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New almost-planar crossing-critical graph families

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1 Crossing-Critical Graphs

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- $\text{cr}(H) \geq k$ and $\text{cr}(H - e) < k$ for all edges $e \in E(H)$.

We study crossing-critical graphs to understand what structural properties force the crossing number of a graph to be large.
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**Remarks:**
- 1-crossing-critical graphs are $K_5$ and $K_{3,3}$ (up to vertices of degree 2).
- Infinite classes of 3, 2-crossing-critical graphs, [Širáň 84, Kochol 87].
- Many infinite classes of crossing-critical graphs are known today, and they tend to have similar “global” structure.
Constructing crossing-critical graphs

Twisted Möbius band: (a classical idea)

forcing a "twist"

↓ many homogeneous planar “tiles”
Constructing crossing-critical graphs

**Twisted Möbius band:**
(a classical idea)

![Diagram of twisted Möbius band]

forcing a “twist”

↓ many homogeneous planar “tiles”

**Crossed planar belt:**
[PH, 2001]

![Diagram of crossed planar belt]

↓ union of $k$ edge-disjoint cycles
Constructing crossing-critical graphs

**Twisted Möbius band:**
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![Diagram showing a twisted Möbius band forcing a “twist”](image)

\[ \downarrow \text{many homogeneous planar “tiles”} \]

**Crossed planar belt:**
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![Diagram showing a crossed planar belt](image)

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**Zip-product:**
[Bokal, 2005] Composing crossing-critical graphs...
Looking at vertex degrees

- [folklore] Infinite families of simple 3-connected crossing-critical graphs can have average degree in $(3, 6]$. (Lower bound by connectivity and graph minors, upper via Euler.)

1993 [Richter and Thomassen]: Are there infinite families of simple 5-regular crossing-critical graphs?
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2006 **[Bokal]**: Are there **infinite** families of simple 3-connected crossing-critical graphs having arbitr. number of vertices of degrees **other than** 3, 4, 6?
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- The case of odd degrees > 3 remains open...
2 Our “Belt” Construction

• Constructing simple 3-connected \textit{almost-planar} crossing-critical graphs (such that deleting \textit{one edge} leaves them planar).

• \textbf{Extending} the previous construction [PH, 2001] much further...
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- Constructing simple 3-connected *almost-planar* crossing-critical graphs (such that deleting one edge leaves them planar).
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**Definition.** Crossed $k$-belt graphs:

- Edge-disj. planar union $C_1 \cup \ldots \cup C_k$, with a 4-terminal “bridge”.
- Forming many disjoint “radial” paths, separating the bridge terminals.
- No vertex of degree > 4 on $C_k$, that is, $C_k \cap C_{k-2} = \emptyset$. 
Getting high-degree vertices

- We start with a “crossed fence” from [PH 2001],
Getting high-degree vertices

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- The following modif. produce vertices of degrees $2k - 2$, $2k - 4$, …

— the resulting graphs are all crossed $k$-belt graphs.
Proposition 1. Let \( k \) be fixed. For every integer \( m \) there is a crossed \( k \)-belt graph which is simple 3-connected and which contains more than \( m \) vertices of each of even degrees \( \ell = 4, 6, 8, \ldots, 2k - 2 \).
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Theorem 2. For $k \geq 3$, every crossed $k$-belt graph is $k$-crossing-critical.

Proof. By induction on $k$:

- $k = 1 \rightarrow$ a subdivision of nonplanar $K_{3,3}$ ($k = 2$ – a false statement),
  $k = 3$ — the base case follows from the $k = 1$ case.
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- If $C_1$ is crossed, then remove it, forming a crossed $(k - 1)$-belt graph, and continue.

- If $C_1$ is not crossed, then it forms a face in the optimal drawing. Then the radial paths witness $2k - 2$ $C_1$-ears separating the bridge terminals on $C_1$, forcing too many crossings.
3 Average Degrees in $[4, 6)$

**Theorem 3.** For every odd $k > 3$ there are infinitely many simple 3-connected crossed $k$-belt graphs with the average degree equal to any rational value in the interval $[4, 6 - \frac{8}{k+1})$. 
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**Proof.** We start with the following belt, and apply suitably local splittings...
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Thank you for your attention... 

Any solutions or counterexamples?