$K_{4,4} - e$ Has No Finite Planar Cover

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Abstract. A graph G has a planar cover if there exists a planar graph H, and a homomorphism $\varphi : H \to G$ that maps the neighbours of each vertex bijectively. Each graph that has an embedding in the projective plane also has a finite planar cover. Negami conjectured the converse in 1988.
This conjecture holds as long as no minor-minimal non-projective graph has a finite planar cover. From the list there remain only two cases not solved yet—the graphs $K_{4,4} - e$ and $K_{1,2,2,2}$. We prove the non-existence of a finite planar cover of $K_{4,4} - e$.

1 Introduction

We consider the following generalization of planarity of graphs: the *planar covering* of graphs. A planar graph H *covers* a graph G if there exists a graph homomorphism φ from H to G such that for each vertex v of H its neighbours are mapped bijectively to the neighbours of $\varphi(v)$. The set $\varphi^{-1}(v)$ is called the *fiber* above v. If G is connected (and H is finite), then the size of each fiber is a constant called the *fold* number of the covering, and the cover is called k-fold where k is the fold number.

Every planar graph has a 1-fold planar cover by definition, and every graph has an infinite planar cover by an infinite tree. As a non-trivial example we mention a 2-fold planar cover of non-planar K_5 (see the right-hand side of Figure 1), obtained from its projective drawing.

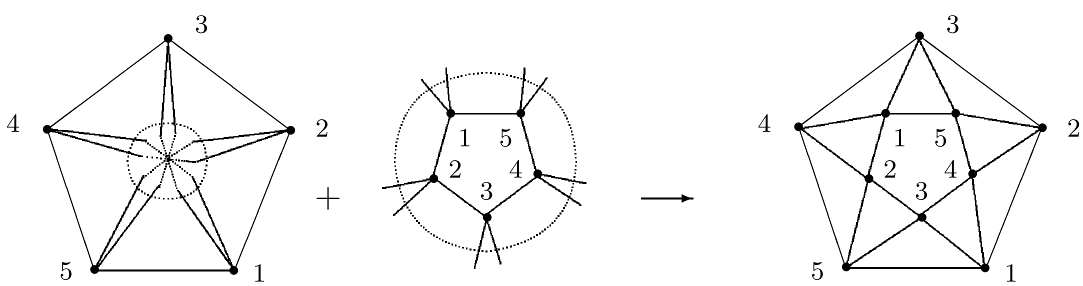

Fig. 1. A 2-fold planar cover of K_5, constructed from two copies of its projective drawing

This method can be easily generalized as follows: Suppose a graph G that has an embedding in the projective plane, realized as a drawing in the normal plane with one cross-cap. Take the drawing twice, and replace the edges going through the cross-caps by new edges connecting the vertices of one copy to those of the other copy. Clearly, the result is a planar graph that double-covers G.

Negami [4] conjectured that this can also be reversed:

Conjecture. (Negami, 1988) *A graph has a finite planar cover if and only if it has an embedding in the projective plane.*

Since the property of having a planar cover is hereditary under the minor ordering, it is sufficient to prove that none of the finite list of minor-minimal non-projective-planar graphs [1] has a planar cover. Some of the cases have been done directly, some reduce to others. Known results are due to Archdeacon, to Fellows [2] and to Negami [3]. The two remaining cases are the graphs $\boldsymbol{K}_{4,4}-e$ and $\boldsymbol{K}_{1,2,2,2}$.

The result of this paper is:

Theorem 1. *The graph $\boldsymbol{K}_{4,4}-e$ has no finite planar cover.*

So to prove the conjecture, it now remains only to solve it for the graph $\boldsymbol{K}_{1,2,2,2}$.

2 A view of the planar cover

2.1 The triangle cover

We show here how to handle a supposed planar cover of the graph $\boldsymbol{K}_{4,4}-e$. Let the vertices of $\boldsymbol{K}_{4,4}-e$ be denoted by $s, t, a, b, c, 1, 2, 3$ as presented in Figure 2, and $\boldsymbol{H}$ be the finite planar graph that covers it (for a contradiction). We label each vertex v of $\boldsymbol{H}$ with the name of the vertex of $\boldsymbol{K}_{4,4}-e$ that v covers. For example, v is labelled 1 iff $\varphi(v) = 1$. To distinguish labels from the names of vertices in a picture, we shall draw the labels framed (see also Figure 3).

