Approximating Multiple Edge Insertion and the Crossing Number

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joint work with **Markus Chimani**
Osnabrück University, Germany

0 Bit of History for Start

A WW II story

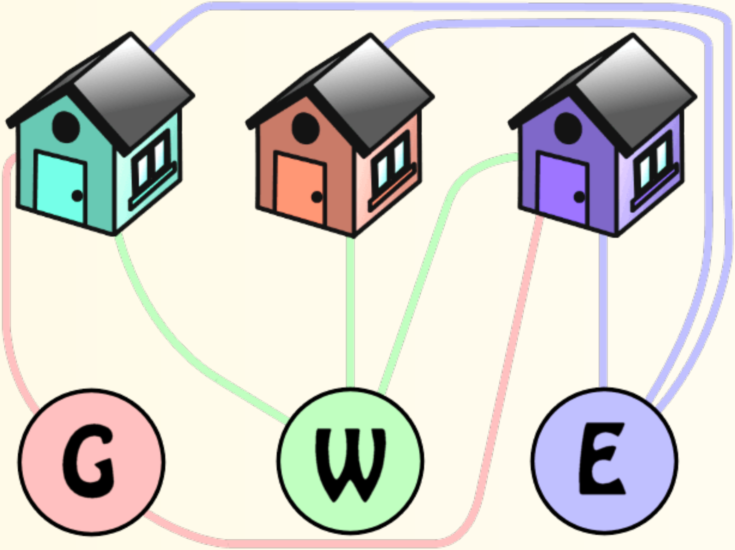
“*There were some kilns where the bricks were made and some open storage yards where the bricks were stored. All the kilns were connected by rail with all the storage yards. The bricks were carried on small wheeled trucks to the storage yards... the work was not difficult; the trouble was only at the crossings. The trucks generally jumped the rails there, and the bricks fell out of them; in short this caused a lot of trouble and loss of time... the idea occurred to me that this loss of time could have been minimized if the number of crossings of the rails had been minimized.*

But what is the minimum number of crossings?

... This problem has become a notoriously difficult unsolved problem.”

Pál Turán, *A note of welcome.*
Journal of Graph Theory (1977)

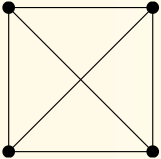
Or, can you avoid all the crossings?

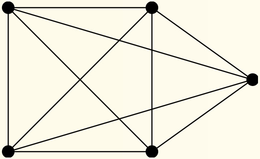


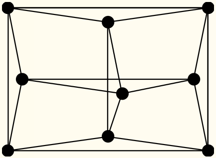
1 Graph Crossing Number

Definition. *Drawing of a graph G*:

- The vertices of G are distinct points,
  and every edge $e = uv \in E(G)$ is a simple curve joining u to v.
- No edge passes through another vertex,
  and no three edges intersect in a common point.







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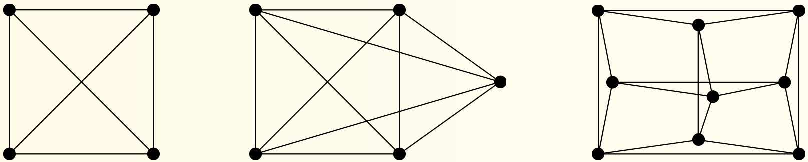

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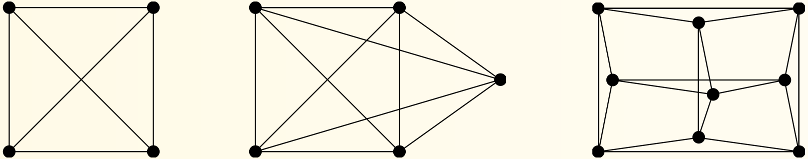

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Warning. There are slight variations of the definition of crossing number, some giving different numbers! Such as counting *odd-crossing pairs* of edges. [Pelsmajer, Schaeffer, Štefankovič, 2005]...

2 How to Compute the Crossing Number

Not easily. . . !

NP-hardness

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- No constant factor $c > 1$ -approximation; [**Cabello**, 2013]

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- Constant-factor for surface-embedded bounded-degree graphs;
  [**Gitler et al**, 2007], [**PH and Salazar**, 2007], [**PH and Chimani**, 2010]

3 Planar Insertion Problems

Keeping "most of" G planar...

Definition. Given a planar graph G and a set F of additional edges (vert.). Find a *drawing of* $G + F$ minimizing the edge crossings *ins*$(\boldsymbol{G}, \boldsymbol{E})$ such that the subdrawing of G is plane.

