

Stars and Bonds in Crossing-Critical Graphs

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Abstract

The structure of all known infinite families of crossing-critical graphs has led to the conjecture that crossing-critical graphs have bounded bandwidth. If true, this would imply that crossing-critical graphs have bounded degree, that is, that they cannot contain subdivisions of $K_{1,n}$ for arbitrarily large n . In this paper we prove two results that revolve around this conjecture. On the positive side, we show that crossing-critical graphs cannot contain subdivisions of $K_{2,n}$ for arbitrarily large n . On the negative side, we show that there are graphs with arbitrarily large maximum degree that are 2-crossing-critical in the projective plane.

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1 Crossing Numbers and Crossing-Critical Graphs

Unless otherwise stated, throughout this paper our notation and terminology follows Diestel [5].

The *crossing number* $cr_{\Sigma}(G)$ of a graph G in a surface Σ is the minimum number of pairwise crossings of edges in a drawing of G in Σ . Whenever the reference to Σ is omitted, it is assumed that Σ is the plane (or, equivalently in the realm of crossing numbers, the sphere).

Calculating the exact crossing number of a graph is a computationally hard problem, and for many years most crossing number papers focused on calculating or estimating the crossing number of interesting families of graphs. This trend has been reversed in the last few years, as questions of a more structural character have been successfully tackled (see for instance [2, 3, 9, 12, 15, 18]).

As with other classical graph theoretical parameters, we gain a great insight into crossing numbers by looking at graphs that are minimal with respect to having a certain crossing number. A graph G is *k-critical* in Σ if $cr_{\Sigma}(G) \geq k$ but $cr_{\Sigma}(G - e) < k$ for each edge e of G . Many interesting questions and results in crossing-critical graphs in the plane are related to the work of Richter and Thomassen [15].

A little over two decades ago, Širáň [17] and Kochol [11] gave nice constructions of crossing-critical graphs. Kochol's family of critical graphs has inspired a good deal of research. These constructions have been generalized by several authors [16, 13, 1]. All such generalizations share one key feature from Kochol's original construction; the infinite families consist of "long and

thin” graphs. This led Salazar and Thomas to conjecture that crossing-critical graphs have bounded path-width (see [6]). This conjecture has been proved in [9].

Thomassen observed that all constructions known, including the slightly different flavoured constructions by Hliněný [8, 10], satisfy the stronger property of having bounded bandwidth. A graph G has *bandwidth* at most k if there is a bijection $\beta : V(G) \rightarrow \{1, \dots, |V(G)|\}$ such that $|\beta(u) - \beta(v)| \leq k$ for each edge $e = uv$ in G . This observation has been recorded as a (still open) conjecture by Richter and Salazar in [14].

Conjecture 1.1 *For each integer $k > 0$ there is a number $B(k)$ such that if G is k -crossing-critical, then the bandwidth of G is at most $B(k)$.*

The following weaker form of Conjecture 1.1 remains also open:

Conjecture 1.2 *For each integer $k > 0$ there is a number $D(k)$ such that if G is k -crossing-critical, then the maximum degree of G is at most $D(k)$.*

In this paper we present two results inspired by Conjecture 1.2. In the direction supporting this conjecture, we show the following.

Theorem 1.3 *For each integer $k > 0$, there is a $f(k)$ such that if G is k -crossing-critical, then G does not contain a subdivision of $K_{2,f(k)}$. In particular, $f(k) \leq 30k^2 + 200k$.*

Supporting the viewpoint that Conjecture 1.2 is false, we show that its projective plane version (and consequently the projective plane version of Conjecture 1.1) is false.

Theorem 1.4 *There is an infinite family of simple 3-connected graphs H_k , $k \geq 3$, such that each H_k is 2-crossing-critical in the projective plane and has a vertex of degree $6k$.*

The rest of this paper is organized as follows. In Section 2 we study the structure of $\{u, v\}$ -bridges in crossing-critical graphs, where u, v are distinct vertices. This is used in Section 3 to prove Theorem 1.3. In Section 4, we give the construction to prove Theorem 1.4. We close with some concluding remarks and open questions in Section 5.

2 2 -cuts and $\{u, v\}$ -bridges in crossing-critical graphs

Recall that a *drawing* \mathcal{D} of a graph G is a mapping of G into the plane (or the surface Σ), such that the vertices of G are points and the edges are simple curves joining their endvertices. Moreover, it is required that no edge passes through a vertex (except at its ends), and no three edges cross in a common point. Throughout this section, we consider graphs drawn in the *plane* \mathbb{R}^2 (equivalently, the sphere). We are mostly interested in *optimal drawings* of a k -crossing-critical graph G , i.e. in drawings \mathcal{D} of G such that $\text{cr}(\mathcal{D}) = \text{cr}(G)$.

To show that no large $K_{2,n}$ subdivisions exist in a k -crossing-critical graph (k is fixed), we first take any two vertices u, v , thinking of them as the degree- n vertices in a $K_{2,n}$ subdivision in G . We wish to analyze the $\{u, v\}$ -bridges in G . We use “bridge” in the sense of Tutte: a $\{u, v\}$ -*bridge* in G is either a single edge with endpoints u and v , including u and v (a *trivial* bridge), or a subgraph of G obtained by adding to a component H of $G - u - v$ the vertices u and v and all edges attaching H to u and v .

Our major aim in this section is to prove Lemma 2.4, which claims that if u, v are the degree- n vertices of a large $K_{2,n}$ subdivision in G , then a large number of u - v paths are drawn (in every optimal drawing of G) inside a closed disc Δ bounded by two u - v paths, in such a way

that the chunk of G drawn in Δ is crossing-free and remains connected even after the removal of u and v . This is a central prerequisite in the proof of Theorem 1.3.

To prove Lemma 2.4 we need two preliminary results. First we study the implications of a large enough number of $\{u, v\}$ -bridges: in an optimal drawing of such a (not necessarily crossing-critical) graph, distinct $\{u, v\}$ -bridges are disjointedly drawn, and a single face of the drawing is incident with both u and v . This is the content of Lemma 2.2. Then we show, in Lemma 2.3, that the number of $\{u, v\}$ -bridges is bounded as a function of k .

