Petr Hliněný

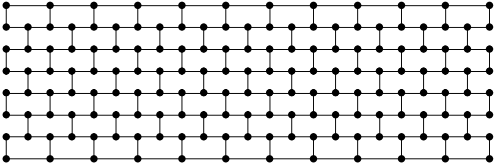
Crossing Number is Hard for Cubic Graphs

Department of Computer Science
FEI, VŠB – Technical University of Ostrava

and

Institute for Theoretical Computer Science
MFF, Charles University, Czech Republic

e-mail: petr.hlineny@vsb.cz
http://www.cs.vsb.cz/hlineny

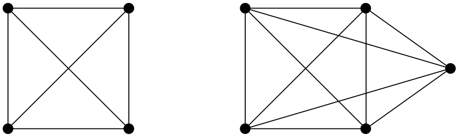


2000 Math Subjects Classification: 05C10, 05C62, 68R10

1 Drawings and the crossing number

Definition. *Drawing of a graph* G:

- The vertices of G are distinct points,
  and every edge $e = uv \in E(G)$ is a simple curve joining u to v.
- No edge passes through another vertex,
  and no three edges intersect in a common point.



Definition. *Crossing number* $\mathrm{cr}(G)$
is the smallest number of edge crossings in a drawing of G.

Origin – Turán's work in brick factory, WW II.

Importance – in graph visualization, VLSI design, etc.

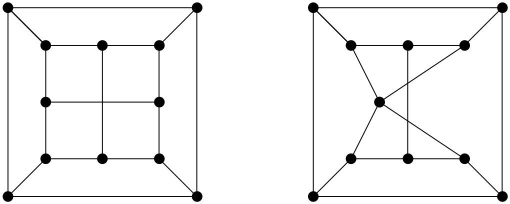
Warning. There are variations of the definition of crossing number, not yet proved to be equivalent. (Like counting crossing or odd-crossing pairs of edges.)

Different versions:

- *Rectilinear* crossing number – requires edges as straight lines. Same up to $\mathrm{cr}(G) = 3$, then much different.
- *Minor-monotone* crossing number – closes the definition down to minors. (Usual crossing number may grow up rapidly when contracting an edge!)

Definition. *Minor-monotone crossing number*

$$\mathrm{mcr}(G) = \min_{H:\, G \le H \ \text{(minor)}} \mathrm{cr}(H)\,.$$



Observation. (Fellows) If a cubic graph F is a minor of G, then F is in G as a subdivision. Hence for cubic F,

$$\mathrm{cr}(F) = \mathrm{mcr}(F)\,.$$

2 Crossing-Critical Graphs

What forces high crossing number?

- Many edges – cf. Euler's formula, and some strong enhancements.
- Structural properties (even with few edges) – but what exactly?

Definition. Graph H is *k-crossing-critical*
– $\mathrm{cr}(H) \geq k$ and $\mathrm{cr}(H - e) < k$ for all edges $e \in E(H)$.

We study crossing-critical graphs to understand what structural properties force the crossing number of a graph to be large.

Notes:

- 1-crossing-critical graphs are K_5 and $K_{3,3}$ (up to vertices of degree 2).
- An infinite class of 2-crossing-critical graphs (first by Kochol).
- Many infinite classes of crossing-critical graphs are known today, and all tend to have similar "global" structure.

Structure of crossing-critical graphs:

1999 **[Salazar]**: A k-crossing-critical graph has bounded tree-width in k.

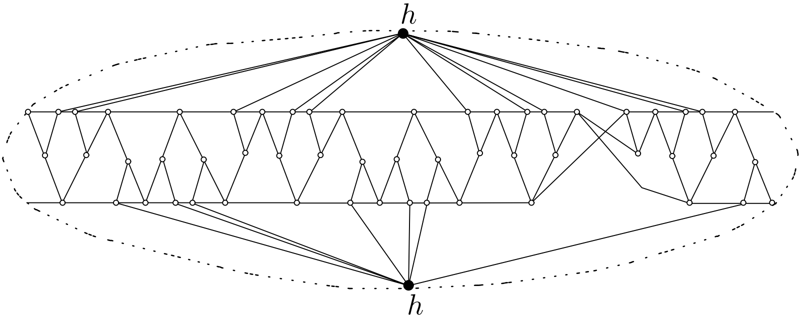
- Conjecture **[Salazar and Thomas]**: an analogue holds for *path-width*.

