Approximating the Crossing Num. of Toroidal Graphs

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Overview

1. **Drawings and the Crossing Number**
   Basic definitions, and an overview of related computational complexity results and questions.

2. **Drawing Toroidal Graphs with few Crossings**
   Natural approaches to planar drawing of toroidal graphs, constructions of Böröczky, Pach and Tóth; Djidjev and Vrt’o. Our refinement and analysis.

3. **Lower-bounding the Crossing Number**
   How to obtain a precise lower bound on the crossing number of a toroidal graph. Proving the approximation ratio.

4. **Conclusion and Future Steps**
1 Drawings and the Crossing Number

**Definition.** *Drawing of a graph* $G$:

- The vertices of $G$ are distinct points, and every edge $e = uv \in E(G)$ is a simple curve joining $u$ to $v$.
- No edge passes through another vertex, and no three edges intersect in a common point.
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![Graphs with different drawings](image)

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Warning. There are slight variations of the definition of crossing number, some giving different numbers! (Like counting odd-crossing pairs of edges.)
Computational complexity

**Remark.** It is practically very hard to determine the crossing number.

**Observation.** The problem $\text{CrossingNumber}(\leq k)$ is in $NP$: Guess a suitable drawing of $G$, then replace crossings with new vertices, and test planarity.

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**Theorem 2.** [Grohe, 2001], [Kawarabayashi and Reed, 2007] $\text{CrossingNumber}(\leq k)$ is in $FPT$. 
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Theorem 2. [Grohe, 2001], [Kawarabayashi and Reed, 2007] \textsc{CrossingNumber}(\leq k) is in \textit{FPT}.

Theorem 3. [PH, 2004] \textsc{CrossingNumber} is \textit{NP}-hard even on simple 3-connected cubic graphs.

Corollary 4. The minor-monotone version of c.n. is also \textit{NP}-hard.
Looking for “natural parametrizations”

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**Theorem 7.** [PH and GS, 2006] $\text{CrossingNumber}$ can be approximated within factor of $\Delta(G)$ for an almost planar graph $G$ in $O(n)$ time.

**Theorem 8.** [Gitler, Leaños, PH and GS, 2007] $\text{CrossingNumber}$ can be approx. w. factor of $4.5\Delta(G)^2$ for a projective graph $G$ in $O(n\log n)$ time.
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Question 9. Can we get any reasonable FPT algorithm for (approximating, at least?) \textsc{CrossingNumber} based on “how far” the graph is from planarity?

The next step — Toroidal graphs...
2 Drawing Toroidal Graphs with few Crossings

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**Approximation?**

Unfortunately, the above constructions in no way provide approximation algorithms.

The reason — lack of a corresponding lower bound on the crossing number...
Cut-and-redraw a toroidal graph

- We embed $G$ on the torus (linear time by [Mohar 1999]).

- We find a “shortest nonseparating” loop of length $k$ on the torus, using an $O(n \log n)$ algorithm of [Kutz 2006]. ($k = \text{dual edge-width}$ of $G$.)

- Cutting the torus into a cylinder, we “reconnect” the cut edges along a shortest length-$\ell$ dual path, producing $\leq k\ell + k^2/4$ crossings.
3 Lower-bounding the Crossing Number of Toroidal Graphs

For the rest we have $k$ the dual edge-width of $G$ on the torus, and $\ell$ the “dual length” of the cylindrical embedding of $G$ we cut out from our torus.

**Lemma 10.**

$$\text{cr}(G) \geq \left( \frac{1}{3\Delta^2} - o_k(1) \right) \cdot k\ell$$
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- We will find a large toroidal grid minor in $G$, relative to $k, \ell, \text{ and } \Delta$. 
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- We will find a large toroidal grid minor in $G$, relative to $k, \ell, \Delta$.
- If $H$ is a minor of $G$, and $H$ has maximum degree at most 4, then $\text{cr}(G) \geq \frac{1}{4} \text{cr}(H)$.
- The crossing number of the toroidal grid of size $p \times q$, where $p \geq q \geq 3$, is at least $\frac{1}{2}(q - 2)p$. 

