Inserting Multiple Edges into a Planar Graph

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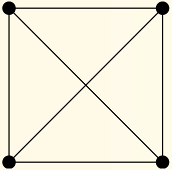
joint work with **Markus Chimani**

Osnabrück University, Germany

1 Drawing Graphs with Crossings

- The **crossing minimization problem**:



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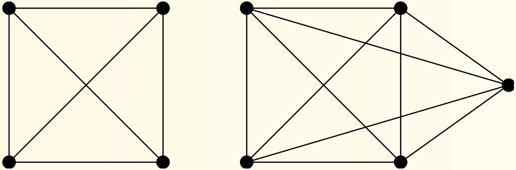

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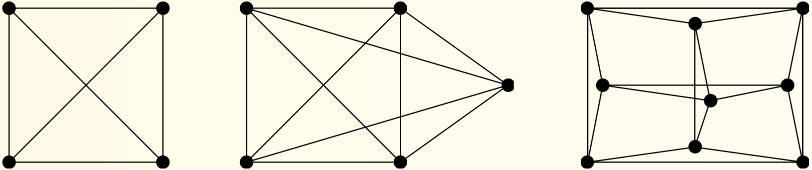

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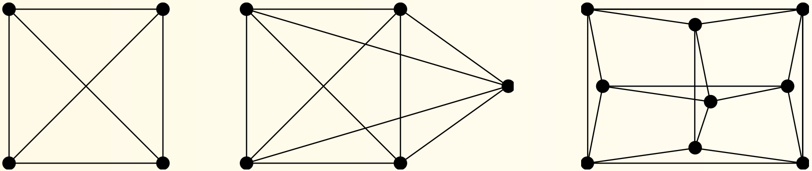

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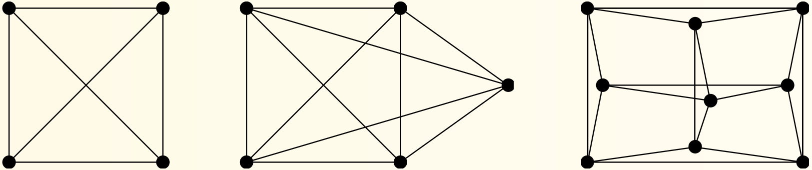

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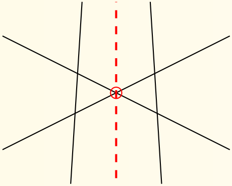

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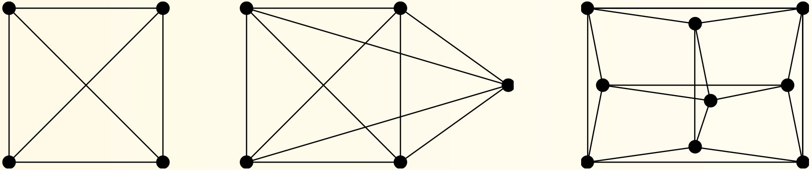

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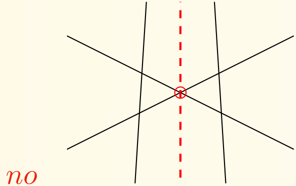

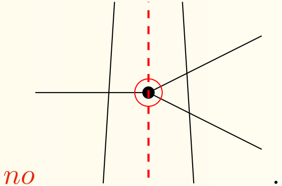

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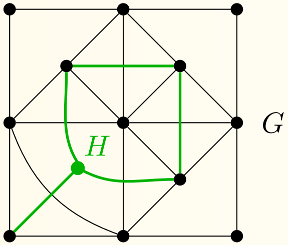

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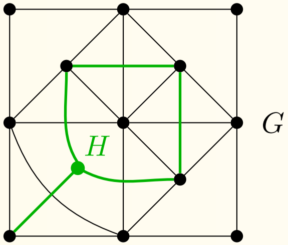

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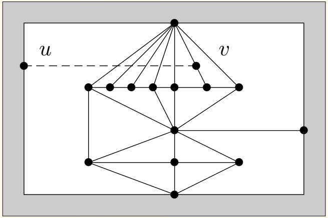

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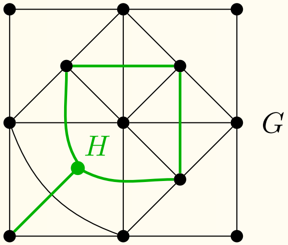

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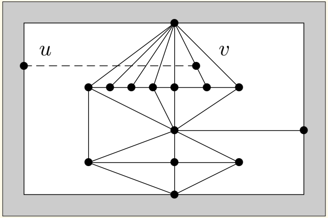

