Addendum to Matroid Tree-Width

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Abstract. Hliněný and Whittle have shown that the traditional tree-width notion of a graph can be defined without an explicit reference to vertices, and that it can be naturally extended to all matroids. Unfortunately their original paper *Matroid tree-width*, European J. Combin. 27 (2006), 1117–1128, as pointed out by Isolde Adler in 2007, contained some incorrect arguments. It is the purpose of this addendum to correct the affected proofs. (All the theorems and results of the original paper remain valid.)

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1 Introduction

In their fundamental work on graph minors, Robertson and Seymour introduced two notions of width for graphs [3], namely *tree-width* and *branch-width*. While the two are qualitatively the same in that a class of graphs has bounded tree-width if and only if it has bounded branch-width, it is undoubtedly tree-width that has proved to be a more popular notion. On the other hand, for matroid theorists, branch-width is the notion since it extends directly from graphs to matroids.

Given this, it is natural to ask if tree-width can also be extended to matroids. It is by no means immediately obvious that this can be done as the definition of graph tree-width makes considerable use of the vertices of a graph. However, Jim Geelen [unpublished] observed that such an extension could be possible. Hliněný and Whittle then proposed in [2] an alternative "matroidal" definition of tree-width. We set forth both these approaches in the next definitions.

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Definition ("Traditional tree-width" [3]).
A *tree-decomposition* of a graph G is a pair (T, β), where T is a tree and $\beta : V(T) \to 2^{V(G)}$ is a mapping (the "bags") that satisfies the following:

- For each edge $e = uv \in E(G)$, there is $x \in V(T)$ such that $\{u, v\} \subseteq \beta(x)$.
- If $x \in V(T)$, and if $y, z \in V(T)$ are two nodes in distinct components of $T - x$, then $\beta(y) \cap \beta(z) \subseteq \beta(x)$ ("interpolation").
- $\bigcup_{x \in V(T)} \beta(x) = V(G)$.

The width of (T, β) is the maximal value of $|\beta(x)| - 1$ over all $x \in V(T)$. The smallest width over all tree-decompositions of the graph G is the *tree-width* of G.

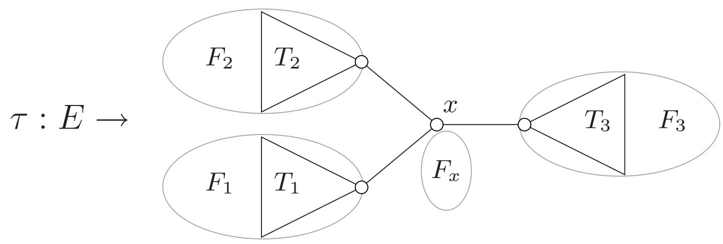

Fig. 1. An illustration of the definition of a matroid tree-decomposition.

Definition ("Matroid tree-width").

(a) A *VF-tree-decomposition* of a graph G is a pair (T, τ), where T is a tree, and $\tau : E(G) \to V(T)$ is an arbitrary mapping of edges to the tree nodes. (VF refers to "vertex-free", for a distinction from traditional tree-width.) For a node x of T, denote the connected components of $T - x$ by $T_1, \dots, T_d$ and set $F_i = \tau^{-1}\big(V(T_i)\big)$. (See in Fig. 1.) The *node-width* of x is defined by

$$|V(G)| + (d-1) \cdot c(G) - \sum_{i=1}^{d} c(G - F_i)\,, \tag{1}$$

where $c(H)$ denotes the number of connected components of a graph H.

(b) A *tree-decomposition* of a matroid M on the ground set $E = E(M)$ is a pair (T, τ) where T is a tree and $\tau : E \to V(T)$ is an arbitrary mapping. For a node x of T, denote the connected components of $T - x$ by $T_1, \dots, T_d$ and set $F_i = \tau^{-1}\big(V(T_i)\big) \subseteq E$. The *node-width* of x is given by

$$\sum_{i=1}^{d} \mathrm{r}_M\left(E - F_i\right) - (d-1) \cdot \mathrm{r}(M)\,. \tag{2}$$

The *width* of the decomposition (T, τ) is the maximal node-width over all the nodes of T, and the smallest width over all tree-decompositions of G or M is the *VF-tree-width* of G or the *tree-width* of M, respectively. The width of an empty tree T is 0.