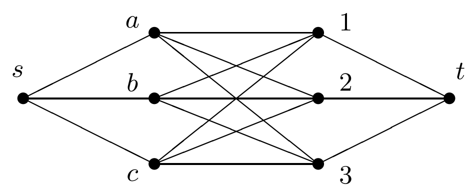

Fig. 2. The graph $\boldsymbol{K}_{4,4}-e$

Realize that $\boldsymbol{H}$ covers $\boldsymbol{K}_{4,4}-e$ if and only if its vertices can be labelled by $s, t, a, b, c, 1, 2, 3$, so that the neighbourhood of each vertex with label s contains exactly three vertices labelled a, b, c, the neighbourhood of each vertex with label t contains exactly three vertices labelled $1, 2, 3$, and each vertex labelled with a, b or c is connected with exactly four vertices labelled $s, 1, 2, 3$, each vertex labelled with 1, 2 or 3 is connected with exactly four vertices labelled t, a, b, c.

For the following constructions, $\boldsymbol{H}$ is considered as a plane graph, i.e. including its planar drawing. In the first approach to handling of $\boldsymbol{H}$, we replace each vertex of label s or t by a triangle (called *st-triangle*) on its three neighbouring vertices. The resulting graph is planar and will be referred to as the *st-triangle cover* of $\boldsymbol{K}_{4,4}-e$, denoted by $\boldsymbol{H}_{st}$. See Figure 3 for an example (a planar cover on the left-hand side, and the resulting st-triangle cover on the right-hand side).

Observation. The st-triangle cover of $\boldsymbol{K}_{4,4}-e$ is a planar graph obtained from a set of disjoint face triangles labelled a, b, c, resp. $1, 2, 3$, by connecting each vertex of a letter labelled triangle with exactly three vertices labelled 1, 2 and 3, and each vertex of a number labelled triangle with exactly three vertices labelled a, b and c.

Corollary 2.1. *The existence of a planar cover of $\boldsymbol{K}_{4,4}-e$ is equivalent to the existence of its st-triangle cover.*

As was observed by the referee, the supposed finite triangle cover is a planar cover of the graph $\boldsymbol{K}_6$ (which *has* a finite planar cover), too. Of course, this is not right in the reversed direction; the necessary pairs of face triangles (abc and 123) are not found in existing finite planar covers of $\boldsymbol{K}_6$.

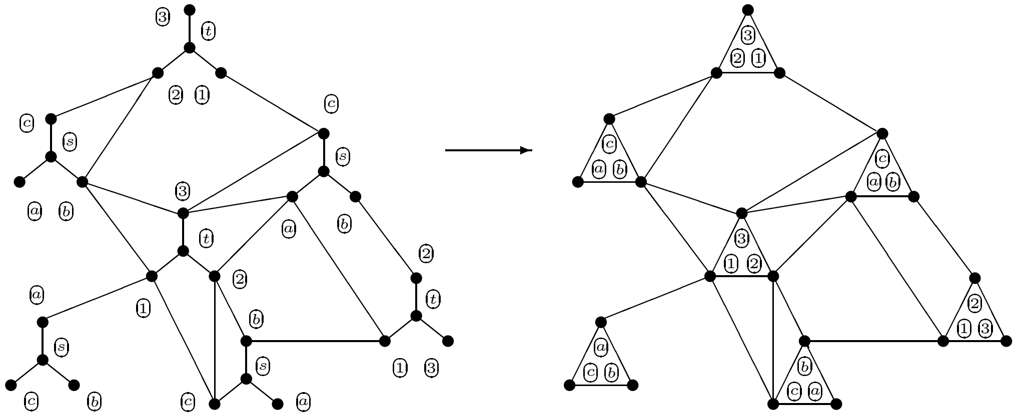

Fig. 3. The construction of the st-triangle cover of $\boldsymbol{K}_{4,4}-e$

2.2 The contraction of the cover

The next approach extracts essential structural information from the st-triangle cover: Each st-triangle of $\boldsymbol{H}_{st}$ is contracted to a vertex, preserving the planar drawing. In the construction, every parallel edges forming faces of size 2 in the planar drawing are replaced by single edges. Notice that there may be parallel edges not forming a face if there is a vertex between them; such edges remain untouched. An example of this construction is shown in Figure 4.

