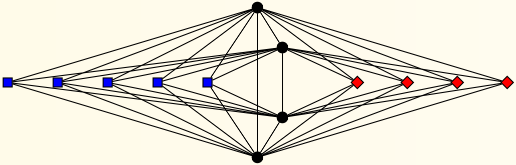

Exact Crossing Number Parameterized by Vertex Cover

Petr Hliněný

Faculty of Informatics, Masaryk University
Brno, Czech Republic

joint work with

Abhisekh Sankaran

Department of Computer Science and Technology
University of Cambridge, UK

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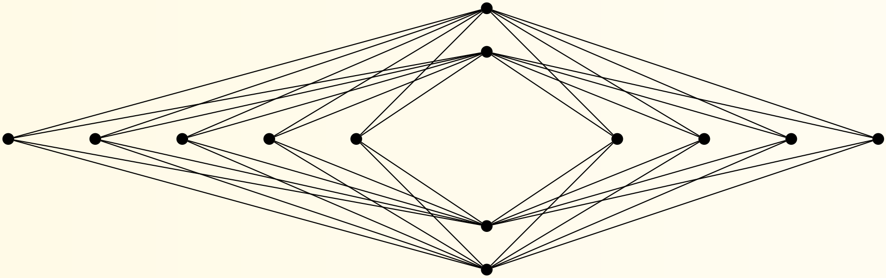

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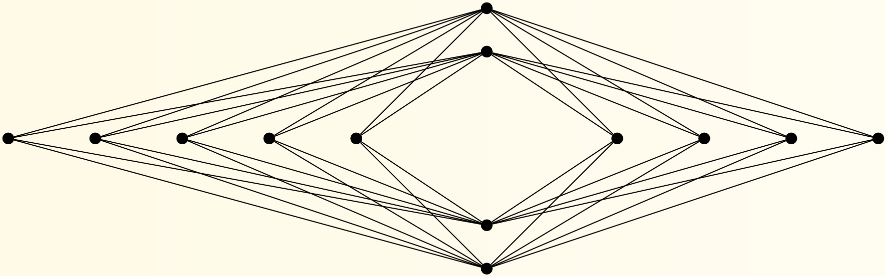

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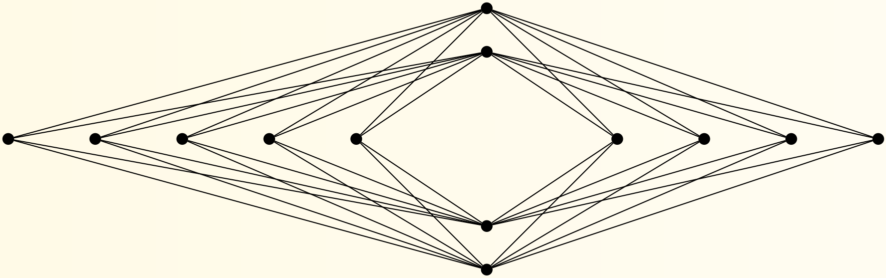

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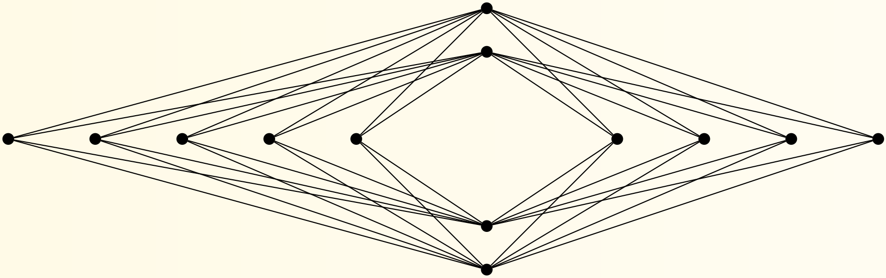

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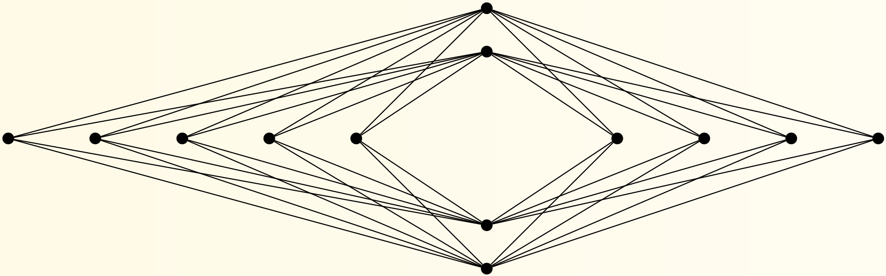

