Matroid Tree-Width

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Abstract. We show that the tree-width of a graph can be defined without reference to graph vertices, and hence the notion of tree-width can be naturally extended to matroids. (This extension was inspired by an original unpublished idea of Jim Geelen.) We prove that the tree-width of a graphic matroid is equal to that of its underlying graph. Furthermore, we extend the well-known relation between the branch-width and the tree-width of a graph to all matroids.

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1 Introduction

In their fundamental work on graph minors [13], Robertson and Seymour introduced two notions of width for graphs, namely *tree-width* and *branch-width*. While the two are qualitatively the same in that a class of graphs has bounded tree-width if and only if it has bounded branch-width, it is undoubtedly tree-width that has proved to be a more popular notion, with many important applications in both graph theory and theoretical computer science. For an overview of these applications see for example [1, 2].

On the other hand, for matroid theorists, it is branch-width that has proved to be the more useful. This is because, unlike tree-width, branch-width extends directly to matroids. Moreover, in recent years a number of interesting matroid-structure results analogous to parts of the graph-minors project [13] have been found, for example [3, 6]. Also the great success of applying graph tree-width in algorithm design has been extended to algorithmic results on representable matroids. See for example [9, 7, 10]. All of those results make essential use of matroid branch-width.

Given this, it is natural to ask if tree-width can also be extended to matroids. It is by no means immediately obvious that this can be done as the definition of

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graph tree-width makes considerable use of the vertices of a graph. However, Jim Geelen [private communication] observed that it is possible to define a notion of tree-width without explicit reference to the vertices, and via this, it is also possible to extend the definition of graph tree-width to matroids.

In this paper we set forth a (somewhat modified) version of Geelen's ideas. The main result of the paper, Theorem 3.2 proves that graph tree-width and matroid tree-width are the same in that, if G is a graph, then the tree-width of G is equal to that of its cycle matroid $M(G)$. In particular, this result immediately provides a "vertex-free" definition for the (classical) graph tree-width, cf. Theorem 2.1. It is interesting to note that the analogous question for branch-width is still open in that it is not known whether the branch-width of a graph and its cycle matroid are the same. That appears to be quite a difficult problem.

We also present some basic results connecting matroid tree-width and branch-width. We prove in Theorem 4.2, that if M is a matroid of tree-width k and branch-width b, then $b - 1 \leq k \leq \max(2b - 1, 1)$. This means that a class of matroids has bounded branch-width if and only if it has bounded tree-width, and it follows that the main results of [3, 6, 9, 7, 10] hold with tree-width replacing branch-width.

2 Definitions of Tree-Width

We begin by recalling the traditional definition of graph tree-width.

Let G be a graph. A *tree-decomposition* of G is a pair (T, β), where T is a tree and $\beta : V(T) \to 2^{V(G)}$ is a mapping that satisfies the following:

- For each edge $e = uv \in E(G)$, there is $x \in V(T)$ such that $\{u, v\} \subseteq \beta(x)$.
- (IP) If $x \in V(T)$, and if $y, z \in V(T)$ are two nodes in distinct components of $T - x$, then $\beta(y) \cap \beta(z) \subseteq \beta(x)$.
- $\bigcup_{x \in V(T)} \beta(x) = V(G)$.

The width of the decomposition (T, β) of G equals the maximal value of $|\beta(x)| - 1$ over all $x \in V(T)$. The smallest width over all tree-decompositions of the graph G is the *tree-width* of G.

The vertex subsets $\beta(x) \subseteq V(G)$ for $x \in V(T)$ are called *bags*. The condition (IP) is called an *interpolation property*. We say that a decomposition is *optimal* if its width equals the tree-width. Note that the third condition is implied by the first two, unless G has isolated vertices. An example of a tree-decomposition is given in Figure 1.

It is obvious that in the above definition of tree-width, vertex sets (the bags) play an important role. To find a definition that extends to matroids, i.e. avoids a direct reference to vertices at all, we proceed as follows.

A *VF-tree-decomposition* of a graph G is a pair (T, τ), where T is a tree, and $\tau : E(G) \to V(T)$ is an arbitrary mapping of edges to the tree nodes. (The shortcut VF refers to "vertex-free".) For a node x of T, denote the connected components of $T - x$ by $T_1, \ldots, T_d$ and set $F_i = \tau^{-1}\big(V(T_i)\big)$. (See in Fig. 2.)

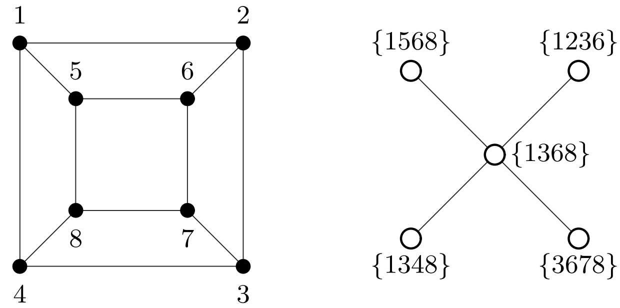

Fig. 1. An example of a tree-decomposition of the cube graph of width 3, where the vertex bags are listed at the tree nodes.