2000 **[PH]**: Yes, a k-crossing-critical graph has bounded path-width in k.

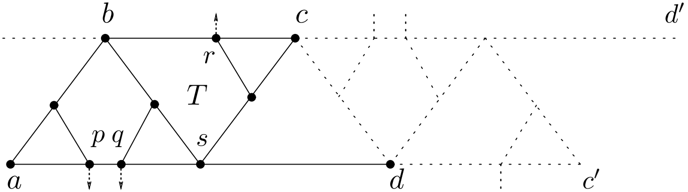
- Conjecture **[Richter, Salazar, and Thomassen]** an. for bandwidth: A k-crossing-critical graph has *bounded bandwidth* in k.
- Is that really true? Bounded bandwidth $\Rightarrow$ bounded max degree...

2003 **[PH]**: The bounded bandwidth conjecture is false in the projective plane. (A construction of a projective crossing-critical family with high degrees.)

A 2-crossing critical graph in the projective plane, max degree $6r$.



A detail of one of the $2r$ "tiles" in the graph:



3 Computational Complexity

Remark. It is (practically) very hard to determine crossing number.

Observation. The problem $\textsc{CrossingNumber}(\leq k)$ is in NP:
Guess a suitable drawing of G, then replace crossings with new vertices, and test planarity.

. . . . . .

Theorem 3.1. [**Garey and Johnson**, 1983] *$\textsc{CrossingNumber}$ is NP-hard.*

Theorem 3.2. [**Grohe**, 2001] *$\textsc{CrossingNumber}(\leq k)$ is in FPT with parameter k, i.e. solvable in time $O(f(k) \cdot n^2)$.*

. . . . . .

Our results:

Theorem 3.3. *$\textsc{CrossingNumber}$ is NP-hard on simple cubic graphs.*

Corollary 3.4. *The minor-monotone version of c.n. is also NP-hard.*

NP-reduction from OLA

CrossingNumber is NP-hard on cubic graphs.

via. . .

OptimalLinearArrangement:

Input: An n-vertex graph G, and an integer a.

Question: Is there a bijection $\alpha : V(G) \to \{1, \dots, n\}$ (a *linear arrangement* of vertices) such that the weight of α is

$$\sum_{uv \in E(G)} |\alpha(u) - \alpha(v)| \ \leq \ a\,? \tag{1}$$

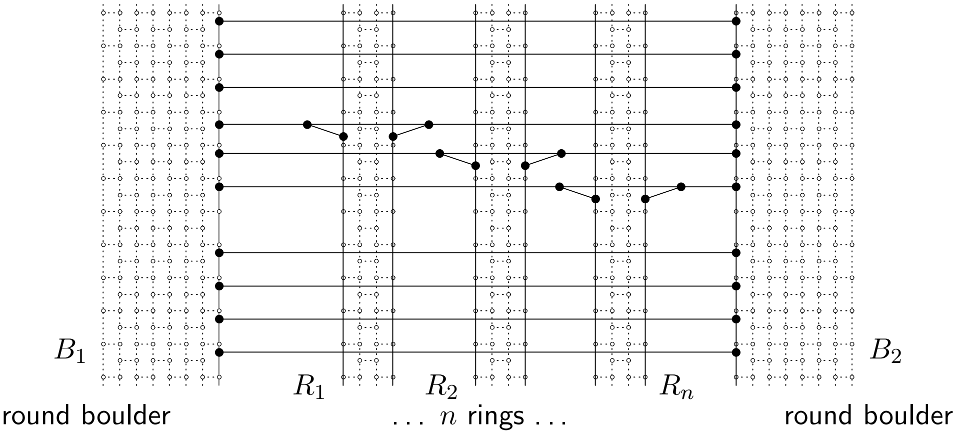
- NP-complete by [Garey and Johnson].
- Used to prove Theorem 3.1 [Garey and Johnson].
- Used (differently) also to prove our Theorem 3.3.