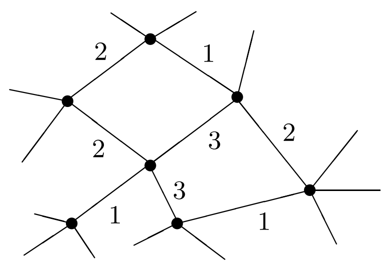

Fig. 4. The Δ-contracted cover of the example from Figure 3

The planar multigraph obtained is called the *Δ-contracted cover* $\boldsymbol{H}_\Delta$, its vertices the *Δ-vertices* and its edges the *Δ-edges*. For each Δ-edge e its *thickness* is defined to be the number of edges of $\boldsymbol{H}_{st}$ that e represents by collapsing faces of size 2. We transfer these concepts of the Δ-contracted cover back to the st-triangle cover, which allows us to consider an st-triangle as a Δ-vertex and speak about its Δ-degree in the multigraph $\boldsymbol{H}_\Delta$, or refer to a collection of edges between two st-triangles as the corresponding Δ-edge. We also introduce the convention that the triangle represented by a Δ-vertex v has its vertices named v_1, v_2, v_3 in positive orientation.

The above construction was proposed by Kratochvíl, who also proved (via personal communication, never published) that the supposed finite planar cover can be always modified to make the multigraph $\boldsymbol{H}_\Delta$ be a simple graph. However, his proof is quite long and our arguments are composed so that they does not need it.

Observation. The Δ-contracted cover $\boldsymbol{H}_\Delta$ of $\boldsymbol{K}_{4,4}-e$ is a bipartite plane multigraph without faces of size 2. The sum of thicknesses of all edges incident with any vertex of $\boldsymbol{H}_\Delta$ equals 9.

Lemma 2.2. *There is no edge of thickness greater than* 3 *in* $\boldsymbol{H}_\Delta$*, and the only possible shapes for Δ-edges of thickness* 1, 2 *or* 3 *are presented in Figure 5.*

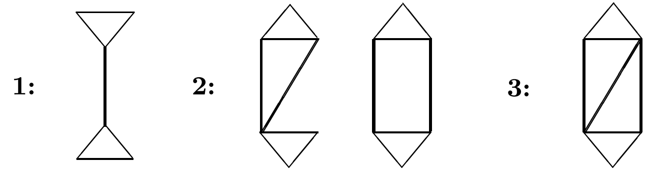

Fig. 5. All possible shapes of Δ-edges between two st-triangles

(Note that there may exist more edges between two Δ-triangles, but then they belong to different Δ-edges.)

Proof. The proof of this lemma is simple, but slightly technical.

Suppose st-triangles $x_1x_2x_3$, $y_1y_2y_3$ have a common Δ-edge e. If there were three edges x_iy_1, x_jy_2, x_ky_3 belonging to the Δ-edge e, each incident with one vertex of the triangle y, then they would divide (together with edges of the triangles) the plane into three regions. Since they form a single Δ-edge, only one of these regions could contain other triangles of the cover. Thus one vertex, say y_2, would not have access to the other triangles, and would be connected with all three x_1, x_2, x_3, but then one of them, say x_2, could not be connected with labels other than that of y_2.

A similar argument applies to a possible four edges (say) x_1y_1, x_1y_2, x_2y_1, x_2y_2 belonging to e, joining two vertices of the triangle x with two vertices of y. In that case, the vertices x_3, y_3 would be separated by a circle (formed by two of the four edges and an edge of one triangle), and since only one region of that circle contains other triangles, the other vertex, say x_3, could not be connected with the label of y_3.