The *node-width* of x is defined by

$$|V(G)| + (d-1) \cdot c(G) - \sum_{i=1}^{d} c(G - F_i)\,,$$

where $c(H)$ denotes the number of components of a graph H. The *width* of the decomposition (T, τ) is the maximal width over all the nodes of T, and the smallest width over all tree-decompositions of G is the *VF-tree-width* of G. (The width of an empty tree T is 0.)

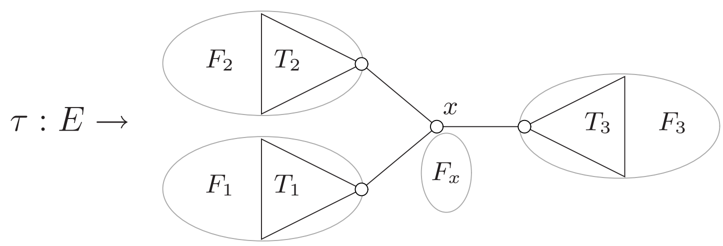

Fig. 2. An illustration to the definition of a VF-tree-decomposition.

At first glance, it may seem surprising that this definition has anything in common with the above traditional definition. It is important to note that the mapping τ is *not* an analogue of the bag mapping β above. Instead, τ replaces the first condition of a tree-decomposition, and the second condition (IP) is "embedded" inside the formula for node-width. (Notice that some edges F_x of G may be mapped to the node x, Fig. 2, and so they appear in the above formula for node-width only implicitly.)

A VF-tree-decomposition is illustrated in Figure 3. Simply speaking, in order to obtain a VF-tree-decomposition of small width, the edges in each single branch of every node should cut off as many new components of the graph as possible.

We now formulate our main result, which is then restated in a matroidal formulation as Theorem 3.2 and proved in Section 5.

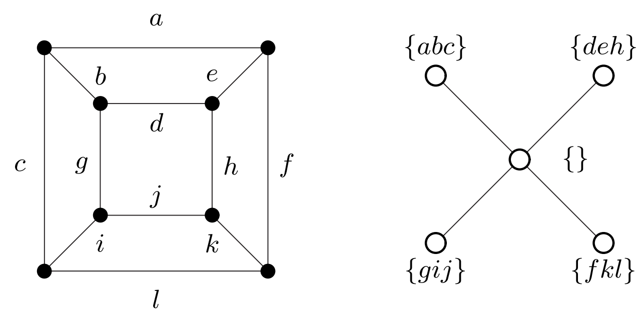

Fig. 3. An example of a VF-tree-decomposition of the cube graph of width 3, where the images of edges under τ are listed at the tree nodes.

Theorem 2.1. *The tree-width of a graph G equals the VF-tree-width of G.*

3 Tree-Width in Matroids

We assume that the reader is familiar with the basics of matroid theory. Our notation follows Oxley [12]. For convenience we briefly recall that the *rank* $\mathrm{r}_M(X)$ of a set X of elements of a matroid equals the maximal cardinality of an independent subset of X, and $\mathrm{r}(M)$ denotes the rank of $E(M)$, the ground set of M. Our matroidal definition of tree-width follows.

Let M be a matroid on the ground set $E = E(M)$. A pair (T, τ), where T is a tree and $\tau : E \to V(T)$ is an arbitrary mapping, is called a *tree-decomposition* of M. For a node x of T, denote the connected components of $T - x$ by $T_1, \dots, T_d$ and set $F_i = \tau^{-1}\big(V(T_i)\big) \subseteq E$ (Fig. 2). We define the *node-width* of x by

$$\text{(MW)} \qquad \sum_{i=1}^{d} \mathrm{r}_M\left(E - F_i\right) \; - (d-1) \cdot \mathrm{r}(M)\,,$$

and the *width* of the decomposition (T, τ) as the maximal width over all the nodes of T. The smallest width over all tree-decompositions of M is the *tree-width* of M.

Some matroid tree-decompositions are illustrated in Figure 4. To assist the reader's understanding of matroid tree-width, we note the following view of our definition. The node-width of x in the above definition can be rewritten as

$$\text{(MW}'\text{)} \qquad \mathrm{r}(M) - \sum_{i=1}^{d} \Big[\, \mathrm{r}(M) - \mathrm{r}_M\left(E - F_i\right) \Big]\,.$$

For a set F of elements of M, the *rank defect* of F is given by $\mathrm{r}(M) - \mathrm{r}_M(E - F)$. The width of a node x is smaller than the rank of M by the sum of rank defects of the sets consisting of the elements in each of the branches of x in the decomposition. So when looking for an optimal tree-decomposition, we want to "maximize rank defects of the branches" at each node.

The following is an easy exercise with the definition of a tree-width.