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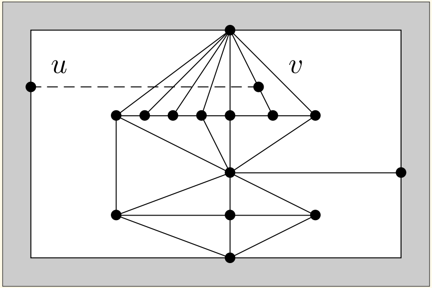
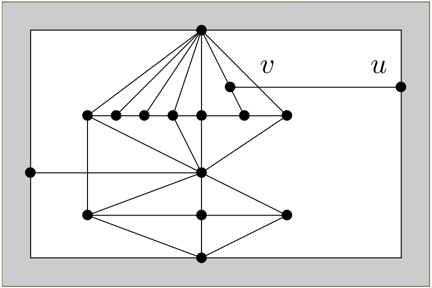
  but we may hope for a special small F... (and there are other ways)

Insertion can be very far from Crossing number

See, e.g., the following example:





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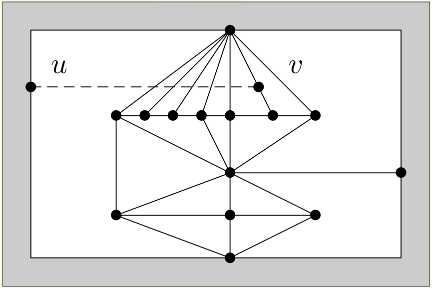

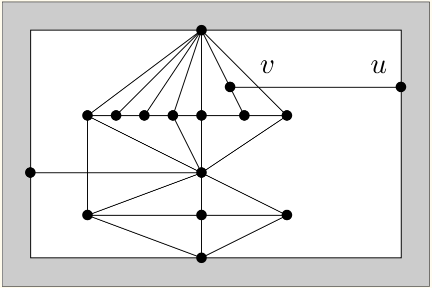

Remark. In cubic planar graphs, edge insertion is optimal for crossing number. [**Riskin**, 1996]

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  - however, how to compute *ins*(G,F)? – enough to approximate!

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Computing $\mathit{ins}(G, F)$ for planar connected G:

- [**Chuzhoy, Makarychev, and Sidiropoulos**, 2011 SODA]

  $$\leq \mathcal{O}(\Delta(G)^3 \cdot |F| \cdot \mathit{cr}(G + F) + \Delta(G)^3 \cdot |F|^2) \text{ crossings,}$$

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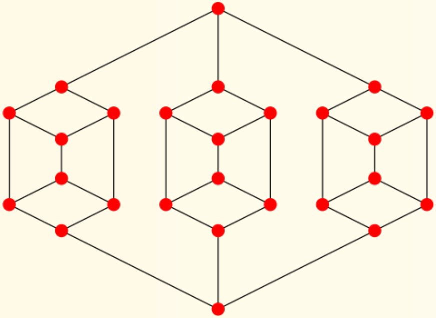
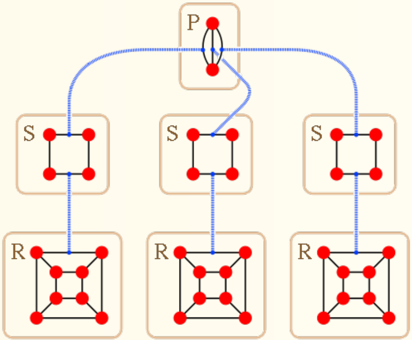
So called *SPQR trees* play key role in both the approaches.

Gentle introduction to SPQR trees





- Graph broken into the *blocks* first.
- Then, for pairwise gluing on *virtual skeleton edges*, we have got
  - *S-nodes* for serial skeletons,
  - *P-nodes* for parallel skeletons,
  - *R-nodes* for 3-connected components.

5 Better Additive Approximation for MEI

Theorem. Given a conn. planar graph G and an edge set F, $F \cap E(G) = \emptyset$, the below Algorithm finds, in $\mathcal{O}(|F|^2 \cdot |V(G)|)$ time, an approximate solution to the MEI problem for G and F with

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Remark. This estimate is assymptotically tight wrt. the difference between $ins(G,F)$ and the sum of individual insertions:

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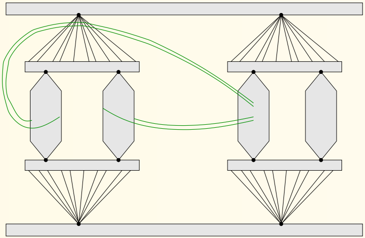

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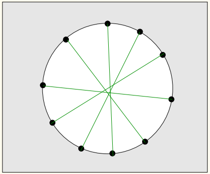

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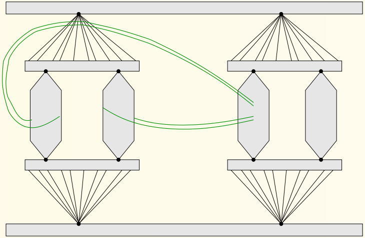

