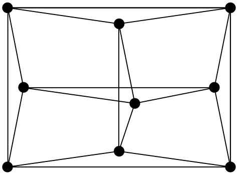
Petr Hliněný

On Crossing-Critical Graphs

Department of Computer Science

e-mail: `petr.hlineny@vsb.cz`
`http://www.cs.vsb.cz/hlineny`



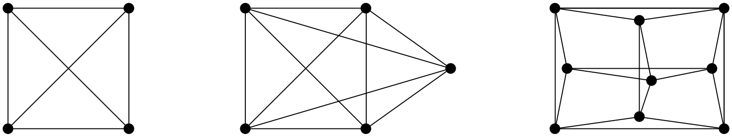
* Supported partly by the program "Information Society" of the Czech Academy of Sciences, project No. 1ET101940420.

1 Drawings and the Crossing Number

Definition. *Drawing of a graph* G:

– The vertices of G are distinct points,
and every edge $e = uv \in E(G)$ is a simple curve joining u to v.

– No edge passes through another vertex,
and no three edges intersect in a common point.



Definition. *Crossing number* $\mathrm{cr}(G)$
is the smallest number of edge crossings in a drawing of G.

Origin – Turán's work in brick factory, WW II.

Importance – in VLSI design [Leighton et al], graph visualization, etc.

Warning. There are variations of the definition of crossing number, not yet proved to be equivalent. (Like counting crossing or odd-crossing pairs of edges.)

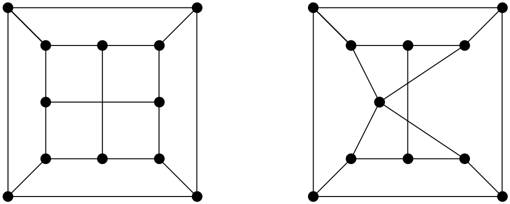
Different versions:

- *Rectilinear* crossing number – requires edges as straight lines.
  Same up to $\mathrm{cr}(G) = 3$, then much different.

- *Minor-monotone* crossing number – closes the definition down to minors.
  (Usual crossing number may grow up rapidly when contracting an edge!)

Definition. *Minor-monotone crossing number*

$$\mathrm{mcr}(G) = \min_{H:\, G \le H \ \text{(minor)}} \mathrm{cr}(H)\,.$$



Observation. (Fellows) If a cubic graph F is a minor of G, then F is in G as a subdivision. Hence for cubic F,

$$\mathrm{cr}(F) = \mathrm{mcr}(F)\,.$$

Computational Complexity

Remark. It is (practically) very hard to determine crossing number.

Observation. The problem $\textsc{CrossingNumber}(\leq k)$ is in NP:
Guess a suitable drawing of G, then replace crossings with new vertices, and test planarity.

. . . . . .

Theorem 1.1. [**Garey and Johnson**, 1983] *$\textsc{CrossingNumber}$ is NP-hard.*

Theorem 1.2. [**Grohe**, 2001] *$\textsc{CrossingNumber}(\leq k)$ is in FPT with parameter k, i.e. solvable in time $O(f(k) \cdot n^2)$.*

. . . . . .

A new result:

Theorem 1.3. [**PH**, 2004] *$\textsc{CrossingNumber}$ is NP-hard on simple 3-connected cubic graphs.*

Corollary 1.4. *The minor-monotone version of c.n. is also NP-hard.*

2 Crossing-Critical Graphs

What forces high crossing number?

- Many edges – cf. Euler's formula, and some strong enhancements [Ajtai, Chvátal, Newborn, Szemeredi, 1982; Leighton].
- Structural properties (even with few edges) – but what exactly?

Definition. Graph H is *k-crossing-critical*
– $\mathrm{cr}(H) \geq k$ and $\mathrm{cr}(H - e) < k$ for all edges $e \in E(H)$.

We study crossing-critical graphs to understand what structural properties force the crossing number of a graph to be large.

Notes:

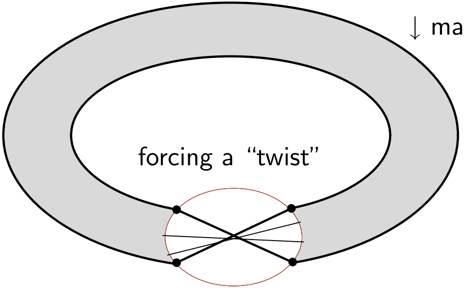
– 1-crossing-critical graphs are K_5 and $K_{3,3}$ (up to vertices of degree 2).

– An infinite class of 2-crossing-critical graphs, first by [Kochol].

– Many infinite classes of crossing-critical graphs are known today, and all tend to have similar "global" structure.
cf. [Oporowski], [Richter, Pinontoan], . . .

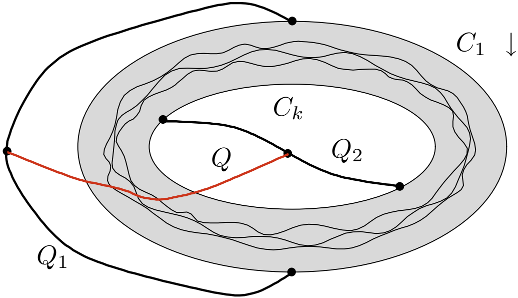
Constructing crossing-critical graphs

Twisted Möbius band:
(a classical idea)



↓ many homogeneous planar "tiles"

Crossed planar band:
[PH, 2001]



C_1 ↓ union k edge-disjoint cycles

Structure of crossing-critical graphs:

1999 **[Salazar]**: A k-crossing-critical graph has bounded tree-width in k.