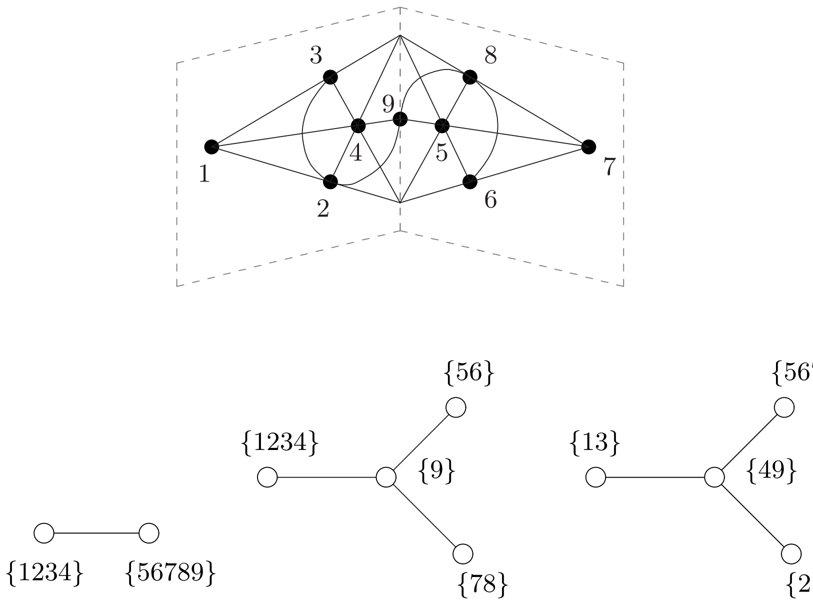

Fig. 4. An example of three optimal tree-decompositions of width 3 of the depicted 9-element matroid, where the elements mapped by τ are listed at the tree nodes. (The solid lines in the matroid picture show dependencies between matroid elements.)

Proposition 3.1. *Let N be a minor of a matroid M. Then the tree-width of N is not larger than that of M.*

Proof. Suppose that $N = M \setminus e$ where e is not a coloop. Then $\mathrm{r}(N) = \mathrm{r}(M)$ and hence the value of node-width (MW) cannot increase after deleting e. Suppose that $N = M/e$ where e is not a loop. Assuming $e \not\in F_2, \dots, F_d$, we rewrite the node-width formula (MW) as $\mathrm{r}_M\left(E - F_1\right) - \sum_{i=2}^{d} \left[\mathrm{r}(M) - \mathrm{r}_M\left(E - F_i\right)\right]$. Since $\mathrm{r}(M) - \mathrm{r}_M\left(E - F_i\right) = \mathrm{r}(N) - \mathrm{r}_N\left(E - \{e\} - F_i\right)$, the node-width cannot increase after contracting e in M, too. ∎

Recall that the *cycle matroid* $M(G)$ of a graph G has the edges of G as the ground set, and the independent subsets are those inducing no cycle in G. Hence the rank $\mathrm{r}_{M(G)}(X)$ equals the size of a spanning forest in the subgraph $H_X \restriction X$ induced by the edges X, i.e. $\mathrm{r}_{M(G)}(X) = |V(H_X)| - c(H_X)$. (Notice that it does not matter whether H_X includes also vertices of G which are isolated wrt. X.) Our main theorem states that the definition of matroid tree-width directly extends the graph tree-width notion.

Theorem 3.2. *Let G be a graph with at least one edge, and let $M = M(G)$ be the cycle matroid of G. Then the tree-width of G equals the tree-width of M.*

The proof is not easy, and so we postpone it till Section 5. However, the relation of matroid tree-width to VF-tree-width, and an equivalence of Theorem 3.2 with Theorem 2.1, are straightforward:

Proposition 3.3. *Let G be a graph with at least one edge, and let M be the cycle matroid of G. Then the VF-tree-width of G equals the tree-width of M.*

Proof. Let $M = M(G)$, and $E = E(G) = E(M)$. We use exactly the same tree-decomposition for the both graphic and matroidal variants of tree-width, and we equivalently rewrite the formula for node-width:

$$\sum_{i=1}^{d} \mathrm{r}_M\,(E - F_i) - (d-1)\cdot \mathrm{r}(M) = \sum_{i=1}^{d} \big(|V(G)| - c(G - F_i)\big) - (d-1)\cdot\big(|V(G)| - c(G)\big) =$$

$$= |V(G)| + (d-1)\cdot c(G) - \sum_{i=1}^{d} c(G - F_i)$$

∎

4 Comparing to Branch-Width

Branch-width is far less known than tree-width despite the fact that branch-width has the attractive property that it can be extended to all structures possessing a reasonable measure of connectivity. We make this idea precise now. Let E be a finite set and λ be an integer-valued function defined on subsets of E. Then, following [5] we say that λ is a *connectivity function* if

1. $\lambda(X) = \lambda(E - X)$ for each $X \subseteq E$ (symmetric), and
2. $\lambda(X) + \lambda(Y) \geq \lambda(X \cap Y) + \lambda(X \cup Y)$ (submodular).