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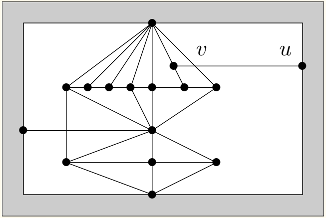

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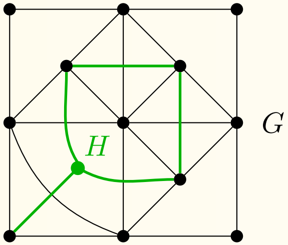

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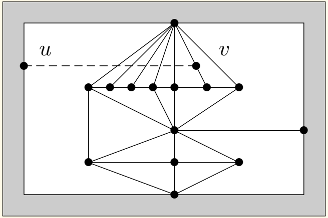

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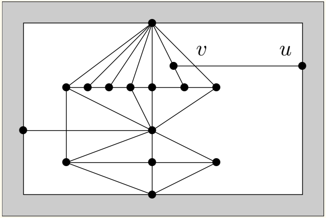

- Though, sometimes useful as an approximation of the crossing number.

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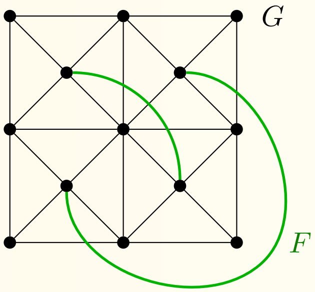
- This problem has a (practically usable!) polynomial time approximation algorithm, with only an additive error depending on $|F|$ and $\Delta(G)$.

  [Chimani and Hliněný, 2011]

2 New Contribution: Exact FPT Algorithm

- Recalling the problem…



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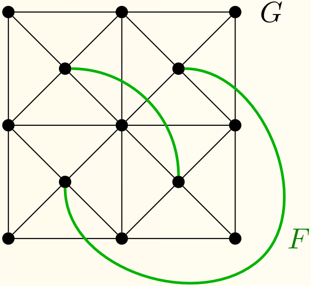

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Theorem. Let G be a 2-connected planar graph and F a set of new edges. The $\mathrm{MEI}(G, F)$ problem is solvable to optimality in FPT time $\mathcal{O}(2^{q(k)} \cdot |V(G)|)$ where q is a polynomial.

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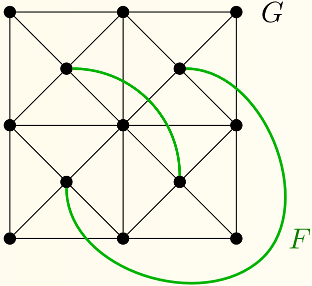

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For connected G the same is true as long as degrees of the cutvertices of G are bounded.

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- Also not comparable to prev. approximation [Chimani and Hliněný, 2011]: the approximation was polynomial-time also in $|F|$...

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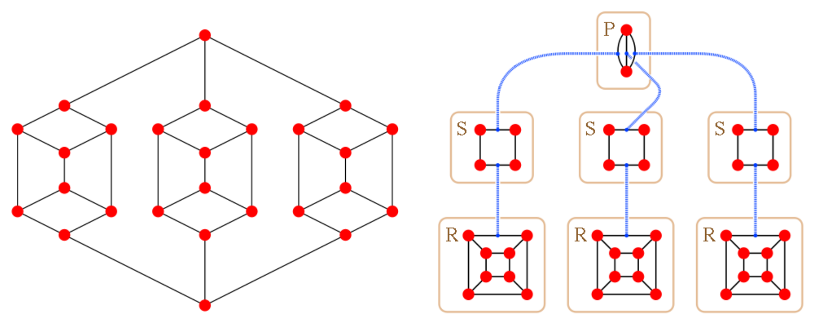

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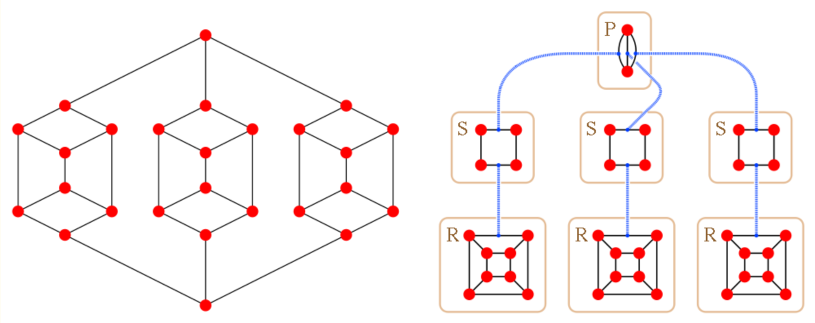

  - G broken into *series, parallel, and rigid* $(3$-conn.$)$ components.
  - Then, G is glued back together along *virtual edges*.