Finally, any possible Δ-edge other than those shown in Figure 5 would clearly contain one of the two impossible cases discussed above. □

Lemma 2.3. *There is no vertex of degree* 1 *or* 2 *in* $\boldsymbol{H}_\triangle$*, while it contains a vertex of degree* 3.

Proof. The first part is clear by Lemma 2.2, and the second one is an easy consequence of Euler's formula for a planar bipartite multigraph without 2-faces. □

3 The proof of the main theorem

3.1 Basic idea

As was observed above, $\boldsymbol{H}_\triangle$ must contain a vertex of degree 3. For each face adjacent to such a vertex we start looking for a "chain" (called a $\diamond$-*chain*) of faces of size 4. That chain goes from the vertex of degree 3 through the graph, ends either at a face of size greater than 4 or at a vertex of degree at least 5, and two distinct chains can cross but cannot merge together (see scheme in Figure 6). Finally, a count argument and Euler's formula show that such chains cannot exist in a planar bipartite multigraph without 2-faces, which implies Theorem 1.

Remember the close correspondence between the st-triangle and Δ-contracted covers. So here we speak about the $\diamond$-chain in $\boldsymbol{H}_\triangle$ where it is easier to describe its general shape, although the chain is, in fact, contained in the graph $\boldsymbol{H}_{st}$ and strongly depends on it.

3.2 One link of the $\diamond$-chain

We define a basis of the $\diamond$-chain, show that the neighbourhood of a Δ-vertex of degree 3 forms such bases, and continue with a lemma proving the existence of a "link" of the chain adjacent to the basis, under certain conditions.

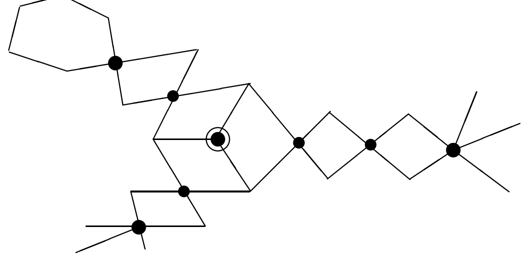

Fig. 6. A scheme of ◇-chains

Definition. We say that two Δ-edges $e_1 = vu$, $e_2 = uw$, in the multigraph $\boldsymbol{H}_\Delta$ *form a ◇-basis* if they lie on a boundary of one face of $\boldsymbol{H}_\Delta$, the Δ-vertices u, v, w appear in positive orientation, the st-triangles $v_1v_2v_3$, $w_1w_2w_3$ (corresponding to the Δ-vertices v, w) are both labelled so that the labels are equal for the pairs v_1, w_1; v_2, w_2; v_3, w_3, and v_2u_2, w_1u_2 are edges in $\boldsymbol{H}_{st}$. (See Figure 7; other possible edges between the considered triangles are not important here, but notice that v_1u_2, w_2u_2 are not in $E(\boldsymbol{H}_{st})$ unless $v = w$, from the properties of a planar cover.)

A ◇-basis is said to be *degenerate* if $v = w$.

