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Matroid Tree-Width and Chordality

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1 Introduction

A matroid $M$ on $E$ is a set system $\mathcal{B} \subseteq 2^E$ of bases, sat. the exchange axiom

$$\forall B_1, B_2 \in \mathcal{B} \; \forall x \in B_1 - B_2, \; \exists y \in B_2 - B_1 : (B_1 - \{x\}) \cup \{y\} \in \mathcal{B}.$$ 

The subsets of bases are called independent.

Representations by graphs and vectors

Cycle matroid of a graph $M(G)$ – on the edges of $G$, where acyclic sets are independent.

Vector matroid of a matrix $M(A)$ – on the (column) vectors of $A$, with usual linear independence.

\[\begin{array}{ccc}
& & a \\
& d & \\
& & b \\
\end{array}\]  

$K_4$  

\[\begin{array}{ccc}
a & f & e \\
d & & b \\
c & & \\
f & & e \\
\end{array}\]  

$M(K_4)$  

\[\begin{array}{ccc}
a & d & f \\
\end{array}\]  

$\begin{bmatrix}
1 \\
0 \\
\end{bmatrix}$  

$\begin{bmatrix}
1 \\
0 \\
\end{bmatrix}$  

$\begin{bmatrix}
1 \\
1 \\
1 \\
\end{bmatrix}$  

$\begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}$  

$\begin{bmatrix}
0 \\
0 \\
1 \\
\end{bmatrix}$
Matroid rank

The *rank function* of a matroid $M$ is $r_M : 2^{E(M)} \to \mathbb{N}$ where

$$r_M(X) = \max \{|I| : \text{independent } I \subseteq X\}.$$ 

Connectivity

The *connectivity function* of $M$ is $\lambda_M : 2^{E(M)} \to \mathbb{N}$ where

$$\lambda_M(X) = r_M(X) + r_M(E - X) - r(M) + 1.$$ 

Geometrically, $\lambda_M(X)$ is the rank of the intersection of the spans of $X$ and $E - X$ plus 1.

(In graphs, $\lambda_G(X)$ equals the number of vertices shared between $X$ and $E - X$.)
2 Matroid TREE-WIDTH

- Introduced [Robertson + Seymour, 80’s] – the “Graph minor” project.

**Definition:** Tree *decomposition* of a graph $G$

- “bags” (subsets) of vertices at the tree nodes,
- each edge of $G$ belongs to some bag, and
- the bags containing some vertex form a subtree (interpolation).

**Tree-width** = $\min_{\text{decomposition}} \max_G \{|B| - 1 : B \text{ bag in a decomp.}\}$.

- Alternative tree-width definitions; some even before R+S…
  (For ex. by linear ordering of vertices, cf. simplicial decomposition.)
- The notion appears in many areas and relations, mainly algorithmic.
  In parametrized computation – linear-time FPT [Bodlaender, 96].
A “vertex-free” definition

– Proposed by [PH + Whittle, 03].

A tree edge decomposition of a graph $G$

– arbitrary $\tau : E(G) \to V(T)$, without further restrictions.

\[
\tau : E \to T_1 \bigtriangledown T_2 \bigtriangledown T_3
\]

$T_1 \bigtriangledown T_2 \bigtriangledown T_3$:

\[
F_2 \bigtriangleup T_2 \bigtriangleup x \bigtriangleup T_3 \bigtriangleup F_3
\]

\[
F_1 \bigtriangleup T_1 \bigtriangleup F_x
\]

Node with of $x = |V(G)| + (d - 1) \cdot c(G) - \sum_{i=1}^{d} c(G - F_i),$

where $F_i$ mapped to the comps. of $T - x$, and $c(\cdot)$ the number of comps.

**VF Tree-width** = $\min_{\text{decomposition } G} \max (\text{node width in a decomp.}).
2.1 A Definition on Matroids

- Introduced [PH + Whittle, 03] (following [Geelen, unpublished]).

Tree *decomposition* of a matroid $M$
- arbitrary $\tau : E(M) \rightarrow V(T)$, *without* further restrictions.

$$
\tau : E \rightarrow \begin{array}{c}
F_2 \\
T_2 \\
F_1 \\
T_1
\end{array} \\
\begin{array}{c}
F_x \\
T_3 \\
F_3
\end{array}
$$

Node width of $x = \sum_{i=1}^{d} r_M (E(M) - F_i) - (d - 1) \cdot r(M)$.

**(M) Tree-width** = \(\min_{\text{decomposition}} M \ max(\text{node width in a decomp.})\).

**Theorem** [PH + Whittle, 03]. If a matroid $M$ has tree-width $k$ and branch-width $b$, then $b - 1 \leq k \leq \max(2b - 1, 1)$.