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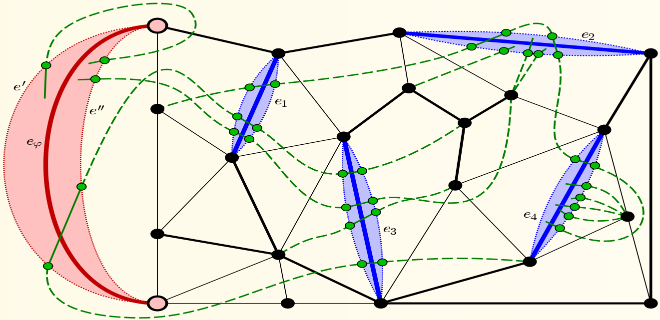
Consider processing one SP(Q)R tree node:

- Embedding flexibility coming from flipping components at 2-cuts.



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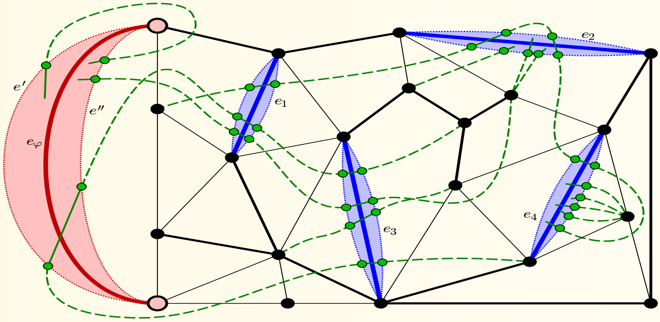

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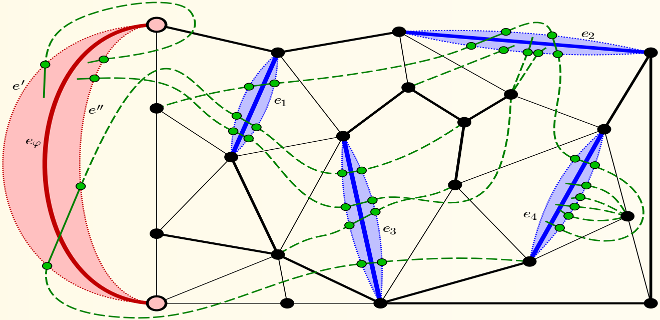

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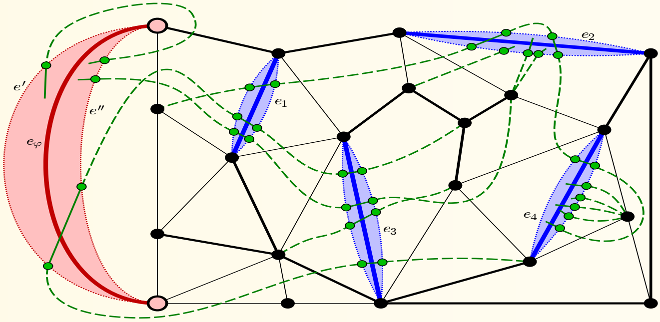

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- Bound the number of crossings of one flip. component as well.
- $\Rightarrow$ At most $f(k)$ rigid cases to consider here, for some (exp.) f.

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- Generalized to cover both the primary case of 3-connected G and the rigid subcases at SPQR…

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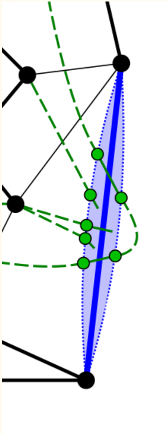

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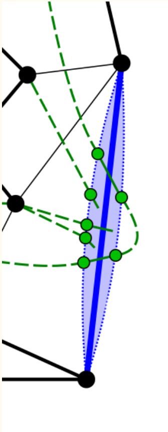

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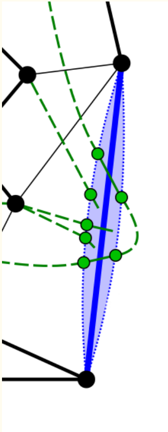

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- Have to find *routes* (dual walks) for the missing segments of F-edges.