We make essential use of a fundamental result by Richter and Thomassen [15], which proves an intuitively appealing yet elusive fact: t -crossing-critical graphs have crossing number bounded by a function of t .

Theorem 2.1 (Richter and Thomassen) *Every t -crossing-critical graph has crossing number at most $2.5t + 16$.*

Our subsequent proofs rely on this upper bound. Any improvement on this bound would immediately imply (marginally, though) sharper bounds in our results.

We start by analyzing drawings with a large number of $\{u, v\}$ -bridges. Having a drawing \mathcal{D} of a graph G , and a subgraph $H \subseteq G$, we shortly denote by $\mathcal{D}(H)$ the corresponding subdrawing of H . We call a *face* of \mathcal{D} a component of $\mathbb{R}^2 \setminus \mathcal{D}$.

Lemma 2.2 *Let G be a 2-connected graph, and let u, v be different vertices of G . Let $t > 0$ be an integer. Suppose that $\text{cr}(G) \leq t$, and that G has at least $t + 2$ $\{u, v\}$ -bridges. Then every optimal drawing \mathcal{D} of G satisfies the following:*

- (i) *No two edges of different $\{u, v\}$ -bridges cross each other in \mathcal{D} .*
- (ii) *There is a face of \mathcal{D} that is incident with both $\mathcal{D}(u)$ and $\mathcal{D}(v)$.*

Proof. Let G, u, v , and t be as in the statement of the claim, and let \mathcal{D} be any optimal drawing of G . Let C_1, C_2, \dots, C_s be all the (u, v) -bridges of G , where, by hypothesis, $s \geq t + 2$.

First we note that, for each C_i , the drawing $\mathcal{D}_i = \mathcal{D}(C_i)$ of C_i induced by \mathcal{D} has a face incident with both $\mathcal{D}_i(u)$ and $\mathcal{D}_i(v)$. For suppose that some C_j does not satisfy this property. Since every C_i contains a u - v path, then every C_i with $i \neq j$ has at least one edge that crosses (in \mathcal{D}) an edge of C_j . But then \mathcal{D} has at least $t + 1$ crossings, a contradiction. We remark that this observation also implies that if \mathcal{D} satisfies (i), then it must satisfy (ii) as well.

A straightforward corollary of the previous paragraph then implies that, for each $i = 1, 2, \dots, s$, there is a drawing \mathcal{D}'_i of C_i such that: (a) $\text{cr}(\mathcal{D}'_i) = \text{cr}(\mathcal{D}_i)$; (b) $\mathcal{D}'_i(u) = (0, 1)$ and $\mathcal{D}'_i(v) = (0, -1)$; and (c) $\mathcal{D}'_i \setminus \{\mathcal{D}'_i(u), \mathcal{D}'_i(v)\}$ is contained in the interior of the region bounded by the parabolas $x = i(1 - y^2)$ and $x = (i + 1)(1 - y^2)$.

The drawings \mathcal{D}'_i can then be combined to yield a drawing \mathcal{D}' of G that clearly satisfies (i) and (ii). Now (a) in the previous paragraph implies that $\text{cr}(\mathcal{D}') \leq \text{cr}(\mathcal{D})$. Moreover, (a) also implies that no edges in different $\{u, v\}$ -bridges cross each other in \mathcal{D} , as otherwise \mathcal{D} would have more crossings than \mathcal{D}' , contradicting the optimality of \mathcal{D} . Thus \mathcal{D} itself satisfies (i). As observed above, this implies that \mathcal{D} also satisfies (ii). ■

Lemma 2.3 *Let $t > 0$ be an integer. Suppose that G is a 2-connected t -crossing-critical graph, and let u, v be any two distinct vertices of G . Then G has at most $2.5t + 18$ $\{u, v\}$ -bridges.*

Proof. Let $\mathcal{C}_{u,v} := \{C_1, C_2, \dots, C_s\}$ denote the collection of all $\{u, v\}$ -bridges of G . We suppose $s \geq 2.5t + 19$, and will show this implies that $\text{cr}(G - e) = \text{cr}(G)$ for some edge e of G , thus contradicting the criticality of G .

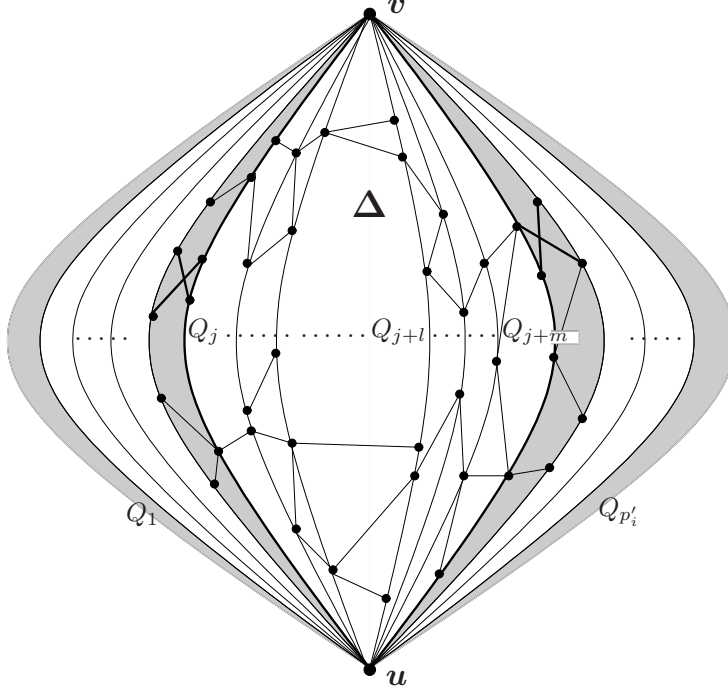


Figure 1: Proof of Lemma 2.4. A collection $Q_1, Q_2, \dots, Q_{p'_i}$ of u - v paths in a $\{u, v\}$ -bridge C_i of G ; its bad indices $j-1, j+m$ (Type I) and p'_i (Type II), depicted with shaded strips, and consecutive m good indices $j, j+1, \dots, j+m-1$ whose strips form Δ .