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- A very hard algorithmic problem, indeed…

Brief complexity status of $CR(k)$

NP-hardness

- The general case (no surprise); [**Garey and Johnson**, 1983]

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Parameterized complexity

- Yes, $CR(k)$ in FPT with parameter k, $\mathcal{O}\big(f(k) \cdot n\big)$ runtime; [**Grohe**, 2001 / **Kawarabayashi and Reed**, 2007]

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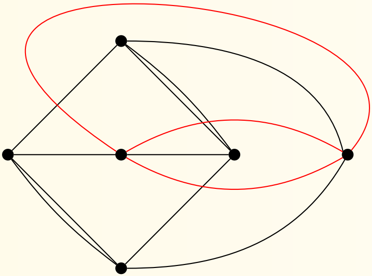
  FPT runtime: $f(k) \cdot n^{\mathcal{O}(1)}$, where $k = |X|$ is the vertex-cover size and f is a computable function (doubly-exponential here).

2 Some Basic Ideas

Inspiration: Crossings and parallel edges



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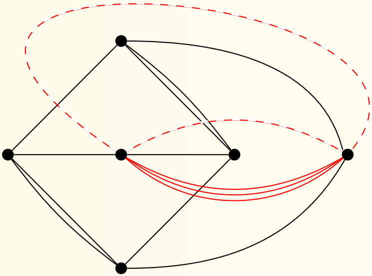

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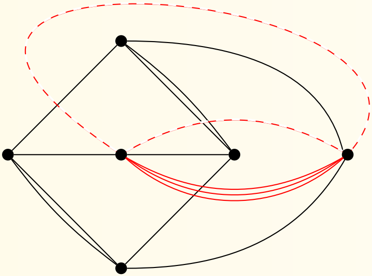

Claim.

A bunch of parallel edges can always be *optimally* drawn as one "thick" edge.

Proof: Draw whole bunch closely along any of its edges with the least crossings.

Vertex cover and neighbourhood clusters

Vertex cover (VC) $\equiv$ min. number of vertices that hit all edges.

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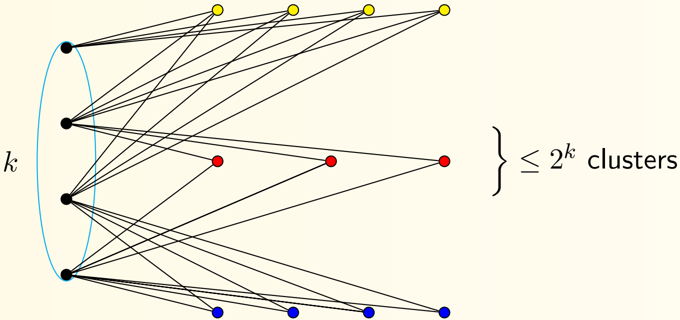

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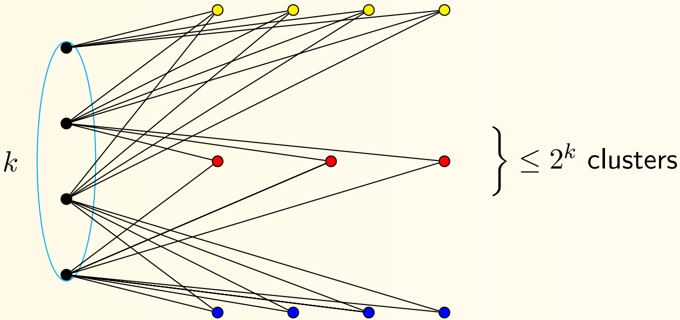

- Can we not now just take one neighbourhood cluster and draw it whole closely along its star with the least crossings?

