposition...

• Actually, the proof complications appear similar to those emerging when proving equality of matroid branch-width to graph branch-width [Hicks & McMurray, 07], [Mazoit & Thomassé]. (No short proof of this statement is known so far.)
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• In response to that, [PH & Whittle, 08] have got an updated, though longer proof.

  We sketch its idea next...
Proof (altering a “disconnected” edge of a VF tree-decomposition $T$ of $G$).

- We assume an edge $e = uv$ of $T$ such that the $G$-edges mapped to the $u$-branch of $T$ form a disconnected subgraph of $G$, and that the edges mapped to the branches of $u$-neighbours (not $v$) stay connected in $G$. 
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- After all, there is a “strictly decreasing” sequence of alterations, leading to the connected case in which both tree-width measures are equal.
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  – We have seen that there exist VF tree-decompositions which do not easily translate to ordinary decompositions of the same width . . .
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• Bringing more properties of graph tree-width to matroids [Azzato 08] . . .
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THANK YOU FOR ATTENTION