- Conjecture **[Salazar and Thomas]**: an analogue holds for *path-width*.

2000 **[PH]**: Yes, a k-crossing-critical graph has bounded path-width in k.

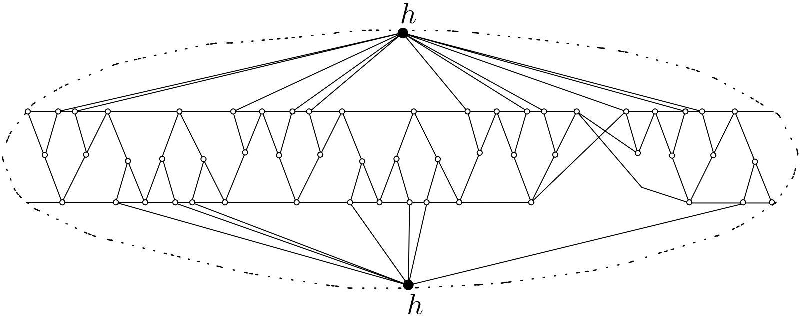
- Conjecture **[Richter, Salazar, and Thomassen]** an. for bandwidth: A k-crossing-critical graph has *bounded bandwidth* in k.
- Is that really true? Bounded bandwidth $\Rightarrow$ bounded max degree...

A new example:

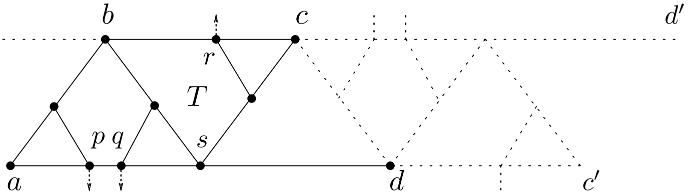
2003 **[PH]**: The bounded bandwidth conjecture is false in the projective plane. (A construction of a projective crossing-critical family with high degrees.)

3 The Example

A 2-crossing critical graph H_r in the projective plane, max degree $6r$.

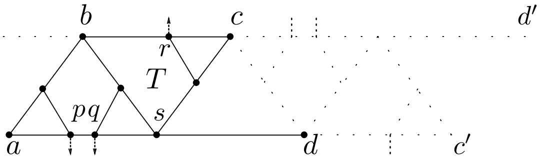


A detail of one of the $2r$ "tiles" in the graph:

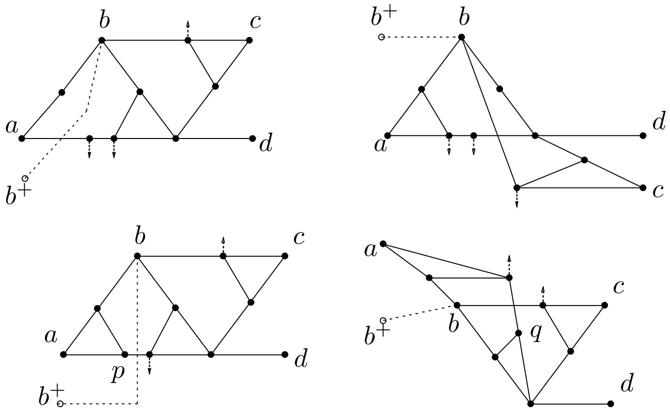


Sketch of Proof

Deleting any edge allows a drawing with ≤ 1 crossing.

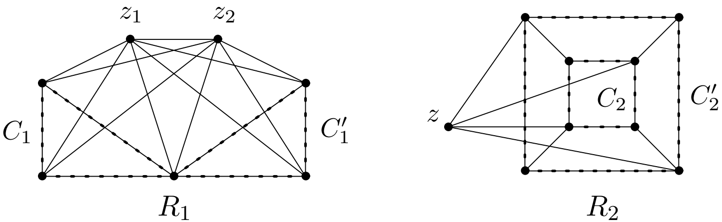


How to save a crossing with the twisted tile?



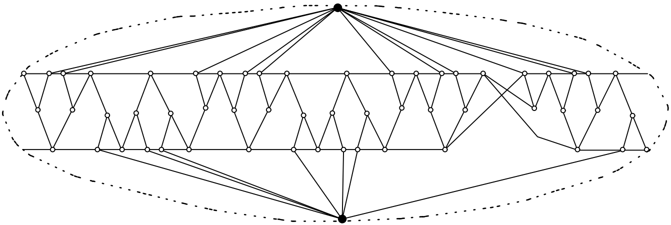
The graph H_r needs ≥ 2 crossings in the projective plane, $r > 2$.

Two graphs – excluded minors for embeddability in the projective plane:



If we "eliminate" one crossing in our graph H_r, the remaining graph is still not projective-planar. $\implies$ $\mathrm{cr}_p(H_r) \geq 2$.

Easily, for most edges f of H_r, the graph $H_r - f$ has an R_2-minor:



In the remaining cases, we instead "subdivide" one of the crossings.
(The subdivided vertices x and x' are identical.)