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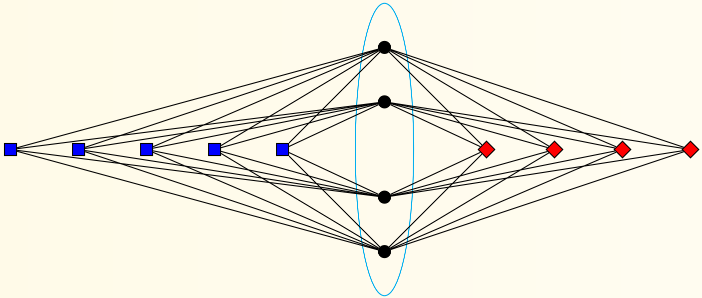

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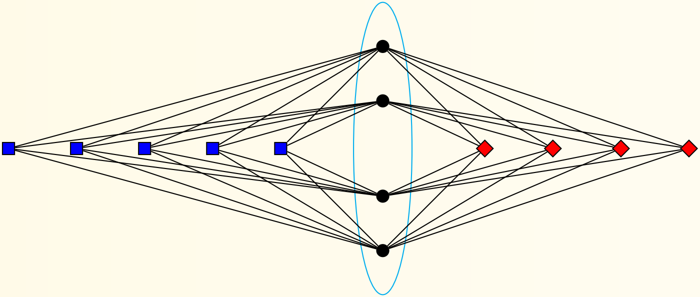

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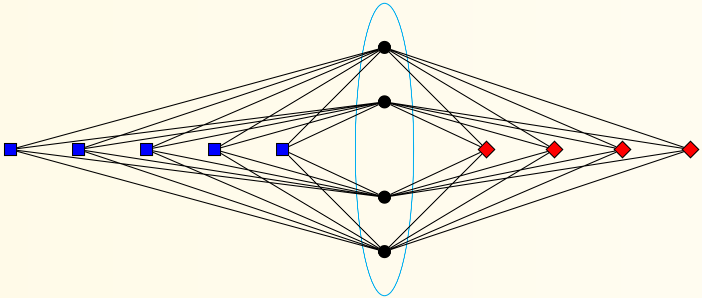

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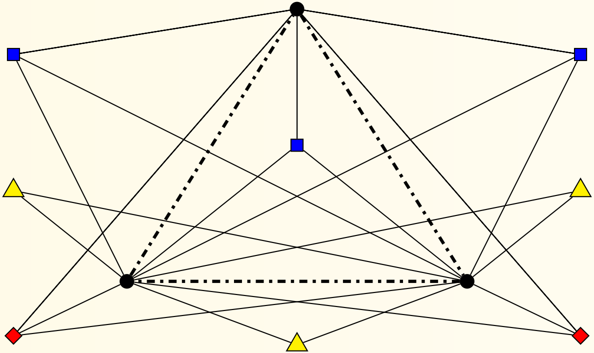
- Rediscovering an idea used for $K_{m,n}$ already by [**Christian, Richter and Salazar**, 2013: Zarankiewicz's Conjecture Is Finite for Each Fixed m].

3 Formal View: Topological Clustering

Topological clusters in a drawing



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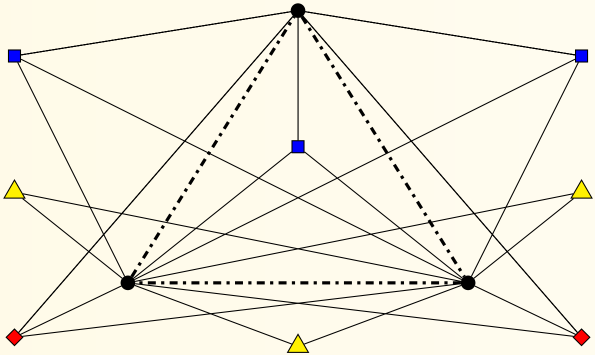
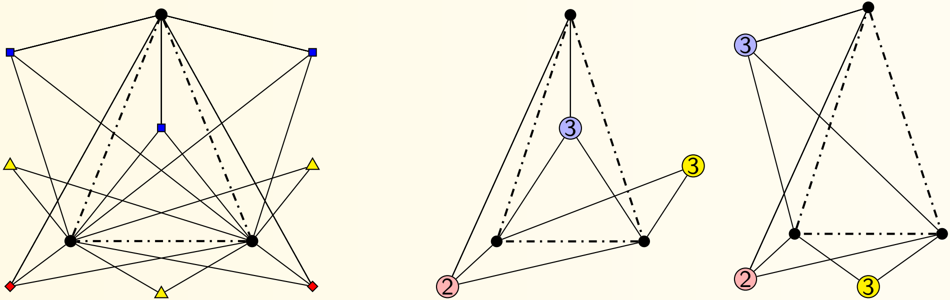


A graph G with a vertex cover X, and its drawing D;

same neighbourhood + same clockwise order in D $\leftrightarrow$ same *topological cluster*

(an equivalence relation on $V(G) \setminus X$).