Let \mathcal{D} be an optimal drawing of G . Since $\text{cr}(G) \leq 2.5t + 16$ by Theorem 2.1, then $s \geq 2.5t + 19 \geq \text{cr}(G) + 3$ and Lemma 2.2 applies. Then it follows from (i) that the number of $\{u, v\}$ -bridges that contain an edge crossed in \mathcal{D} is at most $\text{cr}(G)$. So there is a $\{u, v\}$ -bridge, say C_1 without any loss of generality, such that no edge of C_1 is crossed in \mathcal{D} .

Let \mathcal{D}_1 be any optimal drawing of $G_1 = G - V(C_1 - \{u, v\})$ (or $G_1 = G - e$ in the case C_1 is a single edge e). Note that $\mathcal{C}_{u,v} \setminus \{C_1\}$ is exactly the set of $\{u, v\}$ -bridges in G_1 . Since $\text{cr}(G - C_1) \leq \text{cr}(G)$, and so $|\mathcal{C}_{u,v} \setminus \{C_1\}| \geq 2.5t + 18 \geq \text{cr}(G_1) + 2$, then Lemma 2.2 also applies to G_1 . Moreover, for each $i = 2, 3, \dots, s$, $\text{cr}(\mathcal{D}(C_i)) = \text{cr}(\mathcal{D}_1(C_i))$ since any strict inequality would imply that \mathcal{D} or \mathcal{D}_1 is not optimal. By Lemma 2.2 (i), as applied to both G and G_1 , we get $\text{cr}(\mathcal{D}) = \text{cr}(\mathcal{D}_1)$. Thus $\text{cr}(G) = \text{cr}(G_1)$, and hence $\text{cr}(G - e) = \text{cr}(G)$ for each $e \in E(C_1)$. ■

Lemma 2.4 *Let $t, m > 0$ be integers. Let G be a 2-connected t -crossing-critical graph, and let \mathcal{D} be an optimal drawing of G . Suppose that for some vertices u, v of G , there are at least $(7.5t + 50)m$ pairwise internally-disjoint u - v paths in G . Then there are $m + 1$ such u - v paths, say P_0, P_1, \dots, P_m , such that:*

- (i) $\mathcal{D}(P_0 \cup P_m)$ bounds a closed disc Δ containing no crossings of \mathcal{D} .
- (ii) $\mathcal{D}(P_i) \subseteq \Delta$ for $i = 1, \dots, m-1$, and the subgraph of $G - \{u, v\}$ drawn on Δ is connected.

Note that (i),(ii) then imply that P_0, \dots, P_m all belong to the same $\{u, v\}$ -bridge of G .

Proof. Let $\mathcal{C}_{u,v} := \{C_1, C_2, \dots, C_s\}$ denote the collection of all $\{u, v\}$ -bridges of G , and let p_i be the cardinality of a maximal set of pairwise internally-disjoint u - v paths in C_i . Let r_i be the number of crossings in \mathcal{D} involving edges of C_i , with a crossing counting twice if both edges involved are in C_i .

We first prove that there exists $i \in \{1, \dots, s\}$ such that $p_i \geq m(r_i + 1)$. Indeed, seeking a contradiction, suppose that $p_i \leq m(r_i + 1) - 1$ for each i . Then, since $\sum_{i=1}^s r_i = 2\text{cr}(G) \leq 5t + 32$ by Theorem 2.1, and $s \leq 2.5t + 18$ by Lemma 2.3, we can upper bound the total number of pairwise internally-disjoint u - v paths in G by

$$\sum_{i=1}^s p_i \leq \sum_{i=1}^s (m(r_i + 1) - 1) < m \sum_{i=1}^s r_i + ms \leq m(5t + 32 + 2.5t + 18),$$

contrary to our assumption.

So there is an i such that $p_i \geq m(r_i + 1)$. Consider a maximum-size collection $Q_1, Q_2, \dots, Q_{p'_i}$ of pairwise internally-disjoint u - v paths in G belonging to C_i and not involved in a crossing in \mathcal{D} , labeled in agreement with the clockwise cyclic order in which they appear in \mathcal{D} around u . See Figure 1. Note that $p'_i \geq p_i - r_i$. We say that an index $j \in \{1, \dots, p'_i\}$ is *bad* if either; (*Type I*) the closed disc Δ^j bounded by $\mathcal{D}(Q_j \cup Q_{j+1})$ in the sphere and disjoint from the interior of Q_{j+2} contains a crossing of \mathcal{D} , or (*Type II*) Δ^j contains a face of $\mathcal{D}(C_i)$ incident both with u and v (indices are modulo p'_i).

We argue that the number of bad indices is at most $r_i + 1$. Indeed, as at most r_i of the disks Δ^j contain a crossing of \mathcal{D} involving an edge of C_i , there are at most r_i bad indices of Type I. If Δ^j contains a crossing of \mathcal{D} , but no one involving C_i , then Δ^j contains the drawing of another $\{u, v\}$ -bridge not crossing C_i and so there is a face of $\mathcal{D}(C_i)$ incident both with u and v in Δ^j . Finally, there is at most one such face, as otherwise C_i would not be a single $\{u, v\}$ -bridge. Thus there is at most one bad index of Type II, and therefore at most $r_i + 1$ bad indices in total.

Now $p'_i \geq p_i - r_i \geq (m - 1)(r_i + 1) + 1$, and hence there exist m consecutive *good* (i.e., not bad) indices $j, j + 1, \dots, j + m - 1$ in the cyclic ordering modulo p'_i . Let $(P_0, \dots, P_m) := (Q_j, \dots, Q_{j+m})$. It is now straightforward to verify validity of (i) and (ii) from the definition of a bad index. \blacksquare

3 Large bonds in crossing-critical graphs: proof of Theorem 1.3

In this section we prove Theorem 1.3. We start with a technical dealing with face distance in graph drawings (Lemma 3.1). We then prove Theorem 1.3 for 2-connected graphs (Theorem 3.2), and we finish the section with a proof for the general case.