**Theorem** [PH + Whittle, 03]. Let a graph $G$ has an edge, and $M = M(G)$ be the cycle matroid. Then the tree-width of $G$ equals the tree-width of $M$.

So our VF tree-width definition seems OK...
Computing matroid tree-width

Is tree-width a “good” structural parameter?  
(Can we compute the tree-width and a corresponding decomposition if bounded?)

**Theorem** [PH, 03]. Computing the tree-width of a matroid represented over a finite field is **FPT** in $O(n^3)$.

A sketch:  
tree-width $\rightarrow$ branch-width of the matroid $\rightarrow$ solved in FPT $O(n^3)$ by [PH, 02]  
$\rightarrow$ test the excluded minors for small tree-width.

(A decomposition is computed only approximately...)
Examples of matroid tree decompositions

\[ \{12\ldots9\} \]
\[ \{1234\} \quad \{56789\} \]

widths: 4, 3

3

\[ \{12\ldots9\} \]
\[ \{1234\} \quad \{56\} \]
\[ \{123\} \quad \{49\} \]
\[ \{56\} \]

\[ \{78\} \]

4
3 Matroid CHORDALITY

Simplicial cocircuit ~ simplicial vertex:

— $B \subseteq E(M)$ such that

$M \upharpoonright \text{cl}_M(B) \cong PG(r, q)$ (a projective geometry over $GF(q)$)

(Clique ~ projective geometries)

Matroid $k$-trees

- Defined over a finite field $GF(q)$.

- Formed from
  - rank-\(\leq k\) projective geometries over $GF(q)$ by means of
  - direct sums,
  - "adding" simplicial cocircuits of rank $\leq k$.

Theorem 3.1. A simple $GF(q)$-representable matroid $M$ has tree-width at most $k$ if and only if $M$ is a restriction of a $k$-tree matroid over $GF(q)$.

(Enough to use: cliques ~ non-vertically-separable sets)
3.1 Defining chordal matroids

We want to extend choral graphs to matroids in a nice way that “interplays” with matroid tree-width and $k$-trees...

- The span of every circuit containing another element?
  - uniform matroids $U_{m,n}$ do not look like “chordal”.
3.2 Defining chordal matroids

We want to extend choral graphs to matroids in a nice way that “interplays” with matroid tree-width and \( k \)-trees...

- The span of every circuit containing another element?
  - uniform matroids \( U_{m,n} \) do not look like “chordal”.

- The span of every circuit giving a triangle?
  - is the Fano matroid \( F_7 \) chordal?
    Yes, over \( GF(2) \). But what over larger fields?
3.3 Defining chordal matroids ???

We want to extend choral graphs to matroids in a nice way that “interplays” with matroid tree-width and $k$-trees...

- The span of every circuit containing another element?
  - uniform matroids $U_{m,n}$ do not look like “chordal”.

- The span of every circuit giving a triangle?
  - is the Fano matroid $F_7$ chordal?
    Yes, over $GF(2)$. But what over larger fields?
  - however, what about free swirls? No.

- Combining the previous with a “density” condition?
  - to obtain simplicial cocircuits over a fixed finite field $GF(q)$. 
Proposal – Superchordal Matroids

Chordality

(S1) For every circuit $C$ in $M$; if $e \in C$, then there are two other elements in the closure $f, f' \in \text{cl}_M(C)$ such that $\{e, f, f'\}$ is a triangle in $M$.

Density

(S2) If two elements $e, f$ of $M$ are not vertically separated, then the closure of $e, f$ induces a $(q + 1)$-line; $M \upharpoonright \text{cl}_M(e, f) \simeq PG(1, q) \simeq U_{2,q+1}$.

Conjecture

A simple $GF(q)$-representable matroid $M$ is superchordal over $GF(q)$ if and only if $M$ is a $\leq k$-tree matroid over $GF(q)$. 
4 Conclusions

- Shown that a matroidal (geometric) view can bring new and interesting notions and properties of ordinary graphs – cf. *VF tree-width*.

- Brought up the question of properly defining *matroid chordality*, with “good relations” to tree-width. (If possible...)

- Our research tries to contribute to (nowadays popular) extensions of the Graph-Minor project from graphs to matroids...

- A remark on another interesting question – is the branch-width of a graph equal to the branch-width of its matroid?

- And finally – **MACEK** [PH 01–05], a software tool for practical structural computations with matroids:

  http://www.cs.vsb.cz/hlineny/MACEK

  (Now with a new online interface - TRY IT yourself!)