A straightforward argument shows equivalence between (a) and (b).

Proposition 1.1 ([2, Proposition 3.3]). *Let G be a graph and $M(G)$ be the cycle matroid of G. For any $F_1, \ldots, F_d \subseteq E(G) \neq \emptyset$, the values of (1) and (2) are equal, and hence the VF-tree-width of G equals the tree-width of $M(G)$.*

One of the main results of our paper [2] asserts that "matroidal" VF-tree-width is the same as traditional tree-width on graphs.

Theorem 1.2. *The tree-width of a graph G equals the VF-tree-width of G.*

Regarding this statement, we note that there is a natural way of transforming a traditional tree-decomposition into a VF-tree-decomposition, and vice versa: For each edge e of G we may pick as $\tau(e)$ any of the nodes whose bag contains e, and conversely, we may form bags of the traditional definition from the ends of the mapped edges and some additional vertices to satisfy the interpolation property. The widths of these decomposition, however, are generally different, and hence this theorem requires a nontrivial proof.

Unfortunately, as pointed out [1] by Isolde Adler in 2007, our original paper [2] used some incorrect arguments supporting Theorem 1.2, namely wrong [2, Claim 5.5] (cf. Section 2). It is the purpose of this addendum to provide alternative correct arguments proving our theorems.

2 Correction of Lemma 5.4

The proof of Theorem 1.2 has two directions in view of Proposition 1.1. The easier direction, that the traditional tree-width of a graph G is not smaller than the tree-width of the cycle matroid $M(G)$ of G, has been rigorously proved in [2, Lemmas 5.1 and 5.2]. For the other direction, that the tree-width of a graph G is not bigger than the tree-width of $M(G)$, arguments have been provided in [2, Lemma 5.4]. Unfortunately, there in the proof a wrong intermediate claim appeared, as has been discovered and pointed to us by Adler [1].

To be specific; starting from a tree-decomposition of $M(G)$ or equivalently from a VF-tree-decomposition of G, there is the above sketched obvious translation of it into a traditional tree-decomposition of G. The question is whether the bag at each node of the latter decomposition is not bigger than the respective node-width of the former decomposition plus one. That (false in general) is true if we start from a decomposition possessing certain additional connectivity properties, as proved in [2, Claim 5.6], but preceding [2, Claim 5.5] which originally established the existence of such a decomposition, unfortunately does not hold.

We present an alternative proof for the above assertion in Theorem 2.5 along ideas similar to the original (flawed) one. The new proof is longer, though.

We start first with useful technical results about handling matroid tree-decompositions which did not explicitly appear in [2]. For $F \subseteq E(G)$ we denote by $G \restriction F$ the subgraph of G with edge set F and those vertices incident with edges from F (hence ignoring isolated vertices). To simplify our arguments, we introduce the following notation with respect to a tree-decomposition (T, τ): If e is an edge of T, then let T_e^1, T_e^2 denote the components of $T - e$. Analogously let T_v^i, $i = 1, \ldots, d$ denote the components of $T - v$ where v is a node of T of degree d. Let moreover $F_v^i = \tau^{-1}\big(V(T_v^i)\big)$ and $F_e^j = \tau^{-1}\big(V(T_e^j)\big)$, referring implicitly to the decomposition (T, τ) in consideration.

Proposition 2.1. *Consider a tree-decomposition (T, τ) of a matroid M.*
(a) If a tree T' is obtained by splitting a node x into two nodes x, x' (i.e. contracting xx' in T' gives T), then the width of (T', τ) is not larger than the width of (T, τ).
(b) Assume e is an edge of T, and $C \subsetneq F_e^2$ is a union of connected components of the matroid restriction $M \setminus F_e^1$. If τ' is obtained from τ by arbitrarily re-mapping the elements of C into the nodes of T_e^1, then the node-width of each node of T_e^2 in (T, τ') is not larger than its width in (T, τ).

Notice that, according to Proposition 1.1, we may also write this proposition in a special form suited for our later application to graphs.