We note that both graphs and matroids have natural connectivity functions. For a graph G, we define the function λ_G on subsets of the edges of G as $\lambda_G(F) = |U|$, where $U \subseteq V(G)$ is the subset of vertices incident both with edges in F and edges in $E(G) - F$. It is easily seen that λ_G is a connectivity function in the sense defined above and we say that λ_G is the *connectivity function of* G. For a matroid M, the *connectivity function* λ_M of M is defined by

$$\lambda_M(X) = \mathrm{r}_M(X) + \mathrm{r}_M\big(E(M) - X\big) - \mathrm{r}(M) + 1$$

for all $X \subseteq E(M)$. Again, it is easily seen that λ_M is a connectivity function. If M is represented as a set of vectors in a vector space, then the geometric meaning of matroid connectivity is as follows: A subset $X \subseteq E(M)$ spans a subspace $\langle X \rangle$. The subspace $\langle X \rangle \cap \langle E(M) - X \rangle$ has rank $\lambda_M(X) - 1$.

The connectivity functions of a graph and its associated cycle matroid are related in the sense that $\lambda_G(F) = \lambda_{M(G)}(F)$ provided that both subgraphs of G induced by the edge sets F and $E(G) - F$ are connected.

A tree is *cubic* if every node has degree 1 or 3. Let λ be a connectivity function on ground set E. A *branch-decomposition* of λ is a pair (U, ω) where U is a cubic tree, and ω is a bijection of E onto the leaves of U. For an edge e of the tree U, denote by U_e one of the connected components of $U - e$, and by L_e the set of leaves of U_e. We define the *width* of e as $\lambda\big(\omega^{-1}(L_e)\big)$ (note symmetry of λ), and the *width* of the decomposition (U,ω) as the largest width over all edges of U. The smallest width over all branch-decompositions of λ is the *branch-width* of λ.

The branch-width of a graph G equals the branch-width of its connectivity function λ_G, and the branch-width of a matroid M equals the branch-width of its connectivity function λ_M. See an illustration in Figure 5.

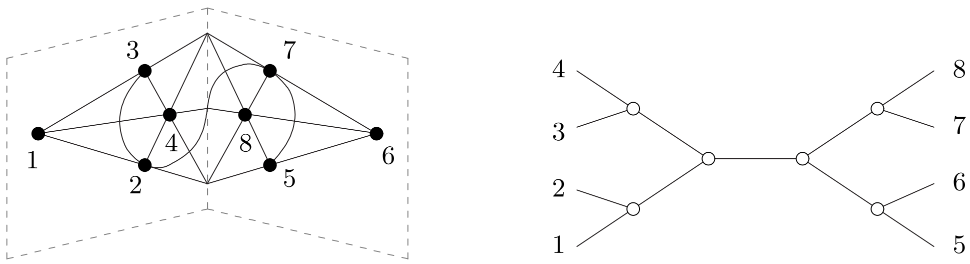

Fig. 5. An example of a branch-decomposition of width 3 of the depicted 8-element matroid (the binary affine cube).

As noted already, the tree-width and branch-width of a graph are closely related to each other. The following basic result is proved in [14]:

Theorem 4.1. *Let G be a graph of tree-width k and branch-width $b>1$. Then*

$$b-1\leq k\leq \left\lfloor \frac{3}{2}b\right\rfloor -1\,.$$

In order to justify our definition of matroid tree-width, we extend this result to all matroids. (Another reason to present the following straightforward proof is to demonstrate the new concept of tree-width in depth, before moving on with more difficult proofs in the next section.) Let us remark that it is not difficult to construct examples showing that the bounds in Theorem 4.2 are both sharp.

Theorem 4.2. *Let M be a matroid of tree-width k and branch-width b. Then*

$$b-1\leq k\leq \max\left(2b-2,1\right).$$

Proof. One direction is quite easy, since a branch-decomposition (U,ω) may be viewed as a tree-decomposition as well. Then the width of a leaf node of U is 1. For a non-leaf node $x\in V(U)$, we denote by U_1,U_2,U_3 the connected components of $U-x$, and by $F_i=\omega^{-1}\big(V(U_i)\big)$, $i=1,2,3$. Notice that $F_1\cup F_2\cup F_3=E(M)$ in this case. By definition the width of a node x equals

$$\mathrm{r}_M(F_1\cup F_2)+\mathrm{r}_M(F_1\cup F_3)+\mathrm{r}_M(F_2\cup F_3)-2\,\mathrm{r}(M)\leq$$

$$\leq \mathrm{r}_M(F_2)+\mathrm{r}_M(F_1\cup F_3)-\mathrm{r}(M)+\mathrm{r}_M(F_1)+\mathrm{r}_M(F_2\cup F_3)-\mathrm{r}(M)\leq$$

$$\leq \lambda_M(F_2) - 1 + \lambda_M(F_1) - 1 \leq 2(b-1)\,.$$

To prove the other direction, we have to modify the tree of an optimal tree-decomposition (T,τ) of M, so that elements of M are mapped to leaves of the new tree. Let T' be obtained from T by subdividing each edge with a new node. We construct a branch-decomposition (W,ω) of M from T' using the following local modifications at each node $x \in V(T)$ of degree d.