Let x, y be vertices of a graph H , and let \mathcal{D} be a drawing of H . A *face chain* is a sequence of (different) faces of \mathcal{D} such that each consecutive pair of them shares (a segment of) an edge. A face chain *joins* x and y if it starts from a face incident with x and ends in a face incident with y . The *face distance* $\text{fd}_{\mathcal{D}}(x, y)$ in \mathcal{D} between x and y is the length (i.e. cardinality) of a shortest face chain that joins x and y .

Lemma 3.1 *Let H be a planar graph, $x, y \in V(H)$, and Q_1, Q_2, Q_3 be three internally disjoint paths connecting x to y in H . Suppose \mathcal{D}_1 is a planar drawing (i.e. without crossings) of H , and \mathcal{D}_2 is an arbitrary drawing of H such that $Q := Q_1 \cup Q_2 \cup Q_3$ is drawn crossing-free in \mathcal{D}_2 . Then $\text{fd}_{\mathcal{D}_2}(x, y) \geq \text{fd}_{\mathcal{D}_1}(x, y)$.*

Proof. A standard network-flow duality argument gives that the face distance $\text{fd}_{\mathcal{D}_1}(x, y)$ is at least $\ell + 1$ if, and only if, there exist ℓ pairwise edge-disjoint cycles C_1, \dots, C_ℓ in H such that each $\mathcal{D}_1(C_i)$, $i \in \{1, \dots, \ell\}$, separates $\mathcal{D}_1(x)$ from $\mathcal{D}_1(y)$. Note that $Q_1 \cup Q_2 \cup Q_3$ is a subdivision of the 3-bond, and thus it is embedded (drawn with no crossings) in the same way under \mathcal{D}_1

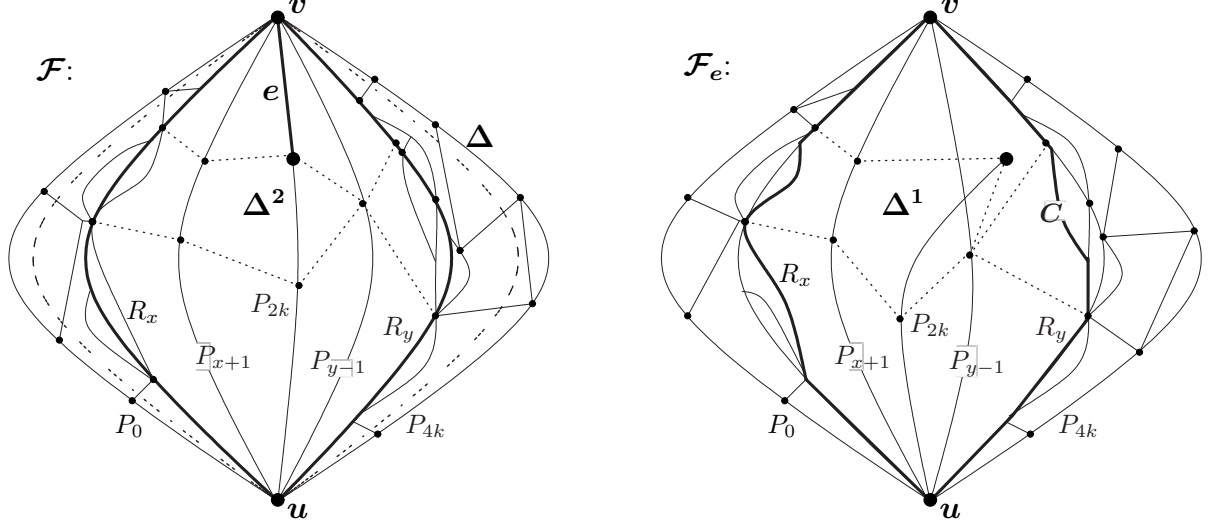


Figure 2: Proof of Theorem 3.2, Claim 1. The drawings \mathcal{F} and \mathcal{F}_e , and the discs Δ^1 and Δ^2 bounded by $\mathcal{F}_e(C)$ and $\mathcal{F}(C)$, respectively.

and \mathcal{D}_2 . Fix any $i \in \{1, \dots, \ell\}$. If in \mathcal{D}_2 there was an arc from $\mathcal{D}_2(x)$ to $\mathcal{D}_2(y)$ that crossed no edge of C_i , then clearly there would be a subpath P of C_i , with one endpoint in Q_j and the other in Q_{j+1} (for some $j \in \{1, 2, 3\}$, reading the indices of Q_j 's modulo 3), internally disjoint from Q , such that P crosses Q_{j+2} , contradicting that Q is drawn crossing-free under \mathcal{D}_2 . It thus follows that for each $i \in \{1, 2, \dots, \ell\}$, $\mathcal{D}_2(C_i)$ also separates $\mathcal{D}_2(x)$ from $\mathcal{D}_2(y)$. This is, in turn, a certificate that $\text{fd}_{\mathcal{D}_2}(x, y) \geq \ell + 1$. \blacksquare

Theorem 3.2 *Let $k > 0$ be an integer and G be a 2-connected graph. Suppose that G contains a subdivision of $K_{2, 30k^2+200k}$. Then G is not k -crossing-critical.*

Proof. Let \mathcal{D} be any optimal drawing of G . By hypothesis, there are distinct vertices u, v such that there are at least $30k^2 + 200k$ pairwise internally disjoint u - v paths in G . Setting $m = 4k$, we obtain using Lemma 2.4 a family P_0, \dots, P_m of u - v paths satisfying (i) and (ii) in Lemma 2.4. Let F denote the $\{u, v\}$ -bridge of G containing all these paths (see remark after Lemma 2.4), and $\mathcal{F} := \mathcal{D}(F)$ be the drawing of F induced by \mathcal{D} . Without any loss of generality, the P_i 's are labeled according to the clockwise cyclic order in which they appear around u in \mathcal{F} .