Proposition 2.1′. *Consider a VF-tree-decomposition (T, τ) of a graph G.*
(a) If a tree T' is obtained by splitting a node x into two nodes x, x', then the width of (T', τ) is not larger than the width of (T, τ).
(b) Assume e is an edge of T, and $C \subsetneq F_e^2$ is a union of edge sets of some connected components of the graph $G \restriction F_e^2$. If τ' is obtained from τ by arbitrarily re-mapping the elements of C into the nodes of T_e^1, then the node-width of each node of T_e^2 in (T, τ') is not larger than its width in (T, τ).

It is, however, more natural to prove Proposition 2.1 in matroidal terms. For a matroid M and arbitrary subsets $F_1, \ldots, F_d$, $d \geq 2$ of its elements, let $\eta_M(F_1, \ldots, F_d) = \sum_{i=1}^{d} \mathrm{r}_M(E(M) - F_i) - (d-1)\,\mathrm{r}(M)$, cf. the node-width formula (2). Proposition 2.1 (a) follows by repeated application of the following:

Lemma 2.3. $\eta_M(F_1, F_2, F_3, \ldots, F_d) \geq \eta_M(F_1 \cup F_2, F_3, \ldots, F_d)$

Proof. By submodularity of the matroid rank function,

$$\eta_M(F_1, F_2, \ldots, F_d) = \sum_{i=1}^{d} \mathrm{r}_M(E(M) - F_i) - (d-1)\,\mathrm{r}(M) \geq$$

$$\geq \mathrm{r}_M\big(E(M) - (F_1 \cup F_2)\big) + \mathrm{r}(M) + \sum_{i=3}^{d} \mathrm{r}_M(E(M) - F_i) - (d-1)\,\mathrm{r}(M) =$$

$$= \eta_M(F_1 \cup F_2, F_3, \ldots, F_d)\,.$$ ∎

Proposition 2.1 (b), on the other hand, follows by an application of the next claim to each node of T_e^2 separately. Recall that $F_1, \ldots, F_d$, $d \geq 2$ are arbitrary subsets of elements of a matroid M.

Lemma 2.4. *Assume $C \subset E(M) - F_1$ is such that $\mathrm{r}_M(C) + \mathrm{r}_M(E - (F_1 \cup C)) = \mathrm{r}_M(E - F_1)$, i.e. C is "disconnected" in the matroid $M \setminus F_1$. Then $\eta_M(F_1, F_2, \ldots, F_d) \geq \eta_M(F_1 \cup C, F_2 - C, \ldots, F_d - C)$.*

Proof. Let $E = E(M)$. By the exchange axiom of matroids there exist independent sets $X_i \subseteq C \cap F_i$ such that it holds $\mathrm{r}_M\big((E - F_i) \cup X_i\big) = \mathrm{r}_M\big(E - (F_i - C)\big) = \mathrm{r}_M(E - F_i) + \mathrm{r}_M(X_i)$, for $i = 2, \ldots, d$. Now we can write

$$\eta_M(F_1, F_2, \ldots, F_d) - \eta_M(F_1 \cup C, F_2 - C, \ldots, F_d - C) =$$

$$= \mathrm{r}_M(E - F_1) - \mathrm{r}_M\big(E - (F_1 \cup C)\big) + \sum_{i=2}^{d} \big[\mathrm{r}_M(E - F_i) - \mathrm{r}_M\big(E - (F_i - C)\big)\big] =$$

$$= \mathrm{r}_M(C) + \sum_{i=2}^{d} \big[\mathrm{r}_M(E - F_i) - \mathrm{r}_M(E - F_i) - \mathrm{r}_M(X_i)\big] = \mathrm{r}_M(C) - \sum_{i=2}^{d} \mathrm{r}_M(X_i)\,.$$

Hence it remains to argue that $\mathrm{r}_M(X_2) + \cdots + \mathrm{r}_M(X_d) \leq \mathrm{r}_M(C)$, which immediately follows if $X_2 \cup \cdots \cup X_d$ is independent. The latter is a consequence of our assumption $\mathrm{r}_M\big((E - F_i) \cup X_i\big) = \mathrm{r}_M(E - F_i) + \mathrm{r}_M(X_i)$ since $E - F_i \supseteq X_2 \cup \cdots \cup X_{i-1} \cup X_{i+1} \cup \cdots \cup X_d$. ∎

Now we are ready for the main task—to repair the proof of [2, Lemma 5.4].

