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Matroid Tree-Width and Chordality

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1 Introduction

A **matroid** M on E is a set system $\mathcal{B} \subseteq 2^E$ of *bases*, sat. the exchange axiom

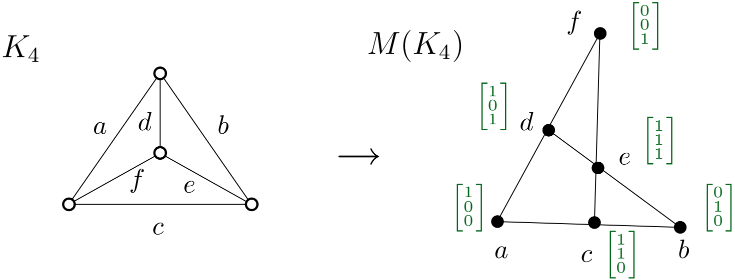
$$\forall B_1, B_2 \in \mathcal{B} \text{ a } \forall x \in B_1 - B_2,\ \exists y \in B_2 - B_1 :\ (B_1 - \{x\}) \cup \{y\} \in \mathcal{B}\,.$$

The subsets of bases are called *independent*.

Representations by graphs and vectors

Cycle matroid of a graph $M(G)$ – on the edges of G, where acyclic sets are independent.

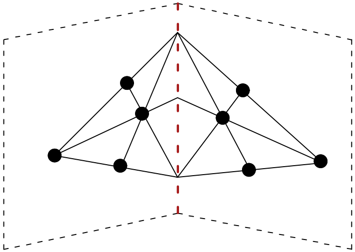
Vector matroid of a matrix $M(\boldsymbol{A})$ – on the (column) vectors of $\boldsymbol{A}$, with usual linear independence.



Matroid rank

The *rank function* of a matroid M is $\mathrm{r}_M : 2^{E(M)} \to \mathbb{N}$ where

$$\mathrm{r}_M(X) = \max \{|I| : \text{ independent } I \subseteq X\} .$$



Connectivity

The *connectivity function* of M is $\lambda_M : 2^{E(M)} \to \mathbb{N}$ where

$$\lambda_M(X) = \mathrm{r}_M(X) + \mathrm{r}_M(E - X) - \mathrm{r}(M) + 1 .$$

Geometrically, $\lambda_M(X)$ is the rank of the intersection of the spans of X and $E - X$ plus 1.

(In graphs, $\lambda_G(X)$ equals the number of vertices shared between X and $E - X$.)

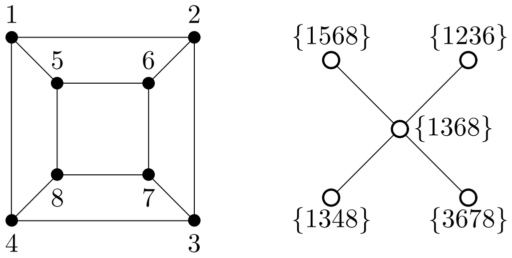
2 Matroid TREE-WIDTH

– Introduced [Robertson + Seymour, 80's] – the "Graph minor" project.

Definition: Tree *decomposition* of a graph G

- "bags" (subsets) of vertices at the tree nodes,
- each edge of G belongs to some bag, and
- the bags containing some vertex form a subtree (interpolation).

Tree-width $= \min_{\text{decomposition } G} \max\{|B|-1 : B \text{ bag in a decomp.}\}$.



– Alternative tree-width definitions; some even before R+S...
  (For ex. by linear ordering of vertices, cf. simplicial decomposition.)

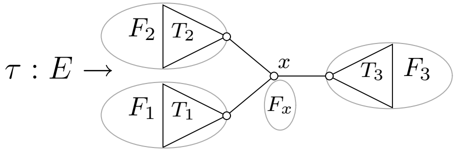
– The notion appears in many areas and relations, mainly algorithmic.
  In parametrized computation – linear-time FPT [Bodlaender, 96].

A "vertex-free" definition

– Proposed by [PH + Whittle, 03].

A tree *edge decomposition* of a graph G

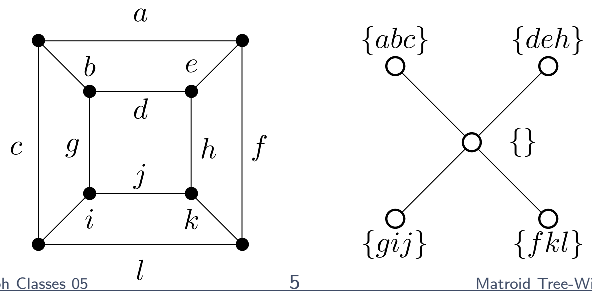
– arbitrary $\tau : E(G) \to V(T)$, without further restrictions.



Node with of x $= |V(G)| + (d-1)\cdot c(G) - \sum_{i=1}^{d} c(G - F_i)$,

where F_i mapped to the comps. of $T-x$, and $c()$ the number of comps.

VF Tree-width $= \min_{\text{decomposition } G} \max(\text{node width in a decomp.})$.

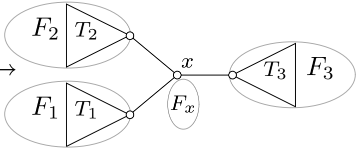


2.1 A Definition on Matroids

– Introduced [PH + Whittle, 03] (following [Geelen, unpublished]).