Question: What about a reduction from Planar 3-SAT?

– Tried (reasonably hard) several times, but yet unsuccessful. . . Why?

4 Sketch of a Proof

For an instance G, a of OLA, construct a graph H_G, and test $\mathrm{cr}(H_G)$...

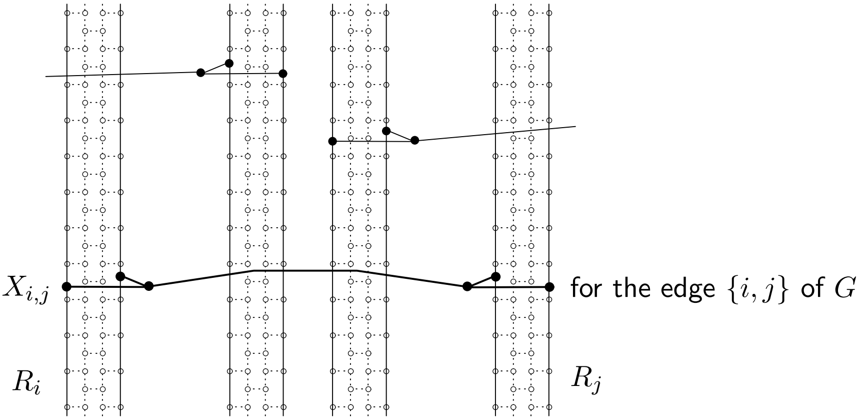


Boulders – huge, and keeping everything "in place".

Rings – one for each vertex of G, their order is (like) α.

Spokes – horizontal connections, force rings to lie "between" the boulders.

Handles for the edges of G are attached to the (above) skeleton as follows:



Lemma 4.1. *Let us, for a given graph G on n vertices and m edges, construct the graph H_G as described above. If G has a linear arrangement of weight A, then*

$$\mathrm{cr}(H_G) \leq (s + rn)nt + 2(A + m)t - 4m\,,$$

where $s + rn$ is the total number of spokes, and t is the thickness of each ring.

Converse direction

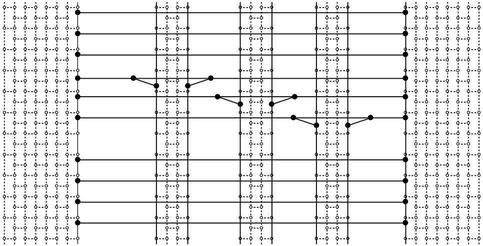
The following statement, together with Lemma 4.1, validates our reduction.

Proposition 4.2. *If an optimal linear arrangement of G has weight A, then*

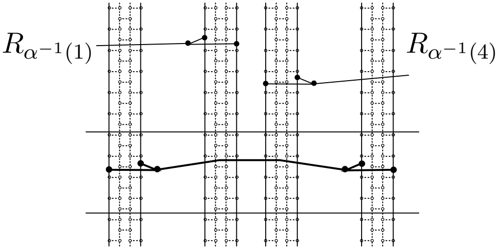
$$\mathrm{cr}(H_G) \geq (s + rn)nt + 2(A + m)t - 8m \,.$$

We proceed the proof along the following sequence of claims:

- In the optimal drawing of H_G, the boulders B_1, B_2 are drawn with no edge crossings.
- In the optimal drawing of H_G, each main cycle of every ring R_i is drawn as a closed curve separating the subdrawing of B_1 from that of B_2.

B_1  B_2

- The rings are drawn (almost) in an order $R_{\alpha^{-1}(1)}, \dots, R_{\alpha^{-1}(n)}$.
- Drawings of the edge handles can be separated (by curves of some spokes) into disjoint areas of the plane.



- The separated edge handles generate (almost) as many edge crossings as expected.

Hence we determine the OLA value A for G from

$$(s + rn)nt + 2(A + m)t - 4m \geq \mathrm{cr}(H_G) \geq (s + rn)nt + 2(A + m)t - 8m\,.$$