Lower Bounds on the Crossing Number of Surface-Embedded Graphs I: Up to Torus.

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joint work with **Gelasio Salazar**

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1 Crossing Number of a Graph

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Warning. There are slight variations of the definition of crossing number, some giving different numbers! (Like counting *odd-crossing pairs* of edges. [Pelsmajer, Schaeffer, Štefankovič, 2005]...)

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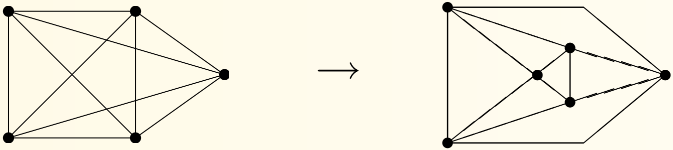
Observation. The problem $\textsc{CrossingNumber}(\leq k)$ is in NP:
Guess a suitable drawing of G, then replace crossings with new vertices, and test planarity.



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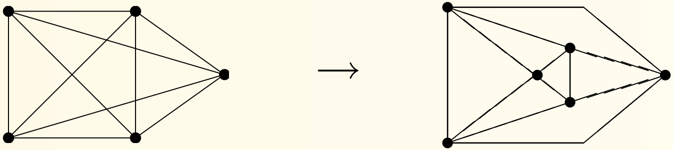

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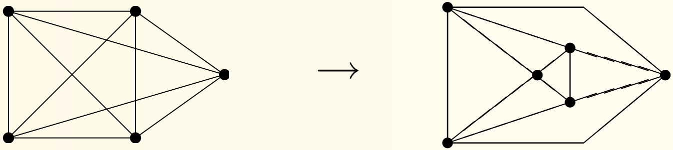

Theorem 1. [**Garey and Johnson**, 1983] *$\textsc{CrossingNumber}$ is NP-hard.*

Fact (sad. . .). We know of no natural graph class with nontrivial and yet efficiently computable $\textsc{CrossingNumber}$ problem.

More on complexity of crossing number

Theorem 2. [**Grohe**, 2001] $\textsc{CrossingNumber}(\leq k)$ *is in* FPT *with parameter* k*, i.e. solvable in time* $O(f(k) \cdot n^2)$*.*

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Theorem 3. [**Even, Guha and Schieber**, 2002] $\textsc{CrossingNumber}$ *can be efficiently approximated:* $\mathrm{cr}(G) + |V(G)|$ *up to a factor of* $\log^3 |V(G)|$ *for graphs G of bounded degree.*

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Theorem 4. [**PH**, 2004] $\textsc{CrossingNumber}$ *is* NP*-complete even on simple* 3*-connected cubic graphs.*

- The reduction by Garey and Johnson created vertices of very high degrees.
- The important cubic case is minor–monotone, and yet the problem remains hard.

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Theorem 5. [**Cabello and Mohar**, 2010]
Given a planar graph G and two non-adjacent vertices $u, v \in V(G)$, it is NP-complete to determine the crossing number of $G + uv$!

- The reduction by Cabello and Mohar uses unbounded vertex degrees. So, what if we also bound the degrees?

- And what about **constant factor approximations**?

Theorem 6. [**PH and GS**, 2006] *Let G be a planar graph and u, v nonadjacent vertices of G. Then there is a planar embedding of G to which the edge uv can be inserted using at most* $\mathbf{\Delta}(G) \cdot \mathrm{cr}(G + uv)$ **crossings**.

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Theorem 6. [**PH and GS**, 2006] *Let G be a planar graph and u, v nonadjacent vertices of G. Then there is a planar embedding of G to which the edge uv can be inserted using at most* $\mathbf{\Delta(G) \cdot cr(G + uv)}$ **crossings**.

  - this gives a useful *lower bound* on $\mathrm{cr}(G + uv)$...
  - improved down to factor $\Delta(G)/2$ by [**Cabello and Mohar**, 2008].

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Drawing idea. [**Gitler, Leaños, PH and GS**, 2007]

- Cut the projective embedding of G at r points (and open it to the plane).
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Now a matching *lower bound* is needed to derive the foll. conclusion. . .

Theorem 7. CrossingNumber *of a (sufficiently dense embedded) projective graph G can be approximated within the factor $4.5\Delta(G)^2$.*

The key: Getting a suitable lower-bound

Theorem 8. [**Gitler, Leaños, PH and GS**, 2007]
If G embeds in the projective plane with face-width at least $r \geq 6$, then the crossing number of G in the plane is at least $r^2/36$.