Tree *decomposition* of a matroid M

– arbitrary $\tau : E(M) \rightarrow V(T)$, without further restrictions.

$\tau : E \longrightarrow$ 

Node width of x $= \sum_{i=1}^{d} \mathrm{r}_M \left(E(M) - F_i\right) - (d-1) \cdot \mathrm{r}(M)$.

(M) Tree-width $=$ $\min_{\text{decomposition } M}$ $\max$ (node width in a decomp.).

Theorem [PH + Whittle, 03]. If a matroid M has tree-width k and branch-width b, then $b - 1 \leq k \leq \max(2b - 1, 1)$.

Theorem [PH + Whittle, 03]. Let a graph G has an edge, and $M = M(G)$ be the cycle matroid. Then the tree-width of G equals the tree-width of M.

So our VF tree-width definition seems OK. . .

Computing matroid tree-width

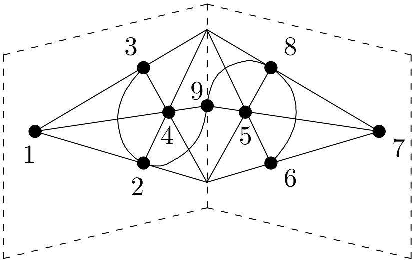
Is tree-width a "*good*" structural parameter?
(Can we compute the tree-width and a corresponding decomposition if bounded?)

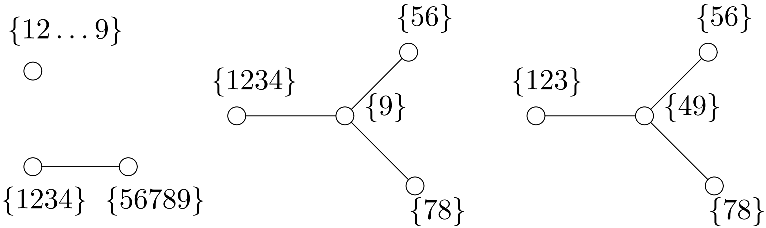
Theorem [PH, 03]. Computing the tree-width of a matroid represented over a finite field is FPT in $O(n^3)$.

A sketch:
tree-width $\rightarrow$ branch-width of the matroid $\rightarrow$ solved in FPT $O(n^3)$ by [PH, 02] $\rightarrow$ test the excluded minors for small tree-width.

(A decomposition is computed only approximately...)

Examples of matroid tree decompositions





widths: 4,3 3 4

3 Matroid CHORDALITY

Simplicial cocircuit $\sim$ simplicial vertex:

— $B \subseteq E(M)$ such that
$M \restriction \mathrm{cl}_M(B) \simeq PG(r,q)$ (a projective geometry over $GF(q)$)

(Cliques $\sim$ projective geometries)

Matroid k-trees

- Defined over a finite field $GF(q)$.
- Formed from
  - rank-$\leq k$ projective geometries over $GF(q)$ by means of
  - direct sums,
  - "adding" simplicial cocircuits of rank $\leq k$.

Theorem 3.1. *A simple $GF(q)$-representable matroid M has tree-width at most k if and only if M is a restriction of a k-tree matroid over $GF(q)$.*

(Enough to use: cliques $\sim$ non-vertically-separable sets)

3.1 Defining chordal matroids ???

We want to extend choral graphs to matroids in a nice way that "interplays" with matroid tree-width and k-trees...

- The span of every circuit containing another element?
  - uniform matroids $U_{m,n}$ do not look like "chordal".

3.2 Defining chordal matroids ???

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- The span of every circuit containing another element?
  - uniform matroids $U_{m,n}$ do not look like "chordal".
- The span of every circuit giving a triangle?
  - is the Fano matroid F_7 chordal?
    Yes, over $GF(2)$. But what over larger fields?

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- The span of every circuit giving a triangle?
  - is the Fano matroid F_7 chordal?
    Yes, over $GF(2)$. But what over larger fields?
  - however, what about free swirls? No.
- Combining the previous with a "density" condition?
  - to obtain simplicial cocircuits over a fixed finite field $GF(q)$.

Proposal – Superchordal Matroids

Chordality

(S1) For every circuit C in M; if $e \in C$, then there are two other elements in the closure $f, f' \in \mathrm{cl}_M(C)$ such that $\{e, f, f'\}$ is a triangle in M.

Density

(S2) If two elements e, f of M are not vertically separated, then the closure of e, f induces a $(q+1)$-line; $M \restriction \mathrm{cl}_M(e, f) \simeq PG(1, q) \simeq U_{2,q+1}$.

Conjecture

A simple $GF(q)$-representable matroid M is superchordal over $GF(q)$ if and only if M is a $\leq k$-tree matroid over $GF(q)$.

4 Conclusions

- Shown that a matroidal (geometric) view can bring new and interesting notions and properties of ordinary graphs – cf. *VF tree-width*.
- Brought up the question of properly defining *matroid chordality*, with "good relations" to tree-width. (If possible. . .)
- Our research tries to contribute to (nowadays popular) extensions of the Graph-Minor project from graphs to matroids. . .
- A remark on another interesting question – is the branch-width of a graph equal to the branch-width of its matroid?

- And finally – **MACEK** [PH 01–05],
  a software tool for practical structural computations with matroids:

  `http://www.cs.vsb.cz/hlineny/MACEK`

  (Now with a new online interface - TRY IT yourself!)