Let e be the edge of P_{2k} that is incident with u . Note that e cannot be the only edge of P_{2k} , as otherwise P_{2k} would be a $\{u, v\}$ -bridge (of G) by itself. Let $G_e := G - e$ and $F_e := F - e$. Let \mathcal{D}_e be any optimal drawing of G_e , and let $\mathcal{F}_e := \mathcal{D}_e(F_e)$ be the drawing of F_e induced by \mathcal{D}_e .

To complete the proof, we assume G is k -crossing-critical, and derive a contradiction. First note that k -crossing-criticality of G implies that $\text{cr}(\mathcal{F}_e) \leq \text{cr}(\mathcal{D}_e) \leq k - 1 < \text{cr}(G)$.

Claim 1. To prove the theorem, it suffices to construct a drawing $\overline{\mathcal{F}}$ of F such that

- (a) $\text{cr}(\overline{\mathcal{F}}) \leq \text{cr}(\mathcal{F}_e)$,
- (b) $\text{fd}_{\overline{\mathcal{F}}}(u, v) \leq \text{fd}_{\mathcal{F}_e}(u, v)$.

Indeed, having such $\overline{\mathcal{F}}$ at hand, we simply replace the subdrawing \mathcal{F}_e in \mathcal{D}_e with $\overline{\mathcal{F}}$, identifying $\overline{\mathcal{F}}(u)$ with $\mathcal{D}_e(u)$ and $\overline{\mathcal{F}}(v)$ with $\mathcal{D}_e(v)$. This yields a drawing \mathcal{D}' of G . By (a), there are no more crossings involving two edges of F in \mathcal{D}' than in \mathcal{D}_e , and by (b) the replacement operation $\mathcal{F} \mapsto \overline{\mathcal{F}}$ may be performed so that there are no more crossings involving one edge

of F and one edge of other $\{u, v\}$ -bridge of G (indeed, in any optimal drawing of G every $\{u, v\}$ -bridge distinct from F gets drawn along a shortest u - v face chain of the drawing of F). Hence $\text{cr}(\mathcal{D}_e) \geq \text{cr}(\mathcal{D}') \geq \text{cr}(G)$, providing the required contradiction.

It remains to construct $\overline{\mathcal{F}}$ with properties (a) and (b) in Claim 1. We start with the drawing \mathcal{F} of F , cf. Figure 2 left, and recall from Lemma 2.4 the closed disc Δ bounded by $\mathcal{F}(P_0 \cup P_m)$ and containing no crossings of \mathcal{F} .

For $j \in \{1, \dots, 2k-1, 2k+1, \dots, 4k-1\}$, let \mathcal{B}_j denote the set of the bridges of $P_j - \{u, v\}$ that do not intersect $P_{j-1} \cup P_{j+1}$, and let R_j denote the subgraph of F induced by $P_j \cup (\bigcup \mathcal{B}_j)$ (informally, we think of R_j as P_j together with its “local bridges”). Note that $\mathcal{F}(R_j)$ is contained in the disc bounded by $(\mathcal{F}(P_{j-1}) \cup \mathcal{F}(P_{j+1})) \subseteq \Delta$. Clearly, $V(R_j) \cap V(R_{j'}) = \{u, v\}$ and $E(R_j) \cap E(R_{j'}) = \emptyset$ whenever $j \neq j'$. We take the minimum index $x \in \{1, \dots, 2k-1\}$ and the maximum index $y \in \{2k+1, \dots, 4k-1\}$ such that neither R_x, R_y are crossed in \mathcal{F}_e . Such indices exist since at most $2\text{cr}(\mathcal{F}_e) \leq 2k-2$ of the R_j 's may be involved in a crossing. Furthermore, there exist $z \in \{x+1, \dots, y-1\} \setminus \{2k\}$ such that also P_z is uncrossed in \mathcal{F}_e , for the same reason.

As neither R_x nor R_y is crossed in \mathcal{F}_e , the subdrawing $\mathcal{F}_e(R_x \cup R_y)$ has a (single) face Δ^1 containing $\mathcal{F}_e(P_j)$ for all $j = x+1, \dots, y-1$, $j \neq 2k$ (recall that $e \notin F_e$), and Δ^1 containing $\mathcal{F}_e(P_{2k} - e)$. By invoking a Riemann stereographic projection argument if necessary, we may assume Δ^1 is a closed disc bounded in \mathcal{F}_e by a cycle $C \subseteq R_x \cup R_y$ (this last follows from 2-connectivity). Now $\mathcal{F}(C)$ bounds a closed disc Δ^2 in the original drawing \mathcal{F} , and the subgraph F_2 of F drawn under Δ^2 contains P_j for all $j = x+1, \dots, y-1$ (in particular, $e \in F_2$).

It follows from the connectivity property (ii) of Lemma 2.4 that F contains at most two bridges of F_2 —one with P_0 , and possible other one with P_{4k} . Moreover, if $B \neq B'$ are two such bridges of F_2 , then B is attached only to vertices of $C \cap R_x$ and B' is attached only to vertices of $C \cap R_y$. Hence, denoting F_1 the graph induced by $(F \setminus F_2) \cup C$, the subdrawing $\mathcal{F}_e(F_1)$ may be (Whitney) flipped, if necessary, so that Δ^1 (bounded by C) becomes also a face of F_1 . See in Figure 2. By removing Δ^1 from $\mathcal{F}_e(F_1)$, and replacing it with a suitable homeomorphic copy of Δ^2 , we obtain a drawing $\overline{\mathcal{F}}$ of F that clearly satisfies (a).

We finish the proof by showing that $\overline{\mathcal{F}}$ also satisfies (b). By the construction, \mathcal{F}_e and $\overline{\mathcal{F}}$ are identical in the boundary of and outside Δ^1 . Transforming for both drawings each crossing in this coinciding region into a degree 4 vertex, $\overline{\mathcal{F}}$ becomes planar, and modified \mathcal{F}_e maintains the property that $P_x \cup P_y \cup P_z$ is not crossed. Thus an application of Lemma 3.1 shows (b) that $\text{fd}_{\mathcal{F}_e}(u, v) \geq \text{fd}_{\overline{\mathcal{F}}-e}(u, v) = \text{fd}_{\overline{\mathcal{F}}}(u, v)$. \blacksquare

It is hardly surprising that Theorem 1.3 follows easily from the 2-connected case:

Proof of Theorem 1.3.