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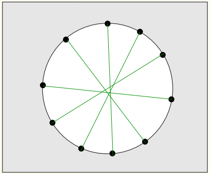
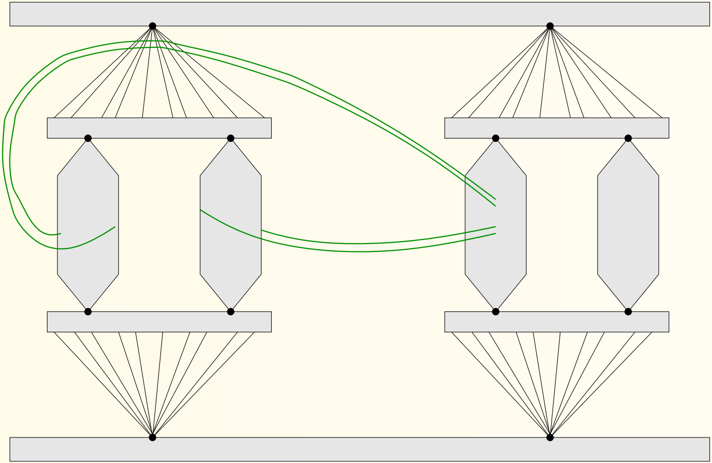


Corollary. The below Algorithm computes a drawing of $G+F$ with crossings

$$\leq \; 2|F| \cdot \lfloor \tfrac{1}{2}\Delta(G) \rfloor \cdot \mathit{cr}(G+F) + 2|F| \cdot \lfloor \log_2 |F| \rfloor \cdot \lfloor \tfrac{1}{2}\Delta(G) \rfloor + \binom{|F|}{2}.$$

$\Omega(|F| \cdot \log_2 |F|)$



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A naive approach...

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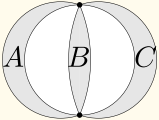
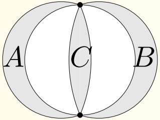
P-node: cyclic permutation  or 

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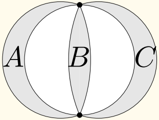
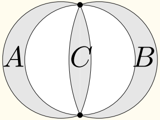
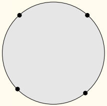
P-node: cyclic permutation 

or 

S-node: just a cycle, but having two faces 

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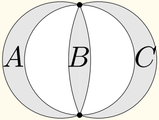
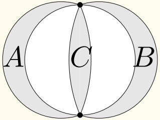
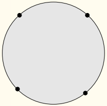
P-node: cyclic permutation 

or 

S-node: just a cycle, but having two faces 

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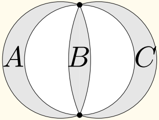
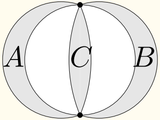
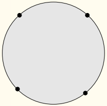
  P-node: cyclic permutation  or 

  S-node: just a cycle, but having two faces 

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sSPQR tree – "*serialized*"; insert dummy S-nodes between all P,R nodes.

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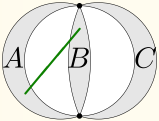

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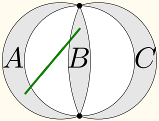

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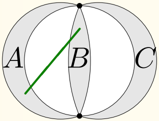

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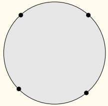

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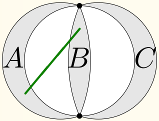

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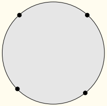

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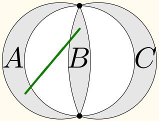

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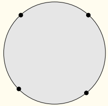

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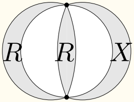
Now making precise!

Tackle nonlocality – big hidden problem of naive preferences;



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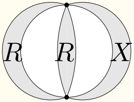

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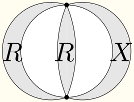

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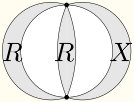

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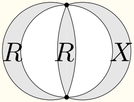

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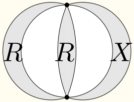

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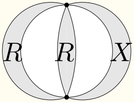

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Final touch - $\log_2 |F|$

A "smart way" of combining embedding preferences in the Algorithm, plus a clever trick in the proof of the approximation guarantee...

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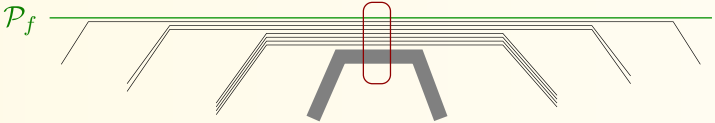

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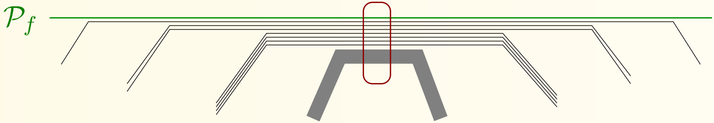

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- And, see Markus' talk...

Thank you for your attention.