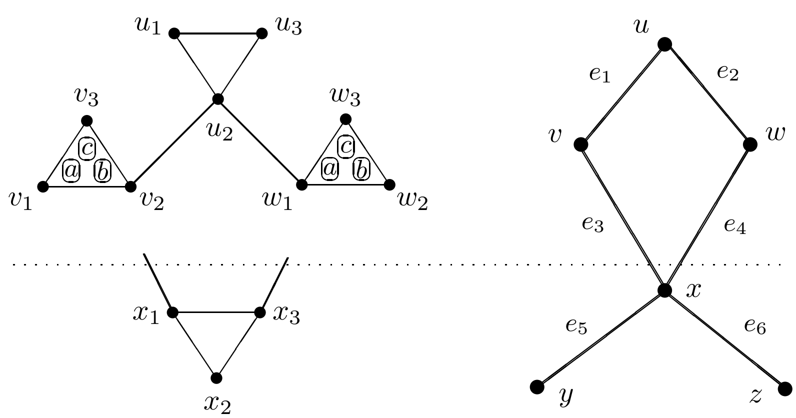

Fig. 7. The ◇-basis and one link of the ◇-chain

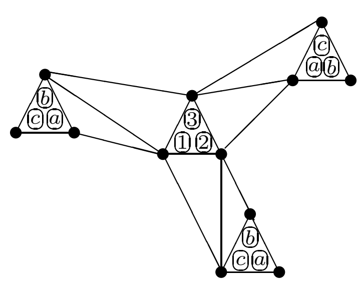

Fig. 8. The unique neighbourhood of an st-triangle of Δ-degree 3

Our ◇-chains start at vertices of degree 3 in $\boldsymbol{H}_\Delta$. From Lemma 2.2, it follows that the Δ-edges of a cubic vertex have unique structure up to an orientation, and the labelling of the neighbouring triangles in $\boldsymbol{H}_{st}$ is uniquely determined up to a possible renaming of symbols a, b, c and $1, 2, 3$ (Figure 8). So each pair of these edges forms a nondegenerate $\diamond$-basis.

Lemma 3.1. *If two Δ-edges vu and uw in $\boldsymbol{H}_\Delta$ form a nondegenerate $\diamond$-basis at a vertex u, and there is a vertex x of degree 4 enclosing a 4-face $\varphi = vuwx$ adjacent to this basis, then*
(a) each of the vertices x_1, x_2, x_3 is connected to at most two labels (of the three labels required) within the two Δ-edges xv, xw lying on boundary of the face φ;
(b) the other two Δ-edges xy, xz of the vertex x form a (possibly degenerate) $\diamond$-basis.

Proof. Let us denote by $e_1 = uv$, $e_2 = uw$, $e_3 = vx$, $e_4 = wx$, $e_5 = xy$, $e_6 = xz$, where e_5, e_6 are the other two Δ-edges incident with the vertex x so that e_3, e_5, e_6, e_4 are in positive orientation (see the right-hand side of Figure 7). Note that y, z are possibly not distinct, they even may be equal to v or w.

The set of all edges of $\boldsymbol{H}_{st}$ corresponding to some of the Δ-edges e_3, e_4 is denoted by E_x. First observe the following facts about these edges:

- None of the edges of E_x is incident with the vertex v_3 or with w_3. Otherwise, if x_1 were connected with v_3, there would be a circle going from x_1 through $v_3 v_2 u_2 w_1$ (and possibly w_2 or x_3, x_2) back to x_1, separating v_1 from other triangles, so it should be connected to the vertices x_1, x_2, x_3 within the Δ-edge e_3, which is impossible due to Lemma 2.2.
- There are at most 2 edges of E_x incident with x_1 (x_2, x_3) because their other vertices are among v_1, v_2, w_1, w_2 having only two distinct labels a, b. This also proves part (a).
- The edges of E_x are incident with only two vertices, say x_1, x_3, of the triangle $x_1 x_2 x_3$. Otherwise, one of these vertices would be separated by a circle formed by the other two and v_2, u_2, w_1, and it could not be connected to a vertex of label c.