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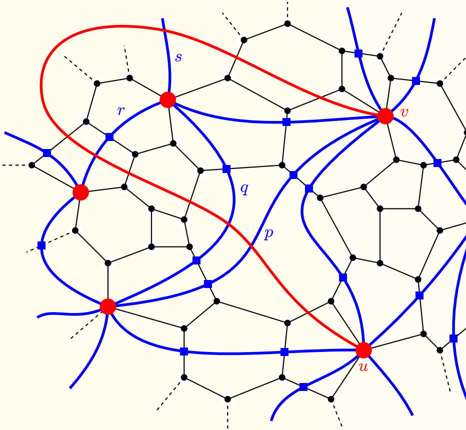

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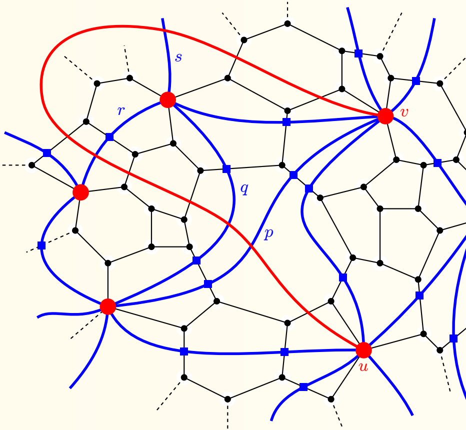

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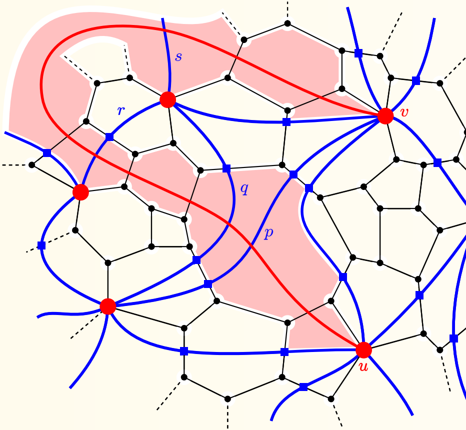

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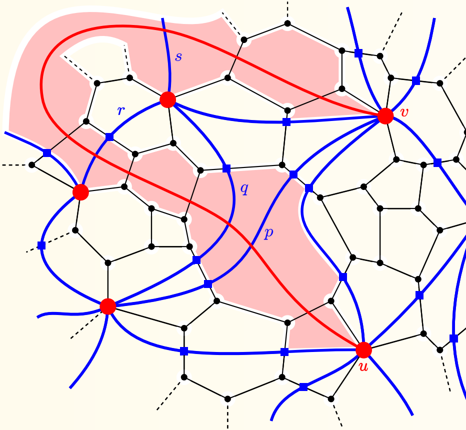

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Lemma. *** In a shortest-spanning trinet, the T-sequence of an optimal $\mathrm{r\text{-}MEI}(G, F)$ solution repeats every triedge at most $8k^4$ times, where $k = |F|$.

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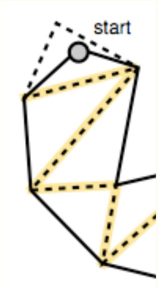

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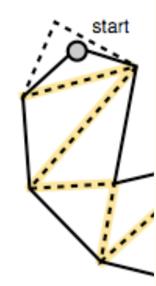

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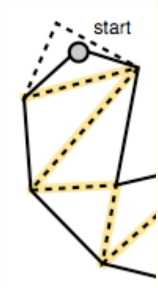

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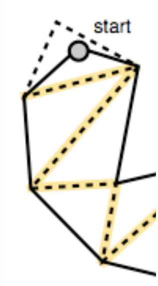

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- Have to similarly check also for "forcing to cross twice"...

The full "rigid" Algorithm

In: plane G, edge weights $w\colon E(G) \to \mathbb{N}_+ \cup \{\infty\}$, new edge set F of $w(f) = 1$.

Out: an optimal solution to (w-weighted) $\mathbf{r\text{-}MEI}(\boldsymbol{G}, \boldsymbol{F})$.

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1. Compute a full trinet (G', T) on the trinodes $N(T) := V(F)$, shortest-spanning;
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4. Pick $\mathcal{P}$ with smallest $\mathit{cr}_{\mathcal{P}} < \infty$.
   Realize routing of all F-edges according to this $\mathcal{P}$, and avoid unforced crossings.

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Thank you for your attention.