Theorem 2.5 ([2, Lemma 5.4]). *Let G be a graph with at least one edge. Then the tree-width of G is not larger than the VF-tree-width of G.*

Proof. Let (T, τ) be a VF-tree-decomposition of G. Without loss of generality, we may assume that G is a connected simple graph. We also recall the notation F_v^i and F_e^j with respect to (now fixed) (T, τ) from the beginning of this section.

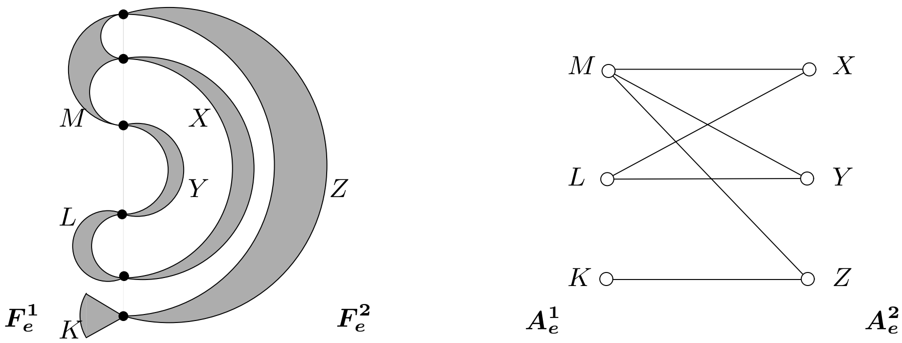

Fig. 2. An illustration of a bipartite component incidence graph (the connected components of $G \restriction F_e^1$ are K, L, M, and the components of $G \restriction F_e^2$ are X, Y, Z).

For any edge $e = v_2 v_2$ of T we define a bipartite *component incidence graph* J_e at e (Fig. 2): The parts A_e^1, A_e^2 of vertices of J_e are the connected components of $G \restriction F_e^1$ and of $G \restriction F_e^2$, respectively, and the edges of J_e are formed by those pairs of components sharing a vertex. Since G is connected, so is the graph J_e for every $e \in E(T)$. If the part A_e^1 has more than one vertex, then we say that the edge e of the decomposition *disconnects the graph G as from v_2* – the other end of e. We denote by k_j the number of edges e of T such that $|V(J_e)| = j$, and by s the largest index such that $k_s \neq 0$. Among all optimal VF-tree-decompositions of G we assume the one with the lexicographically smallest possible *component vector* $(s,\ k_s, k_{s-1}, \dots, k_3)$.

Our aim is to show that the selected decomposition (T, τ) must be *connected*, i.e. that no edge of T disconnects G as from either end. In other words, we aim at showing $s = 2$. Then, as straightforwardly proved in [2, Claim 5.6], there is a derived ordinary tree-decomposition of G of width equal to that of (T, τ). (Though [2, Claim 5.6] spoke about matroid connectivity, graph connectivity is enough in the proof.)

So, seeking a contradiction, we assume that $s > 2$. Since T is a tree, there is an edge $e = uv \in E(T)$ disconnecting G as from v, such that all other edges incident with node u in T do *not* disconnect G as from u. Let (up to symmetry) F_e^1 be the part of $E(G)$ mapped to the subtree of $T - e$ with root u, and denote by d the degree of u in T. Recall that $F_u^1, \dots, F_u^d$ denote the parts of $E(G)$ mapped into the components of $T - u$. We claim that, without loss of generality, one can assume the following:

(i) No element is mapped to u in (T, τ), i.e. $\tau^{-1}(u) = \emptyset$.
(ii) The connected components of $G \restriction F_e^1$ coincide with $F_u^1, \dots, F_u^{d-1}$. (Notice that also $F_u^d = F_e^2$, not necessarily connected.)