Suppose G is k -crossing-critical and contains a subdivision of $K_{2,30k^2+200k}$. Let C_1, C_2, \dots, C_r be the components of G , labeled so that, without any loss of generality, C_1 contains a subdivision of $K_{2,30k^2+200k}$. A routine argument shows that each C_i is r_i -crossing-critical for some r_i (moreover, $\sum_i r_i = k$). Let H_1, H_2, \dots, H_s be the blocks of C_1 , labeled so that, without any loss of generality, H_1 contains a subdivision of $K_{2,30k^2+200k}$. Again, a routine argument shows that each H_i is s_i -crossing-critical for some s_i (moreover, $\sum_i s_i = r_1$). In particular, H_1 is s_1 -crossing-critical for some $s_1 \leq r_1 \leq k$, and contains a subdivision of $K_{2,30k^2+200k}$, and consequently a subdivision of $K_{2,30s_1^2+200s_1}$. This contradicts Theorem 3.2. \blacksquare

4 Projective critical graphs with large maximum degree: proof of Theorem 1.4

We now describe the construction of the graph family H_k , $k = 3, 4, \dots$, claimed in Theorem 1.4. Our construction and its corresponding analysis have been inspired by the fruitful concept of planar tiles [13]. We, however, turn this classical construction “inside out” to produce a planar belt of planar tiles which we consequently force to “twist” in the projective plane by adding an additional high-degree vertex, connected to both sides of the belt. We refer the reader to Figure 3 for a taste of our construction. This is a projective drawing of H_3 , obtained from three copies of the tile T^2 (see Figure 4).

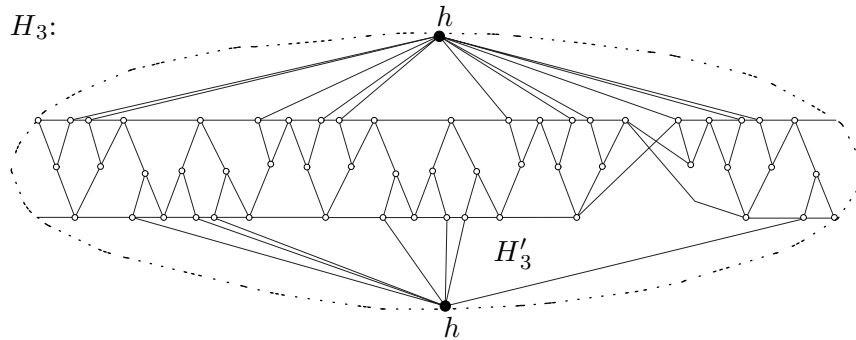


Figure 3: The graph H_3 drawn in the projective plane (the pairs of opposite points on the dashed ellipse get identified). If vertex h is removed, we obtain H'_3 .

The rest of this section is structured as follows. First we give the formal definition of H_k . Then we show that the crossing number of H_k minus any edge is at most 1 (Lemma 4.1). Then comes the hardest part—proving that the projective crossing number of H_k is at least 2 (Lemma 4.3). The main tool for the latter part is a structural analysis: we examine each edge e of H_k and show that, for most choices of e , the subgraph $H_k - e$ still contains one of the forbidden minors for the projective plane (Figure 7)—and hence the crossing number must be $\geq 1 + 1$. For the remaining few edges of H_k , we show that if any two of them cross each other, then the resulting drawing must contain yet another crossing.

Construction of H_k

We let T (the *tile*) denote the graph on 11 vertices and 15 edges depicted in Figure 4 in solid lines, and we distinguish its “boundary” vertices a, b, c , and d . Let T' be a disjoint copy of T , and denote by T^2 the graph obtained from $T \cup T'$ by identifying the vertex c of T with vertex a' of T' , and vertex d of T with vertex b' of T' .

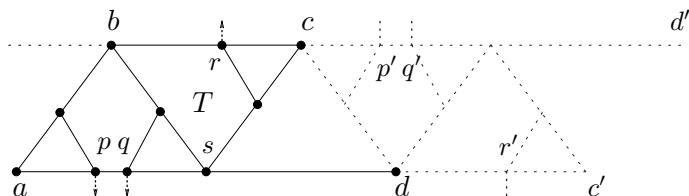


Figure 4: One tile T used in our construction (in solid lines); and another attached copy of T (a scheme in dashed lines), together forming T^2 .

Let T_1^2, \dots, T_k^2 be k disjoint copies of the graph T^2 , where the boundary vertices of T_i^2 are labeled a_i, b_i, c'_i, d'_i . Identify the vertex pair a_{i+1}, c'_i and the pair b_{i+1}, d'_i , for each $i, 1 \leq i \leq k$. Denote the resulting graph by H'_k . Thus, k copies of T^2 are stuck together in a circular fashion to make H'_k . Notice that, unlike in classical crossing-critical constructions, the resulting graph is untwisted (H'_k is a planar graph). Now H'_k is made of $2k$ copies of the tile T . Denote by p_i, q_i, r_i , and p'_i, q'_i, r'_i , for $i = 1, \dots, k$, the copies of the vertices p, q, r (cf. Figure 4) in each T . Finally, let H_k denote the graph that results by adding a new vertex h to H'_k , and joining h to $p_i, q_i, r_i, p'_i, q'_i$, and r'_i for every $i = 1, \dots, k$. Figure 3 illustrates H_3 .

It is clear that every $H_k, k \geq 3$, satisfies the connectivity and degree properties claimed in Theorem 1.4. If we aim to show that our graph H_k is 2-crossing-critical in the projective plane, we need to show that (i) H_k has a projective crossing number equal to 2, and that (ii) if we remove any edge from H_k , then the resulting graph has projective crossing number at most 1. The following statement takes care of (ii) and of the upper-bound part of (i), while subsequent Lemma 4.3 finishes the more difficult direction of (i).

