Structure and Generation of Crossing-critical Graphs, I.

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joint work with

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1 Crossing Number and Crossing-critical

**Crossing number** $cr(G)$: how many *edge crossings* are required to draw $G$ in the plane?
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![Diagram](image-url)
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![Crossing examples](image-url)
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- Many edges – cf. Euler’s formula, and some strong enhancements [Ajtai, Chvátal, Newborn, Szemeredi, 1982; Leighton].
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**Definition.** Graph $H$ is **c-crossing-critical** if $\text{cr}(H) \geq c$ and $\text{cr}(H - e) < c$ for all edges $e \in E(H)$.

We study crossing-critical graphs to understand what structural properties force the crossing number of a graph to be large.
Some starting examples

- Kuratowski (30): The only 1-crossing-critical graphs $K_5$ and $K_{3,3}$.

(Yes, up to subdivisions, but we ignore that...)

\begin{center}
\begin{tikzpicture}[scale=0.7]
\draw[fill=black] (0,0) circle (0.1);
\draw[fill=black] (1,1) circle (0.1);
\draw[fill=black] (2,0) circle (0.1);
\draw[fill=black] (1,-1) circle (0.1);
\draw[fill=black] (0,2) circle (0.1);
\draw[fill=black] (2,2) circle (0.1);
\draw[fill=black] (2,-2) circle (0.1);
\draw[fill=black] (-2,-2) circle (0.1);
\draw[fill=black] (-2,2) circle (0.1);
\draw[fill=black] (-1,1) circle (0.1);
\draw[fill=black] (-1,-1) circle (0.1);
\draw (0,0) -- (1,1) -- (2,0) -- (1,-1) -- (0,2) -- (2,2) -- (2,-2) -- (-2,-2) -- (-2,2) -- (-1,1) -- (-1,-1) -- (0,0);
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Some starting examples

- Kuratowski (30): The only 1-crossing-critical graphs $K_5$ and $K_{3,3}$.

- Širáň (84), Kochol (87): Infinitely many $c$-crossing-critical graphs for every $c \geq 2$, even simple 3-connected.

(Yes, up to subdivisions, but we ignore that...)
2 More Crossing-critical Constructions

- Salazar (03): every edge “drops” $cr(G)$ a lot ($\sqrt{c}$).
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- Salazar (03): every edge “drops” $cr(G')$ a lot ($\sqrt{c}$).

- Hliněný (02): “drop” by 1, but having planarizing edge.
A note on degree properties

- An infinite $c$-crossing-critical family has average degree in $[3, 6]$ (no cheating with degree-2 vertices!).
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- Excluding average degree 3 via Graph minors...
- Excluding avg. deg. 6 – Hernández-Vélez, Salazar and Thomas (12).
- Getting average degree close to 3 – Bokal’s (10) staircase strip.
• Construction with arbitrary even degrees – PH (11),
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• and with arbitrary odd degrees – Bokal, Bračič, Derňár, PH (15).
\[ \text{... and a bit of surprise} \]

- Dvořák, Mohar (10): A \( c \)-crossing-crit. graph with unbounded degree, \( c \geq 171 \).
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  Fully described 2-crossing-critical graphs up to fin. small exceptions.
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- Dvořák, Hliněný, Mohar, Postle (11, not published):
  A $c$-crossing-critical graph cannot contain a deep nest, and so it has bounded dual diameter.
What kinds of crossing-critical graphs do we have?

Informally, “thin and long” bands, joined together, and huge faces around…

+ combinations of these together
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II. There are well-defined local operations (replacements) that can reduce any large $c$-crossing-critical graph to a smaller one.
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II. There are well-defined local operations (replacements) that can reduce any large $c$-crossing-critical graph to a smaller one.

III. There are finitely many well-defined building bricks that can produce all $c$-crossing-critical graphs from a finite set of base graphs.
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4 To be continued...