To show (i), see in the definition that creating a new leaf adjacent to u for each element in $\tau^{-1}(u)$ does not change the with of u and of the whole decomposition. Ad (ii), notice that no F_u^j may intersect two of the components of $G \restriction F_e^1$ since the edges from u other than e do not disconnect G. Hence each component is a union of some F_u^j's, and via applying Proposition 2.1 (a) we may assume that each component of $G \restriction F_e^1$ actually is a single part among $(F_u^1, \dots, F_u^{d-1})$.

As noted above, the bipartite component incidence graph J_e at e is connected. Recall that $A_e^1 \cup A_e^2 = V(J_e)$ are the vertex parts of J_e where A_e^1 is in correspondence with the u-end of e.

If A_e^2 has only one vertex, i.e. $G \restriction F_e^2$ is connected, then we make a new VF-tree-decomposition by contracting e in T. Denoting by h the degree of v in T, we can simply estimate the node-width of v in the new decomposition as

$$|V(G)| + (h + d - 2) - 1 - \sum_{i=1}^{d-1} c(G - F_u^i) - \sum_{i=1}^{h-1} c(G - F_v^i) =$$

$$= |V(G)| + h + d - 3 - \big[d - 2 + c\big(G - (F_u^1 \cup \dots \cup F_u^{d-1})\big)\big] - \sum_{i=1}^{h-1} c(G - F_v^i) =$$

$$= |V(G)| + h - 1 - c\big(G - F_v^h\big) - \sum_{i=1}^{h-1} c\big(G - F_v^i\big)\,, \tag{3}$$

which is the node-width of v in the former decomposition (T,τ). Hence we have found a new optimal VF-tree-decomposition of G having strictly smaller component vector. This contradiction to our least choice of (T,τ) finishes the proof in the particular case.

Hence A_e^2 has more than one vertex. We first consider the case that

(iii) no vertex of A_e^1 is a cutvertex of J_e.

See that $|A_e^1| \geq 2$. Since $G \restriction F_e^2$ is not connected in this case, we find an arbitrary nontrivial partition $F_e^2 = F_3 \cup F_4$ such that $G \restriction F_3$ is disjoint from $G \restriction F_4$, i.e. that F_3 is a union of some components of $G \restriction F_e^2$.

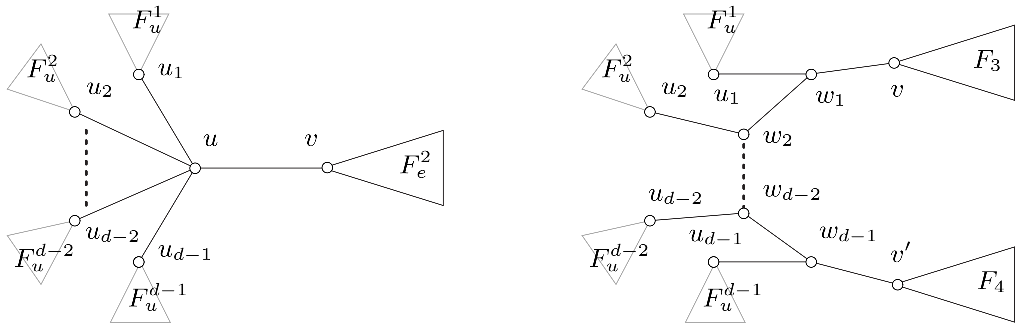

Fig. 3. How to modify a decomposition (T,τ) into new (T',τ') on the right.

Let $u_1, u_2, \ldots, u_{d-1}$, $u_d = v$ denote the neighbours of u in T. For $T_3 = T_e^2$ in T, we make T_4 a disjoint copy of T_3. Then we delete u from T, and for $i = 1, \ldots, d-1$ we add a new vertex w_i adjacent to u_i. We add an edge $w_1 v$, edges $w_i w_{i+1}$ for $i = 1, \ldots, d-2$, and an edge $w_{d-1}v'$ where v' is the copy of v in T_4. This results in a tree T', see Fig. 3. We define τ' as follows: If $x \in E(G) - F_4$, then $\tau'(x) = \tau(x)$. For $x \in F_4$, we set $\tau'(x) = t'$ where t' is the copy of $t = \tau(x) \in V(T_3)$ in the subtree T_4.

