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# On Matroid Representability and Minor Problems

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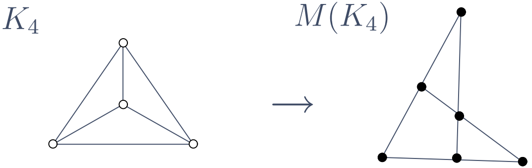
# 1 Introduction

* What really are **matroids**?

  - A common combinatorial generalization of graphs and finite geometries.
  - A new look at structural graph properties (cf. Graph Minors, 1985 +).




* And what can matroids bring into theoretical CS?

  - Important in combinatorial optimization (MST, or Edmonds 70–80's).
  - So far, not of general interest among computer scientists. . .
  - But, some interesting (and even surprising) applications back in graph theory and graph algorithms has been found recently, like in the graph rank-width (Oum and Seymour).

## Definition

A **matroid** $M$ on $E$ is a set system $\mathcal{B} \subseteq 2^E$ of *bases*, sat. the exchange axiom

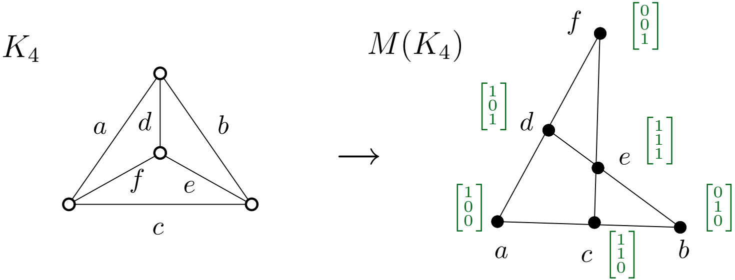
$$\forall B_1, B_2 \in \mathcal{B} \text{ a } \forall x \in B_1 - B_2,\ \exists y \in B_2 - B_1 :\ (B_1 - \{x\}) \cup \{y\} \in \mathcal{B}\,.$$

The subsets of bases are called *independent*.

## Representations by graphs and vectors

*Cycle matroid of a graph* $M(G)$ – on the edges of $G$, where acyclic sets are independent.

*Vector matroid of a matrix* $M(\boldsymbol{A})$ – on the (column) vectors of $\boldsymbol{A}$, with usual linear independence. $\rightarrow$ **geometric view of matroids**:



## 2 Minor Testing

**Theorem 1. (Robertson and Seymour, 1995)**
*Testing for a fixed minor in a graph can be always done in cubic time.*

Actually, the minor notion comes from matroids! (Wagner, 1940's)

*Contracting* a matroid element is dual to deleting it; a geometric interpretation is in a projection from this element. A *minor* is obtained by a sequence of deletions and contractions, the order of which does not matter.

**Matroid minor testing**
Fixed minor $N$ in a vector matroid $M = M(\boldsymbol{A})$ over a field $\mathbb{F}$:

- $\mathbb{F}$ finite field and the branch-width of $M$ bounded $\rightarrow$ cubic time [PH].
- $\mathbb{F}$ finite field and $N$ a planar graph $\rightarrow$ in cubic time, too [GGW + PH].
- $\mathbb{F}$ finite and $N$ arbitrary $\rightarrow$ interesting **open question** [Geelen et al].

* **[new]** For $\mathbb{F} = \mathbb{Q}$, the $N$-minor problem is **$NP$-complete**,
even when the branch-width of $M(\boldsymbol{A})$ is three and $N$ is a planar graph.

([GGW] – assorted works of Geelen, Gerards, and Whittle.)

# 3 Representability of Matroids

A matroid is $\mathbb{F}$-*representable* if it has a vector representation over the field $\mathbb{F}$ (such as, a binary matroid over $GF(2)$).

It seems that matroids representable over finite fields play similar important role in struct. matroid theory as graphs embeddable on a surface play in struct. graph theory.

## Graph embeddability

- Kuratowski theorem and its generalizations,
- efficient algorithms on every (fixed) surface (linear in the plane).