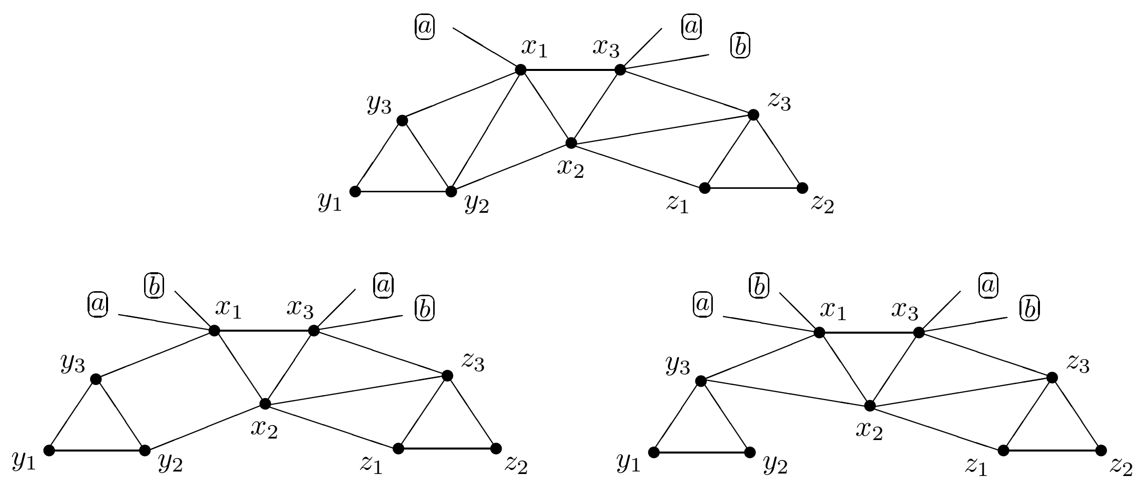

Fig. 9. A new $\diamond$-basis at the Δ-vertex x

From the above facts, $|E_x| \leq 4$, while $|E_x| \geq 3$, because there are only two other Δ-edges incident with x. Moreover, the edges of E_x connect one of the vertices of the triangle $x_1 x_2 x_3$ (say x_3) with two labels a, b, and the second x_1 either with one of these two labels (say a) or with both of them. Therefore, using Lemma 2.2, there are three possible shapes of a neighbourhood of the triangle $x_1 x_2 x_3$, presented in Figure 9. It is enough to discuss each of these possibilities, showing that b) holds:

- In the first case (the top figure) the vertex z_3 must be labelled c, so y_2 or z_1 must have label a, but y_2 is connected with x_1 which already has a neighbour labelled a. Thus z_1 is labelled by a, z_2 by b, and y_1 by a, y_2 by b. Consequently, e_5, e_6 form a $\diamond$-basis.

– The second case (the bottom left figure) is similar; y_3, z_3 must be labelled by c, and then either y_1 has label a and y_2 label b, so z_1 has label a and z_2 label b, or the labels a, b are swapped. Again, we get a ◇-basis.
– The third case (the bottom right figure) is impossible, because y_3 and z_3 would be labelled by c, but they are both connected with x_2.

□

Now, if the ◇-basis formed by e_5, e_6 at x is nondegenerate, we can repeat our arguments, continuing the ◇-chain. In such case we say that the face $e_1e_2e_3e_4$ forms one *link* of the chain.

3.3 End of the ◇-chain

The proof of Lemma 3.1 for a ◇-basis vu, uw was based on these two assumptions: First, $v \neq w$ and there must be a vertex x such that $vuwx$ form a face of size 4; second, the degree of x must be 4. It is easy to see that the vertex x (if it exists) can never have degree 3, because the third of its Δ-edges would have thickness at least 4. So there are three possible reasons why a ◇-chain is not continued:

– the face φ adjacent to the last ◇-basis vu, uw has size greater than 4;
– there is a 4-face $\varphi = xvuw$ adjacent to that ◇-basis, but the degree of x is at least 5; or
– the ◇-basis is degenerate.

The question now is how many ◇-chains can end at one "large face" or at one "high-degree vertex" of $\boldsymbol{H}_\Delta$. Here we need one more lemma about degree-5 vertices.

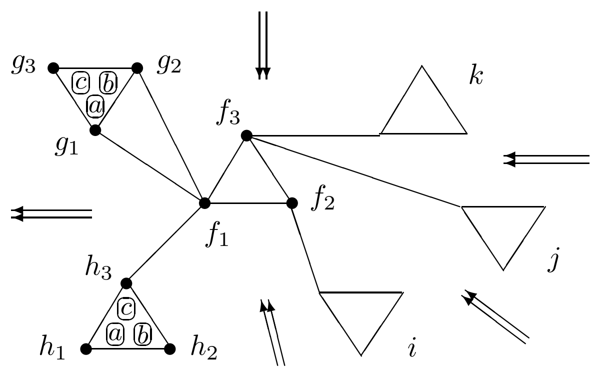

Fig. 10. ◇-chains ending at a vertex of degree 5

Lemma 3.2. *At most* 4 *◇-chains can end at one Δ-vertex f of degree* 5*, and if exactly* 4 *◇-chains end at f, then there is one new ◇-chain starting at f.*