We again aim for a contradiction, showing that the width of (T',τ') is not larger than the width of (T,τ), and that the component vector decreases.

Claim 2.6. The width of (T',τ') is at most the width of (T,τ).

Proof. First of all, notice that Proposition 2.1 (b) is applicable to both subtrees T_3 and T_4 (as "copies of" T_e^2, for $C = F_4$ and $C = F_3$, respectively). So the node-widths of nodes of $T_3 \cup T_4$ in (T',τ') do not exceed the width of (T,τ).

It remains to argue about the node-width of w_j where $j = 1, 2, \ldots, d-1$. We denote by $U \subseteq V(G)$ the set of those vertices that are incident both with an edge of F_e^1 and an edge of F_e^2. Notice that by (ii) above, every vertex in $V(G) - U$ is counted exactly once in $\sum_{i=1}^{d} c(G - F_u^i)$ (as an "isolated" component). If we denote by $G \div F = G \restriction \big(E(G) - F\big)$, then we can write in (T, τ) by (1),

$$\text{node-width}(u) = |V(G)| + d - 1 - \sum_{i=1}^{d} c(G - F_u^i) =$$

$$= d - 1 + |U| - \sum_{i=1}^{d} c(G \div F_u^i) = d - 1 + |U| - \sum_{i=1}^{d-1} 1 - (d-1) = |U| - d + 1\,.$$

The previous equality is the only(!) place where we use the assumption (iii), to argue that $c(G \div F_u^i) = 1$ for $1 \le i < d$.

To compare the previous with the node-width of new w_j, $1 \le j < d$, we have to introduce some notation: Let $H_{a,b} = F_u^a \cup F_u^{a+1} \cup \dots \cup F_u^b$, and for $k = 3, 4$, let $\ell_{a,b}^{k}$ ($\ell_{a,b}^{-k}$) denote the number of those connected components of $G \restriction F_e^1$ that intersect (are disjoint from, respectively) $G \restriction F_k$. Then, by (1) in (T', τ'),

$$\text{node-width}(w_j) =$$

$$= |V(G)| + 2 - c(G - F_u^j) - c\big(G - (F_3 \cup H_{1,j-1})\big) - c\big(G - (F_4 \cup H_{j+1,d-1})\big) \le$$

$$\le 2 - 1 + |U| - \ell_{1,j-1}^{3} - c\big(G \div (F_3 \cup H_{1,j-1})\big) - \ell_{j+1,d-1}^{4} - c\big(G \div (F_4 \cup H_{j+1,d-1})\big) =$$

$$= 1 + |U| - \ell_{1,j-1}^{3} - (\ell_{j,d-1}^{-4} + 1) - \ell_{j+1,d-1}^{4} - (\ell_{1,j}^{-3} + 1) =$$

$$= -1 + |U| - (\ell_{1,j-1}^{3} + \ell_{1,j}^{-3}) - (\ell_{j+1,d-1}^{4} + \ell_{j,d-1}^{-4}) \le$$

$$\le -1 + |U| - (j-1) - (d-1-j) = |U| - d + 1\,.$$

Hence also the node-widths of new $w_1, \dots, w_{d-1}$ in (T', τ') are not larger than the node-width of former u in (T, τ). □

Claim 2.7. The component vector of (T', τ') is strictly lexicographically smaller than that of (T, τ).

Proof. Recall that J_e denotes the component incidence graph at an edge e of (T, τ). For distinction, we analogously denote by J'_e the component incidence graph at e of (T', τ'). If f is an edge of the subtree T_e^1 (the component of $T - e$), explicitly including also the case of f incident with u in T_e^1, then clearly $J'_f = J_f$.

Suppose an edge f of the subtree $T_3 = T_e^2$, and denote by f' the corresponding copy in T_4 (of T'). Since we have "split" the τ'-mapping of elements of $E(G)$ into T_3 and T_4 in a way that $G \restriction F_3$ is disjoint from $G \restriction F_4$, it holds $|V(J'_f)|, |V(J'_{f'})| < |V(J_f)|$, unless $J'_f = J_f$ and $J'_{f'}$ is trivial K_1, or vice versa. The same argument applies with strict inequality also to $e = uv$: $|V(J'_f)|, |V(J'_{f'})| < |V(J_e)|$ where $f = w_1 v$ and $f' = w_{d-1} v'$ correspond to e in T'.