Lemma 4.1 *For every $k \geq 2$, H_k can be drawn in the projective plane with at most two crossings. Moreover, for each edge $e \in E(H_k)$, the graph $H_k - e$ can be drawn in the projective plane with at most one crossing.*

Proof. Figure 3 shows how to draw H_3 in the projective plane with two crossings. This gets easily extended to a projective drawing of H_k with two crossings, for any $k \geq 2$. We wish to formalize this idea, since this will give us the opportunity to introduce concepts that will be used in the second part of this proof.

Let T^+ denote the graph obtained from the tile T by adding a new vertex b^+ adjacent only to b . (Then, if T_0 is a copy of T in H'_k , the vertex b_0^+ of T_0^+ coincides with the s -vertex of the adjacent T -copy in H'_k .) We say that a drawing of a spanning subgraph $G \subseteq T^+$ is *twisted* if the vertices a, b^+, d, c appear on the outer face in this cyclic order. Figure 3 contains a twisted drawing of some copy of T^+ with two crossings, and all other T^+ -copies are there drawn untwisted and crossing free.

We need to show that $H_k - e$, for any edge e of H_k , can be drawn in the projective plane with at most one crossing.

Suppose first that $e \in E(H'_k)$ (recall that $H'_k = H_k - h$). Let T_0 be the T -copy in H'_k that contains e . We say that a vertex $x \in \{p_0, q_0, r_0\}$ of T_0 is *accessible* in a twisted drawing of $T_0^+ - e$ in the plane if x lies on the outer face between a_0, c_0 or between b_0^+, d_0 . (The practical meaning of this notion is that such a twisted drawing can be extended to a drawing of the whole H'_k without additional crossings around a Möbius band, in which the accessible vertices are on the boundary. A small arrow is used in our pictures to indicate accessibility). In particular, a twisted drawing of $T_0^+ - e$ with just one crossing and accessible p_0, q_0 , and r_0 , easily extends to a drawing of the whole $H_k - e$ with one crossing in the projective plane.

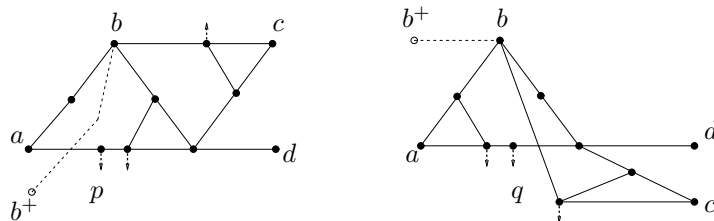


Figure 5: Proof of Lemma 4.1. Some twisted drawings of $T^+ - e$.

So the task is to find a twisted drawing of the graph $T_0^+ - e$ with at most one crossing such that all three vertices p_0, q_0, r_0 are accessible. That is a matter of an elementary, straightforward case checking. Say, if $e = pq$, then (even) a planar twisted drawing of $T_0^+ - e$ clearly exists. For the other two cases of e incident with p , the required twisted drawings of $T_0^+ - e$ are as illustrated in Figure 5 on the left hand side; while the right hand side of this figure illustrates one of the two cases of e incident with q . The rest of cases of e in T_0 are dealt with analogously.

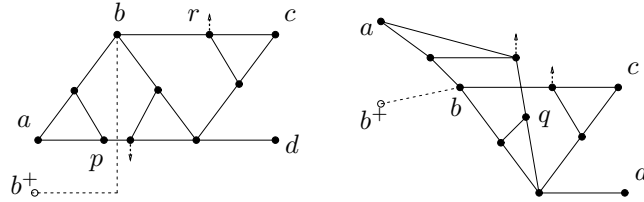


Figure 6: Proof of Lemma 4.1. Twisted drawings of T^+ with inaccessible p or q , resp.

Finally, we assume that e is incident with the vertex h , and denote by T_0 the copy of T in H'_k incident with the other end of e . Up to symmetry, there are now three cases to consider: $e = hq_0$, $e = hp_0$, and $e = hr_0$, where the latter two are analogous with each other. We show in Figure 6 that there always is a twisted drawing of T_0^+ with one crossing such that the two vertices of p_0, q_0, r_0 that are not incident with e are accessible. As observed above, such a drawing easily extends to a drawing of $H_k - e$ with no additional crossings. ■

Projective crossing number of H_k

Our final task to prove the criticality of the graphs H_k is to show that the projective crossing number of H_k is at least (and, in view of Lemma 4.1, equal to) 2.

Recall that an *embedding* is a drawing with no edge crossings, and that a *minor* of a graph is obtained from a subgraph by means of contracting edges. We need a result that follows from [7] plus the observation that being embeddable in the projective plane is closed under taking minors.

Lemma 4.2 [7] *A graph that contains as a minor either R_1 and R_2 (Figure 7) is not embeddable in the projective plane.*

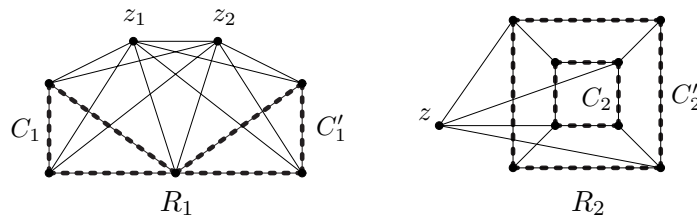


Figure 7: The graphs $R_1 (\simeq K_7 - C_4)$ and $R_2 (\simeq K_{4,5} - M_4)$, two of the “forbidden minors” for the projective plane.

Lemma 4.3 *For each $k \geq 3$, the graph H_k has projective crossing number at least 2.*

Proof. Let $F \subset E(H_k)$ consist of those edges that are *not* copies of the edges (in T) $e_p = pq$, $e_r = br$, or $e_s = sd$ (cf. Figure 4).

at least one of them none of the solid edges in the picture is incident with the vertex x' .