Proof. If each vertex of the *st*-triangle $f_1f_2f_3$ corresponding to f were connected with three different *st*-triangles, the Δ-degree of f would be at least 6. Thus one vertex, say f_1, has two edges connecting it to the vertices g_1, g_2 of one *st*-triangle $g_1g_2g_3$ and the third edge connects it to the vertex h_3 of an other *st*-triangle $h_1h_2h_3$, see the scheme in Figure 10 (the *st*-triangles i, j, k are only for illustration, and they may be connected to f in a different way). Suppose that the vertices g_1, g_2 are labelled a, b, respectively. Then the vertex h_3 (also adjacent to f_1) must have label c. By Lemma 3.1(a), there is no ◇-chain coming into f through the face between g and h.

Observe that in every ◇-chain the two side triangles are labelled in the same cyclic order. So if there are 4 ◇-chains coming into f through the faces between h and i, i and j, j and k,

k and g, all the corresponding pairs of st-triangles must have the same cyclic order of labels a, b, c, therefore h_1 must be labelled by a and h_2 by b. That means that the Δ-edges fg and fh form a $\diamond$-basis for a new $\diamond$-chain starting at f. □

Corollary 3.3. *At any vertex of degree* 5*, at most* 3 *$\diamond$-chains end, and the possible fourth incoming $\diamond$-chain can be continued through this vertex.*

Observations. For a vertex x of degree d, $d \geq 6$, there are at most d chains ending at it, since each chain comes to x through a different face.
Similarly, if φ is a face of size $2k$, $k \geq 3$ (remember the multigraph is bipartite), at most $2k$ chains can end at φ, each one coming to a different vertex.

The last case remaining to discuss is the one of a $\diamond$-chain ending in a degenerate $\diamond$-basis vu, uw where $v = w$. If there is not a face of size 4 adjacent to that $\diamond$-basis or not a vertex x of degree 4, we end the chain as usual. Otherwise (see Figure 11), we find a special way to terminate the chain so that the validity of the previous observations is not affected.

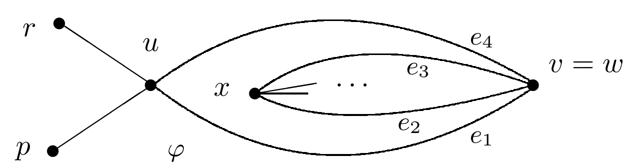

Fig. 11. A degenerate $\diamond$-basis vu, uw

If the vertex $v = w$ has degree 4, the neighbourhood of v looks exactly as in the picture and the face φ surrounding edges e_1, e_4 has size greater than 4 (also in the case that u has degree 3 or 5). Thus we may terminate the chain at vertex u of face φ, since there is no chain normally coming through this face. The same situation occurs when additional edges of v are only between e_2 and e_3.

If v has degree 5 and e_5, the fifth of its edges, is between e_1 and e_4, then there can be only two chains normally coming to v, between e_1, e_5 and between e_4, e_5, then the chain is terminated at v. This is correct even if there is another degenerate basis formed by e_2, e_3 at x (of a chain lying "inside" the circle e_2e_3), since such basis would be counted as in the previous paragraph.

And if v has degree at least 6 (and some of its edges lie between e_1, e_4), there are no chains normally ending at v between e_1, e_2 or between e_3, e_4. Then the chain is terminated at v again. Moreover, a possible other chain, ending in a degenerate basis formed by e_2, e_3 at x, may be counted by v as well.