Finally, since the order of $u_1, u_2, \dots, u_{d-1}$ has been irrelevant so far, we may assume without loss of generality that $G \restriction F_u^1$ is incident with $G \restriction F_3$ and that $G \restriction F_u^{d-1}$ is incident with $G \restriction F_4$. Then particularly, for $1 \le i \le d-2$,

$G \restriction F_u^1$ (the component representing a vertex of J_e) is not a separate component of $G \restriction (F_3 \cup H_{1,i})$. Therefore, it again holds $|V(J'_f)| < |V(J_e)|$ where $f = w_i w_{i+1}$.

Altogether, we see that the first changed entry of the component vector gets decreased, at least due to edge e which is in T' replaced by edges of strictly smaller numbers of components, and so the component vector of (T', τ') is strictly lexicographically smaller than that of (T, τ). □

We are left with the case:

¬(iii) There exists a cutvertex $r \in A_e^1$, representing a component R of $G \restriction F_e^1$.

In this case we partition $F_e^2 = F_3 \cup F_4$ such that $G \restriction F_3$ is disjoint from $G \restriction F_4$, and moreover, R is the only connected component of $G \restriction F_e^1$ incident both with $G \restriction F_3$ and $G \restriction F_4$. We again consider the decomposition (T', τ') defined above, see Fig. 3. Then Claim 2.7 applies here, too, since it does not rely on (iii). Unfortunately, Claim 2.6 cannot be used now, and we have to argue differently that the width of (T', τ') is at most the width of (T, τ).

We make a tree T'' from T' by contracting all $w_1, \ldots, w_{d-1}$ into a single vertex w. Analogously to (3) we show that the node-width of w in the decomposition (T'', τ') equals the node-width of former u in (T, τ):

$$|V(G)| + d + 1 - 1 - \sum_{i=1}^{d-1} c(G - F_u^i) \; - c(G - F_3) - c(G - F_4) =$$

$$= |V(G)| + d - \sum_{i=1}^{d-1} c(G - F_u^i) - c\big(G - (F_3 \cup F_4)\big) - 1 = |V(G)| + d - 1 - \sum_{i=1}^{d} c(G - F_u^i)$$

For all other nodes of T'' we argue analogously to Claim 2.6, i.e. referring Proposition 2.1 (b), that their node-widths do not exceed the width of (T, τ). See that Proposition 2.1 (a) is now enough to show that also all $w_1, \ldots, w_{d-1}$ in (T', τ') resulting by splitting of w have node-widths at most the width of (T, τ), and so we are done here.

Once again, we have got to a contradiction of the new optimal decomposition (T', τ') of G with the former least choice of (T, τ). The whole proof is now finished. ∎

3 Correction of Claim 4.3

There is yet another unfortunate small bug in our original paper [2] that has gone unnoticed so far: In the proof of [2, Claim 4.3], an "obvious" inequality was used in the wrong direction. Although this is not a serious problem, and a reader familiar with matroid theory could easily find the correct argument, we take an opportunity to clear out every detail in the addendum. We restate the affected statement and its complete proof now.[1]

[1] Although the correction is added in short Claim 3.3, we have to repeat the preceding arguments here because of their specific context and notation.

Theorem 3.1 ([2, Theorem 4.2]). *Let M be a matroid of tree-width k and branch-width b. Then* $b-1 \le k \le \max(2b-2, 1)$.

Proof. The (easier) right-hand inequality is proved as in [2].

