# A Tighter Insertion-based Approximation of the Graph Crossing Number

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# 1 A Bit of History for Start

(A WW II story)

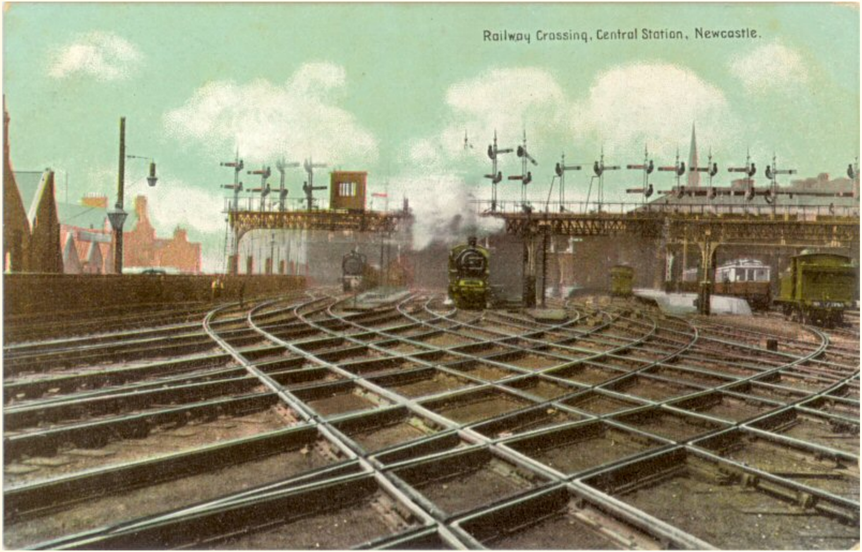
"*There were some kilns where the bricks were made and some open storage yards where the bricks were stored. All the kilns were connected by rail with all the storage yards. The bricks were carried on small wheeled trucks to the storage yards. . . the work was not difficult; the trouble was only at the crossings. The trucks generally jumped the rails there, and the bricks fell out of them; in short this caused a lot of trouble and loss of time. . . the idea occurred to me that this loss of time could have been minimized if the number of crossings of the rails had been minimized.*

**But what is the minimum number of crossings?**

. . . This problem has become a notoriously difficult unsolved problem."

Pál Turán, *A note of welcome.*
Journal of Graph Theory (1977)

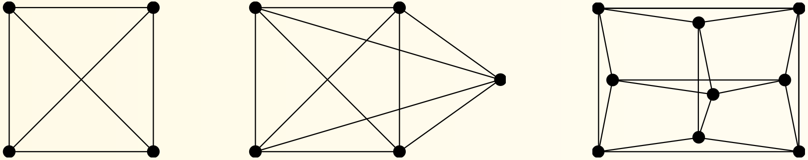
# Crossings. . .



# 2 Graph Crossing Number

**Definition**. *Drawing of a graph $G$*:

- The vertices of $G$ are distinct points,
  and every edge $e = uv \in E(G)$ is a simple curve joining $u$ to $v$.
- No edge passes through another vertex,
  and no three edges intersect in a common point.



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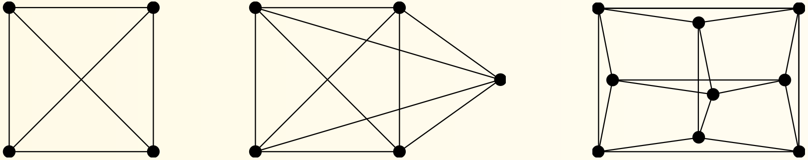

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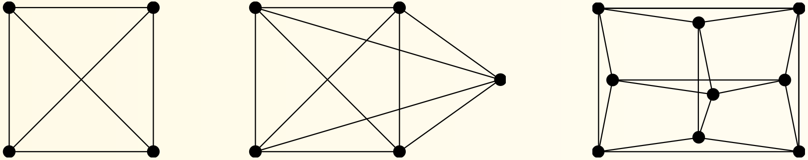

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**Warning.** There are slight variations of the definition of crossing number, some giving different numbers! Such as counting *odd-crossing pairs* of edges. [Pelsmajer, Schaeffer, Štefankovič, 2005]...

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- Much worse – hard already for planar graphs plus one edge ! [**Cabello and Mohar**, 2010]

Can anything be computed efficiently?