Now, in the graph H_k^\bullet , we contract the three solid edges incident with the vertex y , and denote by G_0 the subgraph consisting of the two shaded triangles after the contraction (see Figure 8). Notice that G_0 together with h and the remaining solid edges form a subgraph isomorphic to a subdivision of the graph $R_1 - z_2$, where h corresponds to z_1 and the shaded triangles to the cycles C_1, C'_1 in R_1 . Moreover, there is a collection of paths (in dashed edges of H_k^\bullet) joining the vertex $x = x'$ to all vertices of G_0 , and so $x = x'$ is “good for” z_2 in the graph R_1 . Hence the graph H_k^\bullet has a minor isomorphic to R_1 .

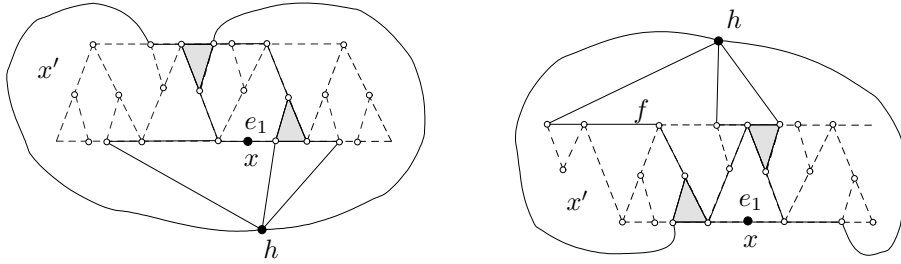


Figure 9: How to get R_1 -minors in the cases (b)–left and (c)–right of Claim 2.

We do similarly in the other two cases (b),(c): e_1, e_2 are again chosen without loss of generality such that the subgraphs drawn in solid lines as fragments of H_k° in Figure 9 are not incident with the vertex x' . In case (c), if $e_2 = f$ happens (where f is as in the picture), then we interchange e_1 and e_2 . Those “solid-line” subgraphs in both cases evidently contain minors isomorphic to the graph $R_1 - z_2$, such that the shaded triangles correspond to C_1, C'_1 in R_1 . Moreover, the vertex $x = x'$ in H_k^\bullet is connected to vertices of the shaded triangles by five paths, and hence the desired R_1 -minor follows in both cases. ■

5 Concluding Remarks

We note that our Theorem 1.3 cannot be strengthened to claim the nonexistence of a $K_{2,f(k)}$ -minor: Kochol’s 2-crossing-critical family [11] contains arbitrarily large $K_{2,m}$ -minors.

Theorem 1.4 gets generalized to other nonorientable surfaces as follows.

Theorem 5.1 *For any integers $k \geq 3$ and $m \geq 1$, there exists a graph $G_{m,k}$ that is 2-crossing-critical in the nonorientable surface \mathcal{N}_m of genus m , and has a vertex of degree $6k$.*

Proof. Let F denote a graph embeddable into the projective plane such that F is 2-crossing-critical in the ordinary plane, and such that $F - e$ is nonplanar for every edge e of F (K_6 minus an edge is an example of such an F). We construct the graph $G_{m,k}$ for $m > 1$ as the disjoint union of H_k (any H_k satisfying Theorem 1.4) and $m - 1$ copies of F .

We claim that $G_{m,k} - e$ has crossing number at most one in \mathcal{N}_m , for any edge e of $G_{m,k}$. For if e is in H_k , then one crosscap may be used to draw $H_k - e$ with ≤ 1 crossing, and the remaining crosscaps allow us to embed the $m - 1$ copies of G . On the other hand, if e is in a copy F_1 of F , then one can draw planarly $F_1 - e$, use $m - 2$ crosscaps to embed the other copies of F , and use the remaining two crosscaps to embed H_k (Figure 3). Thus, in any case, $G_{m,k} - e$ has crossing number at most one in \mathcal{N}_m , as claimed.

It remains to show that $G_{m,k}$ cannot be drawn into \mathcal{N}_m with only one crossing. By way of contradiction, suppose that such a drawing \mathcal{D} exists. We need to analyze two cases separately.

First, suppose that the only crossing in \mathcal{D} involves two edges in H_k . Then each copy of F is embedded in \mathcal{N}_m , and therefore contains a noncontractible cycle. Cutting along these $m - 1$

disjoint noncontractible cycles leaves a (projective–planar) drawing that contains a (projective–planar) drawing of H_k with exactly 1 crossing, contradicting that H_k is 2–crossing–critical in the projective plane.

Second, suppose that the only crossing in \mathcal{D} involves at least an edge e in a copy F_1 of F . Thus removing e leaves an embedding of the disjoint union of the other copies of F , and of $F_1 - e$ and H_k . Analogously to the previous paragraph, each of the other copies of F , as well as (nonplanar) $F_1 - e$, contain a noncontractible cycle, and cutting along these $m - 1$ disjoint cycles yields an embedding of H_k in the projective plane, again contradicting that H_k is 2–crossing–critical in the projective plane. ■

The proof above illustrates the issues involved in a related open problem posed by DeVos, Mohar, and Šámal [4]: if G is the disjoint union of the (say blue) graph G_1 and the (say red) graph G_2 , is it true that an optimal drawing of G always exist that has no red–blue crossing?

Our construction of H_k in Section 4 makes essential use of the nonorientability of the host surface (precisely, it requires the embeddability of a one–sided loop). It is thus interesting to ask whether Conjecture 1.2 generalizes to orientable surfaces.

Question 5.2 *Are there $h > 0$ and $k = k(h)$ such that there exist k –crossing–critical graphs in the orientable surface S_h of genus h with arbitrarily large maximum degree?*

Although the results in this paper do not constitute a direct step towards the final resolution of Conjecture 1.1, our results shed some new light on the structure of crossing-critical graphs, and they also further illustrate the nontriviality of the conjecture. On the one hand, a positive answer to Conjecture 1.2 could bring closer a possible “nice” structural characterization of k –crossing–critical graphs (k fixed). However, a counterexample like the one in Theorem 1.4 would make such a characterization much harder, if possible at all.

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