To prove the left-hand inequality, we have to modify the tree of an optimal tree-decomposition (T, τ) of M, so that elements of M are mapped to leaves of a new subcubic tree. Let T' be obtained from T by subdividing each edge with a new node. We construct a branch-decomposition (W, ω) of M from T' using the following local modifications at each node $x \in V(T)$ of degree d:

– Let $Y = \{y_1, \dots, y_d\}$ be the set of neighbours of x in T' (yes, not in T), and let $F_0 = \tau^{-1}(x)$. We define U_x to be a cubic tree with a set L of $d + |F_0|$ leaves, such that $Y \subseteq L$ and $U_x - Y$ is disjoint from all other U_y for $y \in V(T)$.
– We define a restriction of a mapping ω onto F_0 as an arbitrary bijection from F_0 to $L - Y$.
– Altogether, we take the tree $W' = \bigcup_{y \in V(T)} U_y$, and denote by W the cubic tree obtained from W' by contracting the degree-2 vertices of T'.

Claim 3.2. The pair (W, ω) defined above is a branch-decomposition of M of width at most $k + 1$.

Proof. Let f be an edge of W incident with $V(U_x)$ for some $x \in V(T)$. Moreover, let $T_1, \dots, T_d$ be the connected components of $T - x$, and let $W_i = \bigcup_{y \in V(T_i)} U_y$ for $i = 1, \dots d$. (Hence W_i, $i = 1, \dots d$ are the connected components of $W' - V(U_x)$.) We denote by W^1, W^2 the connected components of $W' - f$. Notice that, without loss of generality, we may write $W^1 - V(U_x) = W_1 \cup \dots \cup W_c$ and $W^2 - V(U_x) = W_{c+1} \cup \dots \cup W_d$ for some $1 \le c < d$.

We denote by $F^i = \omega^{-1}\big(V(W^i)\big)$ and $F_0^i = F^i \cap \omega^{-1}\big(V(U_x)\big)$ for $i = 1, 2$ (see that $F_0^1 \cup F_0^2 = F_0$ above), and by $F_i = \omega^{-1}\big(V(W_i)\big)$ for $i = 1, \dots, d$. Then $F^1 \cup F^2 = E = E(M)$, and $F^1 = F_0^1 \cup F_1 \cup \dots \cup F_c$ and $F^2 = F_0^2 \cup F_{c+1} \cup \dots \cup F_d$. So the width of the edge f in the branch-decomposition (W, ω) is

$$\lambda_M(F^1) = \mathrm{r}_M(F^1) + \mathrm{r}_M(F^2) - \mathrm{r}(M) + 1 =$$

$$= \mathrm{r}_M\left(E - F_0^2 - \bigcup_{i=c+1}^{d} F_i\right) + \mathrm{r}_M\left(E - F_0^1 - \bigcup_{i=1}^{c} F_i\right) - \mathrm{r}(M) + 1 \le$$

$$\le \mathrm{r}_M\left(E - F_0^2\right) + \sum_{i=c+1}^{d} \mathrm{r}_M(E - F_i) + \mathrm{r}_M\left(E - F_0^1\right) + \sum_{i=1}^{c} \mathrm{r}_M(E - F_i) - d\,\mathrm{r}(M) - \mathrm{r}(M) + 1 \le$$

$$\le \sum_{i=1}^{d} \mathrm{r}_M(E - F_i) - (d-1)\,\mathrm{r}(M) + 1 \le k + 1,$$

where the second step holds by the next claim.

Claim 3.3. Let $X_1, \dots, X_m \subset E = E(M)$ be pairwise disjoint subsets of elements of a matroid M. Then

$$\mathrm{r}_M\left(E - (X_1 \cup \dots \cup X_m)\right) \le \sum_{i=1}^{m} \mathrm{r}_M(E - X_i) - (m-1)\,\mathrm{r}(M).$$

Proof. We proceed by induction on m, the case of $m = 1$ being trivial. Using submodularity of rank,

$$\mathrm{r}_M\left(E-(X_1\cup\cdots\cup X_{m+1})\right) \leq \mathrm{r}_M\left(E-(X_1\cup\cdots\cup X_m)\right)+\mathrm{r}_M\left(E-X_{m+1}\right)-\mathrm{r}(M) \leq$$

$$\leq \sum_{i=1}^{m} \mathrm{r}_M(E - X_i) - (m-1)\,\mathrm{r}(M) + \mathrm{r}_M\left(E - X_{m+1}\right) - \mathrm{r}(M)\,. \qquad \blacksquare$$

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