Let $y_1,\dots,y_d$ be the neighbours of x in T' (yes, not in T), let $Y = \{y_1,\dots,y_d\}$, and let $F_0 = \tau^{-1}(x)$. We define U_x to be a cubic tree with a set L of $d + |F_0|$ leaves, such that $Y \subseteq L$ and $U_x - Y$ is disjoint from all other U_y for $y \in V(T)$. Moreover, we define a restriction of a mapping ω on F_0 as an arbitrary bijection from F_0 to $L - Y$. Altogether, we define the tree $W' = \bigcup_{y\in V(T)} U_y$, and denote by W the cubic tree obtained from W' by contracting the degree-2 vertices created in T' above.

Claim 4.3. The pair (W,ω) defined above is a branch-decomposition of M of width at most $k+1$.

Proof. Let f be an edge of W incident with a subtree U_x for some $x \in V(T)$, and let W^1, W^2 be the connected components of $W - f$. Moreover, let $T_1,\dots,T_d$ be the connected components of $T - x$, and denote by $W_i = \bigcup_{y\in V(T_i)} U_y$ for $i = 1,\dots d$. (Hence W_i, $i = 1,\dots d$ are the connected components of $W' - V(U_x)$.) Notice that neither of W^1, W^2 intersects all of the subtrees $W_1,\dots,W_d$. So it follows that there are distinct $j_1, j_2 \in \{1,\dots,d\}$ such that W^i is disjoint from W_{j_i} for $i = 1,2$. We denote by $F^i = \omega^{-1}\big(V(W^i)\big)$ for $i = 1,2$, and by $F_i = \omega^{-1}\big(V(W_i)\big)$ for $i = 1,\dots,d$. Then the width of the edge f in (U,ω) can be estimated from above as follows

$$\lambda_M\left(F^1\right) = \mathrm{r}_M(F^1) + \mathrm{r}_M(F^2) - \mathrm{r}(M) + 1 \leq$$

$$\leq \mathrm{r}_M(E(M) - F_{j_1}) + \mathrm{r}_M(E(M) - F_{j_2}) - \mathrm{r}(M) + 1 \leq$$

$$\leq \sum_{i=1}^{d} \mathrm{r}_M(E(M) - F_i) - (d-2)\,\mathrm{r}(M) - \mathrm{r}(M) + 1 \leq k+1\,. \qquad \square$$

∎

5 Equivalence of the Tree-Widths

We now present an important geometric link that enables us to connect the traditional graph and new matroid tree-width definitions. Let q be a prime or a prime power. Recall that a *representation* of a matroid M is a matrix $\boldsymbol{A}$ over $GF(q)$ whose columns are labelled by elements of $E(M)$ with the property that a set of columns is linearly independent if and only if their labels form an independent set in M. Note that the matrix is really unnecessary — it is just a convenient device. It is really the *set* of column vectors that represents M. In this spirit we define a *point configuration* to be a labelled multiset of vectors in

$V(r, q)$. Then a point configuration P *represents* a matroid M if it is labelled by $E(M)$ and a subset X of P is linearly independent if and only if the labels of X are independent in M.

The following example, which is classical, illustrates the connection between graphic matroids and certain point configurations. Let $\boldsymbol{J}(G)$ denote the vertex-edge incidence matrix of an arbitrary simple graph G. Then the rank of $\boldsymbol{J}(G)$ over $GF(2)$ is $|V(G)| - c(G)$, where $c(G)$ denotes the number of connected components. It is well known that the columns of $\boldsymbol{J}(G)$ over $GF(2)$ represent the cycle matroid $M(G)$. This simple construction is illustrated in Figure 6.

Fig. 6. An example of a point configuration constructed from the vertex-edge incidence matrix of the graph K_4.

With this example in mind it is easy to see how the definition of graph tree-width may be extended to a definition of tree-width for point configurations. In a graph, a set of edges "span" a set of vertices; in a point configuration, a set of points span a subspace. The idea is to replace a bag consisting of a subset of vertices of a given cardinality by a bag consisting of a subspace of a given rank.

Definition. Let $P \subseteq PG(r, q)$ be a point configuration over $GF(q)$. A *tree-decomposition* of P is a pair (T, Σ), where T is a tree, and Σ is a mapping from $V(T)$ to subspaces of $PG(r, q)$, satisfying the following:

- For each $p \in P$ there is $x \in V(T)$ such that $p \in \Sigma(x)$.
- (IP′) If $v \in V(T)$, $e \in E(T)$ is an edge incident with v, and T_1, T_2 are connected components of $T - e$, then $\langle \Phi_1 \rangle \cap \langle \Phi_2 \rangle \subseteq \Sigma(v)$ where $\Phi_i = \bigcup_{x \in V(T_i)} \Sigma(x)$ for $i = 1, 2$.

The width of the decomposition (T, Σ) of P equals the maximal rank of a subspace $\Sigma(x)$ for $x \in V(T)$. The smallest width over all tree-decompositions of the point configuration P is the *tree-width* of P.