## Matroid $\mathbb{F}$-representability

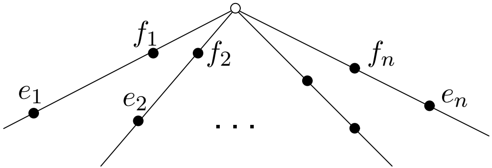
- For $\mathbb{F} = GF(2)$ it is polynomial, though nontrivial [Seymour 1981].
- For $\mathbb{F} = GF(3)$ still **open**.

* **[new]** For $\mathbb{F} = GF(q)$ where $q \geq 4$, it is $co$-$NP$-**complete** (using a nontrvivial $co$-$NP$ membership theorem by [GGW]).

Beware (concerning $NP$ membership), that verifying $\mathbb{F}$-representability requires evaluation of (all?) subdeterminants of the matrix!

# 4 Some Proof Ideas

**Spikes** – (vector) matroids of the form:




$$
\begin{array}{ccccccccccc}
e_1 & e_2 & \dots & e_{n-1} & e_n & & f_1 & f_2 & \dots & f_{n-1} & f_n \\
1 & 0 & \cdots & 0 & 0 & & x_1 & 1 & \cdots & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & & 1 & x_2 & 1 & 1 & 1 \\
\vdots & 0 & \ddots & 0 & \vdots & & \vdots & 1 & \ddots & 1 & \vdots \\
0 & 0 & 0 & 1 & 0 & & 1 & 1 & 1 & x_{n-1} & 1 \\
0 & 0 & \cdots & 0 & 1 & & 1 & 1 & \cdots & 1 & x_n
\end{array}
$$

Their structure is determined by all the "diagonal" subdeterminants

$$
\begin{vmatrix}
x_{i_1} & 1 & \cdots & 1 \\
1 & x_{i_2} & \cdots & 1 \\
\vdots & & \ddots & \vdots \\
1 & 1 & \cdots & x_{i_k}
\end{vmatrix}
= \left( \prod_{j=1}^{k} (x_{i_j} - 1) \right) \cdot \left( 1 + \sum_{j=1}^{k} \frac{1}{x_{i_j} - 1} \right) .
$$

$\longrightarrow$ a straightforward relation with Partition / Knapsack problems.

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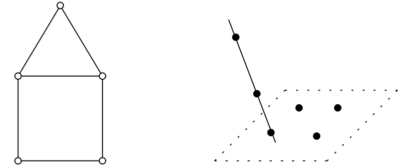
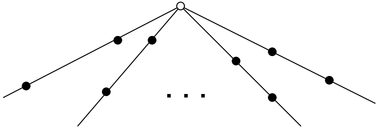
## Spikes to hardness of minor testing

A *free spike* $\iff$ no zero "diagonal" subdet. $\iff$ no solution of Knapsack;

$\iff$ no minor (planar) $N_6 =$



in our spike



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Hence we get:

**Theorem 2.**
*Testing for an $N_6$-minor in a given $\mathbb{Q}$-represented spike is $NP$-complete.*

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## Spikes to hardness of representability

Consider a non-prime finite field $\mathbb{F} = GF(p^r)$ now.

**Claim.** The free spikes are always $\mathbb{F}$-representable.

In the other direction,

**Claim.** If a $\mathbb{Q}$-represented non-free spike is also $\mathbb{F}$-representable, then the associated Knapsack problem has a "small" solution.

$$\begin{vmatrix} x_{i_1} & 1 & \cdots & 1 \\ 1 & x_{i_2} & \cdots & 1 \\ \vdots & & \ddots & \vdots \\ 1 & 1 & \cdots & x_{i_k} \end{vmatrix} = \left( \prod_{j=1}^{k} (x_{i_j} - 1) \right) \cdot \left( 1 + \sum_{j=1}^{k} \frac{1}{x_{i_j} - 1} \right)$$

and so we could search for such a "small" solution in polynomial time. Hence there is a polynomial reduction from Knapsack to $\mathbb{F}$-representability:

**Theorem 3.** *Testing for $GF(p^r)$-representability of a given $\mathbb{Q}$-represented spike is co-$NP$-complete.*

For prime finite fields $\mathbb{F}$ we do similarly with so called "swirls"...