Finally, we summarize our knowledge about $\diamond$-chains: There are three chains starting at every vertex of degree 3, the chains continue through the graph and end either at a vertex of degree at least 5 or at a face of size at least 6. The number of chains ending at such vertex or face is bounded by the above observations. Of course, it may happen that some $\diamond$-chain has zero length, for example, if it starts at a vertex of degree 3 which lies on a boundary of a face larger than 4. It can be easily checked that two distinct $\diamond$-chains cannot merge together (although they may cross one another). Two $\diamond$-chains also cannot collide—if it happened so, they would continue against each other until one reached the starting vertex of the other. The start is either at a vertex of degree 3, or it is a special case at degree-5 vertex discussed in Lemma 3.2, but neither possibility allows an incoming chain, producing a contradiction.

3.4 Conclusion of the proof

It is now enough to show that the $\diamond$-chains found above cannot exist. This is the aim of the next technical lemma, whose proof is, in fact, easier than its formulation.

Lemma 3.4. *Let $\boldsymbol{G}$ be a bipartite plane multigraph without faces of size 2, all of whose vertices have degree at least 3. Let the set of all vertices of $\boldsymbol{G}$ of degree i, $i \geq 3$, be denoted by $V_i \subset V(\boldsymbol{G})$, and the set of all faces of size $2j$, $j \geq 2$, by F_{2j}.*
Then it is impossible to define a directed graph $\boldsymbol{D}$ on the vertex set $V(\boldsymbol{D}) = V_3 \cup (\bigcup_{i\geq 5} V_i) \cup (\bigcup_{j\geq 3} F_{2j})$, so that the outdegree of each of the vertices from V_3 is 3 and the indegree is 0, the indegree of each vertex from V_5 is at most 3, the indegree of each vertex from V_i, $i \geq 6$ is at most i, and the indegree of each vertex from F_{2j}, $j \geq 3$ is at most $2j$.

Realize that the graph $\boldsymbol{D}$ only "counts" the vertices and faces of $\boldsymbol{G}$. It generally has nothing in common with the structure of $\boldsymbol{G}$. (For one thing, it need not be planar.)

Proof. If we denote by v the number of vertices of $\boldsymbol{G}$, e the number of its edges, f the number of its faces, and specially $v_i = |V_i|$, $f_j = |F_j|$, we can write $e = \frac{1}{2}(3v_3 + 4v_4 + 5v_5 + \ldots)$, and also $e = \frac{1}{2}(4f_4 + 6f_6 + 8f_8 + \ldots)$. By Euler's formula,

$$0 < 2 = v + f - e = v - \frac{1}{2}e + f - \frac{1}{2}e = \sum_{i=3}^{\infty} v_i - \frac{1}{4}\sum_{i=3}^{\infty} iv_i + \sum_{j=2}^{\infty} f_{2j} - \frac{1}{4}\sum_{j=2}^{\infty} 2jf_{2j}$$

$$= \frac{1}{4}\left(v_3 - v_5 - \sum_{i=6}^{\infty}(i-4)v_i - \sum_{j=3}^{\infty}(2j-4)f_{2j}\right) . \tag{1}$$

On the other hand, the existence of the directed graph $\boldsymbol{D}$ would imply

$$3v_3 \leq 3v_5 + \sum_{i=6}^{\infty} iv_i + \sum_{j=3}^{\infty} 2jf_{2j} \ , \text{ i.e. } \quad v_3 - v_5 - \sum_{i=6}^{\infty}\frac{i}{3}v_i - \sum_{j=3}^{\infty}\frac{2j}{3}f_{2j} \leq 0 \ ,$$

a contradiction to (1), since $\frac{i}{3} \leq i - 4$ and $\frac{2j}{3} \leq 2j - 4$ in the above sums. □

We are ready to prove the main theorem:

Proof of Theorem 1. If $\boldsymbol{K}_{4,4}-e$ had a finite planar cover $\boldsymbol{H}$, we would use Lemma 3.4 directly for $\boldsymbol{G} = \boldsymbol{H}_\vartriangle$ and $\boldsymbol{D}$ defined by replacing each $\diamond$-chain in $\boldsymbol{H}_\vartriangle$ with a directed edge starting at its starting vertex of degree 3, and ending at its ending vertex or ending face. Thus the existence of a finite planar cover of $\boldsymbol{K}_{4,4}-e$ would imply a contradiction. □

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