The relation of this definition to the traditional definition of graph tree-width is almost straightforward: The points of P are the edges of a graph, and the subspaces $\Sigma(x)$ correspond to the bags of graph vertices. The second condition (IP′) is a version of the interpolation property. The third condition is meaningless in a geometric setting. (Notice that, unlike for graphs, there is no "$|\cdot| - 1$" in the previous definition of tree-width. That is natural since the "rank" of an n-vertex set is actually $n-1$.) Yet, surprisingly, it seems that a natural short proof relating these two definitions works only in one of the directions:

Lemma 5.1. *Let G be a simple connected graph on at least 2 vertices, and let P be the point configuration given by the columns of the matrix $\boldsymbol{J}(G)$ over $GF(2)$. Then the tree-width of P is not larger than the tree-width of G.*

Proof. Notice that the sum of all rows of the matrix $\boldsymbol{J}(G)$ is the zero vector over $GF(2)$, and so all points of P belong to the hyperplane Ψ of $PG(n,2)$ which is orthogonal to the vector $\boldsymbol{1}$ of all ones. Let (T,β) be a tree-decomposition of the graph G, and let $\boldsymbol{u}_v$, for a vertex $v \in V(G)$, denote the unit vector with 1 as the entry in the row of v in the incidence matrix $\boldsymbol{J}(G)$. We define $\Sigma(x) = \big\langle \{\boldsymbol{u}_v : v \in \beta(x)\} \big\rangle \cap \Psi$ for each node $x \in V(T)$. Then the rank of $\Sigma(x)$ equals $|\beta(x)| - 1$ since none of $\boldsymbol{u}_v$ belongs to Ψ. It is now straightforward to verify that (T,Σ) is, indeed, a tree-decomposition of the point configuration P. ∎

As we see later, it is not an essential restriction that we consider only simple connected graphs. Next, we prove equality between the tree-widths of a point configuration and of the represented matroid in Lemma 5.2, and then we "get back" from a tree-decomposition of a graphic matroid to a tree-decomposition of the underlying graph in Lemma 5.4.

Lemma 5.2. *Let M be a simple $GF(q)$-representable matroid, and let P be a point configuration representing M over $GF(q)$. Then the tree-width of M is equal to the tree-width of P.*

Proof. Let (T,τ) be a tree-decomposition of M. For a node $x \in V(T)$, let $T_1,\dots,T_d$ be the connected components of $T - x$. We denote by $P_0 \subseteq P$ the points representing the elements of $F_0 = \tau^{-1}(x)$, and by $P_i \subseteq P$, $i = 1,\dots,d$, the points representing the elements of $F_i = \tau^{-1}\big(V(T_i)\big)$. We set

$$\text{(BS)} \qquad \Sigma(x) = \Psi = \left\langle P_0 \cup \bigcup\nolimits_{i=1}^{d} (\langle P_i\rangle \cap \langle P - P_i\rangle) \right\rangle .$$

Notice that the space $\Sigma(x)$ is spanned by points of P. It is now easy to see that (T,Σ) is a tree-decomposition of P.

Claim 5.3. For an arbitrary partition $(F_0,F_1,\dots,F_d)$ of $E(M)$, the corresponding partition $(P_0,P_1,\dots,P_d)$ of P, and the subspace Ψ defined as in (BS), the rank of Ψ equals

$$\eta_M(F_1,\dots,F_d;F_0) = \sum_{i=1}^{d} \mathrm{r}_M(E(M) - F_i) - (d-1)\,\mathrm{r}(M)\,.$$

Proof. We prove the claim by induction on d. For $d = 1$, $\Psi = \langle P_0\rangle$ and the rank of P_0 equals $\eta_M(F_1;F_0) = \mathrm{r}_M(F_0)$. For $d > 1$, we contract the set F_d in

M, which corresponds to contracting the subspace $\langle P_d\rangle$ in P. Then the rank of $\Psi/\langle P_d\rangle$ equals $\eta_{M/F_d}(F_1,\dots,F_{d-1};F_0)$ by the inductive assumption for $d-1$. So, for $E=E(M)$ and $E'=E-F_d$, the rank of Ψ equals

$$\sum_{i=1}^{d-1} \mathrm{r}_{M/F_d}(E'-F_i)-(d-2)\,\mathrm{r}(M/F_d)+r\big(\langle P_d\rangle\cap\langle P-P_d\rangle\big)=$$

$$=\sum_{i=1}^{d-1}\Big(\mathrm{r}_M(E-F_i)-\mathrm{r}_M(F_d)\Big)-(d-2)\Big(\mathrm{r}(M)-\mathrm{r}_M(F_d)\Big)+$$

$$+\,\mathrm{r}_M(F_d)+\mathrm{r}_M(E-F_d)-\mathrm{r}(M)=$$

$$=\sum_{i=1}^{d-1}\big(\mathrm{r}_M(E-F_i)-\mathrm{r}_M(F_d)\big)-(d-1)\,\mathrm{r}(M)+(d-1)\,\mathrm{r}_M(F_d)+\mathrm{r}_M(E-F_d)=$$

$$=\eta_M(F_1,\dots,F_d;F_0)\,. \qquad \square$$

We see that the width of (T,Σ) is the same as the width of (T,τ). Hence the tree-width of P is not larger than the tree-width of M.