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Actually, without asymptotic terms our lower bound reads $\text{cr}(G) \geq \frac{1}{4\Delta^2} \cdot k\ell$, provided that $k \geq 16 \lfloor \Delta/2 \rfloor$. 
For the rest we have $k$ the dual edge-width of $G$ on the torus, and $\ell$ the “dual length” of the cylindrical embedding of $G$ we cut out from our torus.

Hence we need to prove:

**Theorem 11.** $G$ contains a minor isomorphic to the toroidal grid of size

$$\max \left( \left\lfloor \frac{2k}{3 \left\lceil \Delta/2 \right\rceil} \right\rfloor, \left\lceil \frac{\ell}{\left\lfloor \Delta/2 \right\rfloor} \right\rceil \right) \times \left\lfloor \frac{2k}{3 \left\lfloor \Delta/2 \right\rfloor} \right\rfloor.$$
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Proof outline:

- Using [de Graaf and Schrijver, 1994] we get a toroidal grid minor of size

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Proof outline:

- Using [de Graaf and Schrijver, 1994] we get a toroidal grid minor of size \( \left\lfloor \frac{2}{3} \left\lfloor \frac{k}{\Delta/2} \right\rfloor \right\rceil \times \left\lfloor \frac{2}{3} \left\lfloor \frac{k}{\Delta/2} \right\rfloor \right\rceil \) in \( G \).

- We obtain another collection of \( \left\lfloor \frac{\ell}{\left\lfloor \Delta/2 \right\rfloor} \right\rceil \) pairwise disjoint cycles of \( G \) on our cylinder, using a network-flow duality argument.
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Proof outline:

- Using [de Graaf and Schrijver, 1994] we get a toroidal grid minor of size $\left\lfloor \frac{2}{3} \left\lfloor \Delta/2 \right\rfloor \right\rfloor \times \left\lfloor \frac{2}{3} \left\lfloor \Delta/2 \right\rfloor \right\rfloor$ in $G$.

- We obtain another collection of $\left\lfloor \frac{\ell}{\Delta/2} \right\rfloor$ pairwise disjoint cycles of $G$ on our cylinder, using a network-flow duality argument.

- We will then combine one collection of $\left\lfloor \frac{2}{3} \left\lfloor \Delta/2 \right\rfloor \right\rfloor$ cycles in $G$ with the latter collection to form a new toroidal grid minor of the required size.
Our main theoretical contribution actually is the following:

**Theorem 12.** Suppose a toroidal graph $H$ contains a collection $C$ of $p$ pairwise disjoint pairwise freely homotopic cycles, and an analogous collection $D$ of $q$ cycles, such that $D$ is not homotopic to an iteration of $C$.

Then $H$ contains a $p \times q$ toroidal grid minor.
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Unfortunately, the two cycle collections can interact in really nasty ways on the torus, and the proof requires a detailed technical analysis (proceedings).
4 Conclusion and Future Steps

Main result. We have got an $O(n \log n)$ time algorithm that approximates \textsc{CrossingNumber} on toroidal graphs up to a factor of $6\Delta(G)^2$,

provided that the graph embeds with dual edge-width at least $8\Delta(G)$. 
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Possible extensions. For graphs embedded on a higher orientable surface $\Sigma_g$. (Assume bounded $g$ and $\Delta$.)

- Repeat the algorithm of Section 2 for $g$ steps until $\Sigma_g$ is cut down to a plane. Denote by $k_i$ and $\ell_i$ the “dual lengths” obtained at step $i$.
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- It is straightforward to show that one gets $O(\max_{i=1,\ldots,g} k_i \cdot \ell_i)$ crossings.
- The same lower-bound proof now shows $\text{cr}(G) \geq \Omega(k_g \times \ell_g)$;
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- The same lower-bound proof now shows $\text{cr}(G) \geq \Omega(k_g \times \ell_g)$; but we need to prove $\text{cr}(G) \geq \Omega(\max_{i=1,\ldots,g} k_i \cdot \ell_i)$, which is still open (work in progress), and it does not seem easy to finish...