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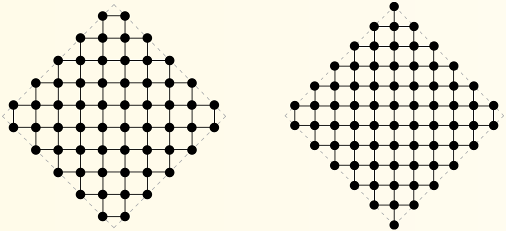
To prove this theorem, we argue:

Claim. Every graph that embeds in the projective plane with face-width r has a minor isomorphic to the *projective diamond grid P_r*:



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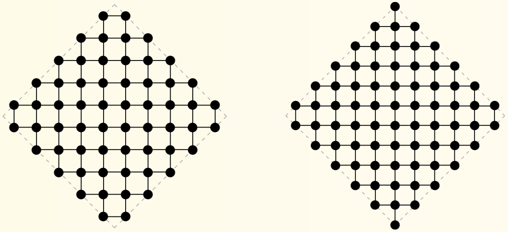
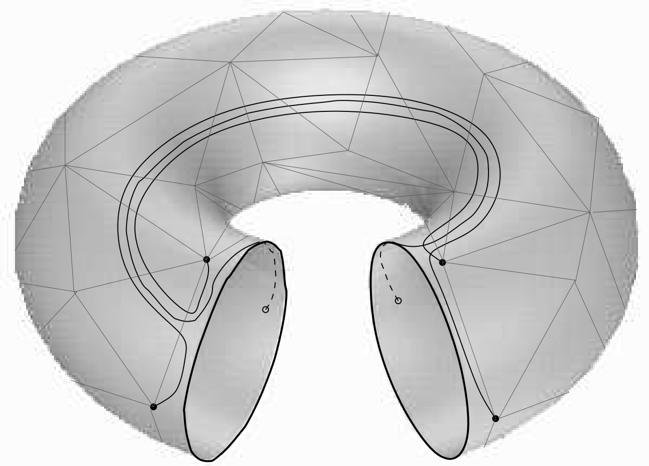


Claim. If G has an H-minor and $\Delta(H) = 4$, then $\mathrm{cr}(G) \geq \frac{1}{4}\,\mathrm{cr}(H)$.

Drawing toroidal graphs – the next step



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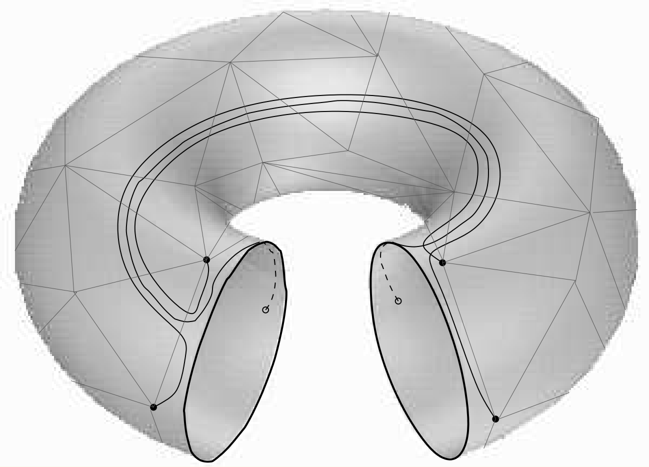

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- After turning the torus into a cylinder, reconnect the cut edges "through", producing $\leq (k\ell + k^2/4) \cdot \lfloor \Delta/2 \rfloor^2$ crossings (so, $\leq (3\Delta^2/8) \cdot k\ell$).

Again: Getting a suitable lower-bound

Theorem 9. *Respecting the above sketch of redrawing a toroidal graph into the plane,*

$$\mathbf{cr}(G) \;\geq\; \frac{1}{16} \cdot k\ell \;, \qquad \textit{provided } k \geq 16.$$

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Proof outline:

- Find a large *toroidal grid minor* H in G, relative to k and ℓ. Precisely,

$$\max\left(\lfloor 2k/3\rfloor,\, \ell\right) \times \lfloor 2k/3\rfloor\,.$$

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Theorem 10. CrossingNumber *of a (sufficiently dense embedded) toroidal graph G can be approximated within the factor* $\mathbf{6\,\Delta(G)^2}$.

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Theorem 11. [**de Graaf, Schrijver**, 1994] *A graph embedded in the torus contains a minor isomorphic to the* $s \times s$*–toroidal grid for* $s = \lfloor 2fw(G)/3 \rfloor \geq 5$.

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Then G contains a minor isomorphic to the $p \times q$–toroidal grid.

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- Thus, our lower-bound Thm 9 follows...

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The idea is to iteratively modify the two collections of cycles, until they cross in an "orderly fashion". This gives the minor.

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TO BE CONTINUED...

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