Conversely, let (T,Σ') be a tree-decomposition of P. We define a function τ by letting $\tau(e)=x$ for $e\in E(M)$ and $x\in V(T)$, where x is chosen such that the point representing e in P belongs to $\Sigma'(x)$. Then (T,τ) is a tree-decomposition of M. We use (T,τ) and the above construction (BS) to define (another) tree-decomposition (T,Σ) of P. Clearly, $\Sigma(x)\subseteq\Sigma'(x)$ for all $x\in V(T)$, and hence the tree-width of M is not larger than the tree-width of P. ∎

Lemma 5.4. *Let G be a graph with at least one edge, and let $M=M(G)$ be the cycle matroid of G. Then the tree-width of G is not larger than the tree-width of M.*

Proof. Without loss of generality, we may assume that G is a connected simple graph. If G is not 2-connected, then we may compose its tree-decomposition from the tree-decompositions of its 2-connected components. So we may assume that G is 2-connected, and that M is connected. Moreover, $E(M)=E(G)$. The main problem is that we need a tree-decomposition of M in which the subtrees induce connected matroid restrictions. In some ideas this proof is similar to the proof of Lemma 5.2.

Let (T,τ) be a width-k tree-decomposition of the matroid M. We denote by n_i, $i\in[0,k]$ the number of vertices in T of width exactly i. For another width-k tree-decomposition (T',τ') of M, we analogously define n_i', $i\in[0,k]$. We say that (T,τ) is *lexicographically smaller* than (T',τ') if there is $j\in[0,k]$ such that $n_i=n_i'$ for $i\in[j+1,k]$ and $n_j<n_j'$. Let $e=xy\in E(T)$ be an edge of the decomposition (T,τ), and let us denote by T_1 the connected component of $T-e$ containing the node y. We say that the edge $e=xy$ in (T,τ) is disconnected at y if the restriction $M\restriction\tau^{-1}(V(T_1))$ is not connected. We say that the decomposition (T,τ) is *connected* if no edge $e=xy\in E(T)$ is disconnected at x or y.

Claim 5.5. Let (T, τ) be the lexicographically smallest optimal tree-decomposition of the matroid M. Then (T, τ) is connected.

Proof. Suppose not. So there is an edge $e = xy \in E(T)$ which is disconnected at y. We denote by T_1 the connected component of $T - e$ containing y, by $M_1 = M \restriction \tau^{-1}(V(T_1))$ the restriction of M, and by (F_1, F_2) a 1-separation of M_1 witnessing that M_1 is not connected. We define a new tree-decomposition (T', τ') as follows: Let T_1', T_1'' be two disjoint copies of T_1, and let $T' = \big(T - V(T_1)\big) \cup T_1' \cup T_1'' \cup \{xy', xy''\}$ where y', y'' are the corresponding copies of y in T_1', T_1''. For an element $f \in F_1$, we set $\tau'(f) = z'$ where z' is the node corresponding to $z = \tau(f)$ in T_1'. For an element $f \in F_2$, we set $\tau'(f) = z''$ where z'' is the node corresponding to $z = \tau(f)$ in T_1''. We set $\tau'(f) = \tau(f)$ for the remaining elements.

Informally speaking, we have split the branch T_1 in T into two branches according to the 1-separation (F_1, F_2). It is clear that the width of (T', τ') is not larger than the width of (T, τ). Moreover, (T', τ') is lexicographically smaller than (T, τ), which contradicts the assumption of Claim 5.5. □

For a tree-decomposition (T, τ) of M, we define a tree-decomposition (T, β) of the underlying graph G as follows: Let $\beta'(x) \subseteq V(G)$ for $x \in V(T)$ be the set of all endvertices of the edges from $\tau^{-1}(x)$. Let $\beta''(x) \subseteq V(G)$ be the minimal set containing all intersections $\beta'(z) \cap \beta'(z')$ where z, z' belong to distinct connected components of $T - x$ (cf. the interpolation property (IP)). We set $\beta(x) = \beta'(x) \cup \beta''(x)$ for $x \in V(T)$.

Claim 5.6. (T, β) is a tree-decomposition of G. If (T, τ) is a connected tree-decomposition of M, then the width of (T, β) equals the width of (T, τ).

Proof. The decomposition (T, β) clearly satisfies the tree-decomposition conditions from page 2. If $F \subseteq E(M)$ is a subset of M such that $M \restriction F$ is connected, then the edges of F in G induce a 2-connected subgraph on a vertex subset $X \subseteq V(G)$, and $\mathrm{r}_M(F) = |X| - 1$ since the matroid rank equals the size of a spanning forest.