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- Up to factor $\log^3 |V(G)|$ ($\log^2 \cdot$) for $cr(G)+|V(G)|$ with bounded degrees; [**Even, Guha and Schieber**, 2002]
- Constant factors for surface-embedded bounded-degree graphs;

  [**Gitler et al**, 2007], [**PH and Salazar**, 2007], [**PH and Chimani**, 2010]

# 3 Planar Insertion Problems

**Definition**. Given a planar graph $G$ and a set $F$ of additional edges (vert.?). Find a *drawing of* $G + F$ minimizing the edge crossings *ins*$(\boldsymbol{G}, \boldsymbol{E})$

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**Remark.** Difficulty of insertion problems comes from possible inequivalent embeddings of $G$.

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- Multiple edge insertion $\leftrightarrow$ graph $G + F$ (a very general case)
  - *cr*$(G + F)$ approximated by *ins*$(G, F)$; [**Chimani, PH, and Mutzel**, 2008]
  - however, *ins*$(G, F)$ is NP-complete! (as well as finding $F$)

# 4 Approximating MEI up to Additive Factor

- [**Chuzhoy, Makarychev, and Sidiropoulos**, 2011 SODA]
  Using MEI, a solution to $cr(G+F)$ for given planar $G$ and $F$, with
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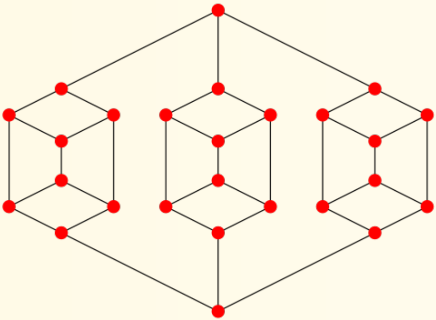
**Theorem 1.** *Given a planar graph $G$ and an edge set $F$, $F \cap E(G) = \emptyset$, Algorithm 2 described below finds, in*

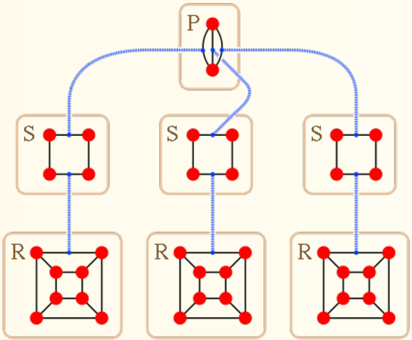
$$O(|F| \cdot |V(G)| + |F|^2) \text{ time,}$$

*an approximate solution to the MEI problem for $G$ and $F$ with*

$$\leq ins(G,F) + (\lfloor \tfrac{1}{2}\Delta(G) \rfloor + \tfrac{1}{2}) \cdot (|F|^2 - |F|) \text{ crossings.}$$

## Gentle introduction to SPQR trees






- Graph broken into the *blocks* first.
- Then, for pairwise gluing on *virtual skeleton edges*, we have got
  - *S-nodes* for serial skeletons,
  - *P-nodes* for parallel skeletons,
  - *R-nodes* for $3$-connected components.

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3. Fix an *embedding $\Gamma$ of $G$* by suitably combining the embedding preferences from step 2 (at least one happy con-chain per node).
4. Independently compute the *insertion paths* for each edge $e \in F$ into the fixed embedding $\Gamma$, as shortest dual paths.

## Proof sketch

A very informal one, neglecting all technical obstacles. . .


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- Two con-chains can split/merge twice, hence $\leq 2\binom{|F|}{2}$ dirty passes.
- Every dirty pass is associated with a $1$- *or* $2$-*cut*, and the inserted edge needs $\leq \lfloor \Delta(G)/2 \rfloor$ crossings to "pass by" it. Altogether

$$\leq \; \mathit{ins}(G,F) + \left(2\left\lfloor \frac{\Delta(G)}{2} \right\rfloor + 1\right) \cdot \binom{|F|}{2}. \qquad \square$$

# 5 Consequences

**Theorem 3.** *Given a planar graph $G$ and an edge set $F$, $F \cap E(G) = \emptyset$, Algorithm 2 finds an approximate solution to cr$(G+F)$ with*

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- Can the MEI $(G, F)$ problem have, say, an FPT algorithm wrt. $|F|$?