For a node x of T, denote by $T_1, \ldots, T_d$ the connected components of $T - x$. (See the definition of matroid tree-width on page 4.) Let $X_0 = \beta(x)$, and $F_i = \tau^{-1}(V(T_i))$, $X_i = \bigcup_{z \in V(T_i)} \beta(z)$ for $i \in [1, d]$. By the definition of β and the connectivity assumption for (T, τ), we see that the edge set $E(G) - F_i$ for $i \in [1, d]$ induces a 2-connected subgraph of G on the vertex subset $X_0 \cup \bigcup_{j \in [1,d]-\{i\}} X_j$. Therefore

$$\sum_{i=1}^{d} \mathrm{r}_M \big(E(M) - F_i\big) - (d-1) \cdot \mathrm{r}(M) =$$

$$= \sum_{i=1}^{d} \Big(\big|X_0 \cup \bigcup\nolimits_{j \in [1,d]-\{i\}} X_j\big| - 1 \Big) - (d-1) \cdot \big(|V(G)| - 1\big) = |X_0| - 1 \,,$$

since the sets X_0 and $X_i - X_0$ for $i = 1, \ldots, d$ form a partition of $V(G)$. Hence the width of x in the graph decomposition (T, β) equals the width of x in the matroid decomposition (T, τ). □

The whole proof is now finished. ∎

We are ready to conclude the proof of our main result.

Proof of Theorem 3.2. If G is not simple, then any tree-decomposition of the simplification of G is also a tree-decomposition of G itself. If G is not connected, then the tree-width of G equals the largest tree-width over the connected components of G. Thus, without loss of generality, we may consider only simple connected graphs.

So, for a graph G, the point configuration P given by the incidence matrix $\boldsymbol{J}(G)$ over $GF(2)$, and the cycle matroid $M(G)$ represented by P, we get from Lemmas 5.1, 5.2, and 5.4

$$\mathrm{tw}(G) \geq \mathrm{tw}(P) = \mathrm{tw}(M) \geq \mathrm{tw}(G)\,,$$

where tw() denotes the tree-width in the respective definition. Hence an equality holds here. In particular, the statement is true even when the only edges in G are loops, in which case the tree-width of G is zero. ∎

6 Concluding Remarks

The main contribution of our paper lies in a novel way of defining tree-width which readily extends from graphs to matroids, and more generally to all structures where a "geometric" notion of rank can be found or defined. Namely, our new definition eliminates the need (often uncomfortable, but so far necessary) to refer to graph vertices when dealing with tree-width. Such an extension parallels the situation with branch-width which extends from graphs to all structures where a connectivity function can be defined.

In particular, we are able to define a robust notion of matrix tree-width that is invariant on standard matrix row operations, or in other words, invariant on the projective equivalence of point configurations. The hope is that matrices of small "width" are much easier to handle than general matrices, and that fast algorithms may exist for problems involving these matrices. The *tree-width* of a matrix $\boldsymbol{A}$ equals the tree-width of the matroid $M(\boldsymbol{A})$ represented by the columns of $\boldsymbol{A}$. Moreover, since matroid representations are in one-to-one correspondence with linear codes, we can also give a definition of tree-width for linear codes.

Let us mention that some authors have already used another "matrix tree-width" parameter defined as follows. For a matrix $\boldsymbol{A} = [a_{i,j}]_{i,j=1}^{n}$, let G_A be the graph on the vertex set $\{1, \dots, n\}$ and the edge set consisting of all $\{i, j\}$ such that $a_{i,j} \neq 0$ or $a_{j,i} \neq 0$. Then the graph tree-width of the matrix $\boldsymbol{A}$ is given by the tree-width of G_A. This definition was, perhaps, inspired by Choleski factorization of sparse symmetric matrices which is related to the graph tree-width of the matrix. (See [1] for more details.) However, such a notion of a matrix width is not robust in the above sense — applying a row operation to a matrix $\boldsymbol{A}$ may dramatically change the tree-width of G_A, while the geometric configuration represented by $\boldsymbol{A}$ is still the same.

Another remark and question concerns so called excluded-minor characterizations. It is proved in [4] that a matroid M of branch-width k, but with all proper minors of M of branch-width less than k (called an excluded minor for branch-width $< k$), has size at most $\left(6^{k+1}-1\right)/5$. Hence there is a finte number of them for each k. In future research it would be interesting to investigate whether an analogous statement holds also for matroid tree-width. A positive answer to that would imply, together with Theorem 4.2, a uniform FPT algorithm for finding the tree-width of a matroid represented by a matrix over a finite field [8].

We conclude the paper with a note on an interesting question of Seymour; whether tree-widths of a planar graph and of its dual differ by at most one. (An easy example that tree-width is not self-dual is given by the cube and the octahedron graphs.) This has been recently proved true by Lapoire [11]. A natural generalization would be to ask the following:

Problem 6.1. What is the maximal value of the tree-width difference $|\operatorname{tw}(M) - \operatorname{tw}(M^*)|$ among all matroid-dual pairs M, M^*?

It follows from Theorem 4.2 and self-duality of branch-width that the tree-widths of M and M^* are within a multiplicative constant of each other. So far, we have found no matroid-dual pair with tree-width difference greater than 1, but we have not tried hard yet.

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