Topological clustering of a drawing



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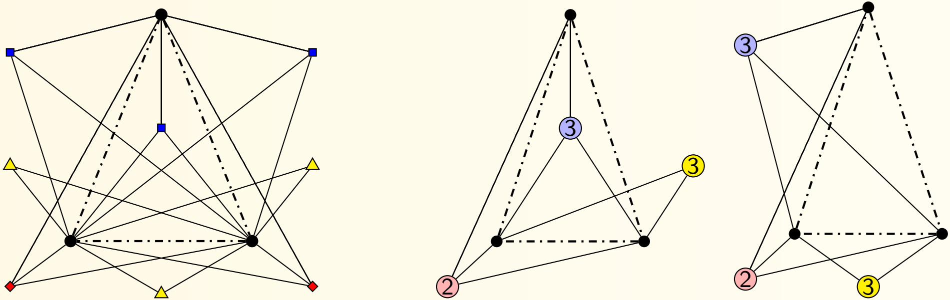

Topological clustering $\equiv$ an induced subdrawing of D s.t.

- we pick exactly one representative from each topological cluster of D,
- and remember the size of each cluster as the *weight* of the representative.

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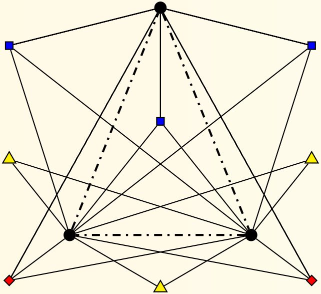

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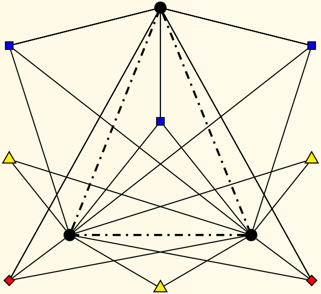

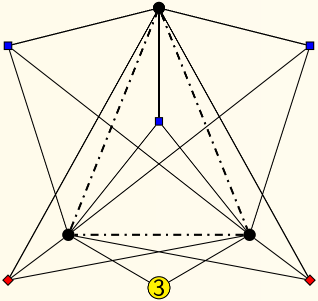

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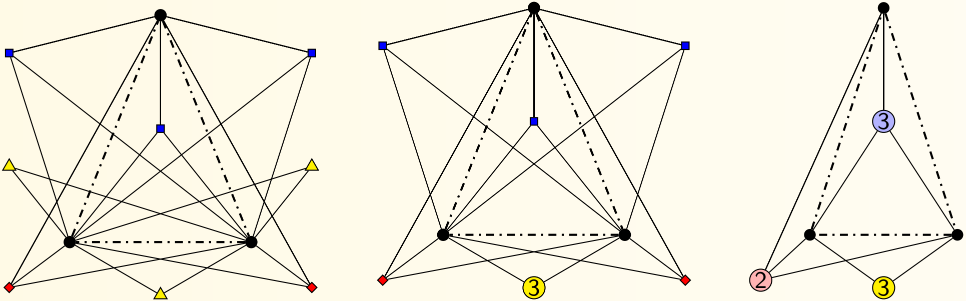

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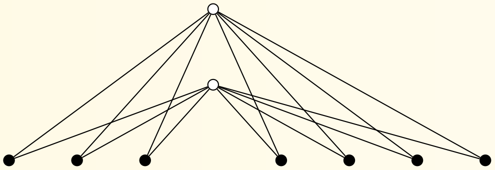

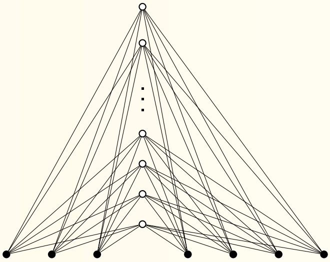

Lemma. [**Christian, Richter and Salazar**, 2013]
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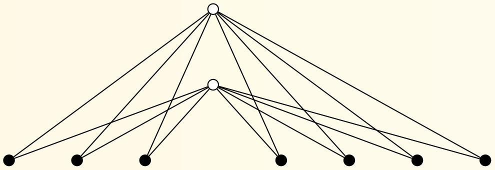

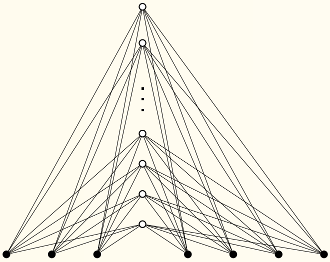

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Corollary. Any topological cluster of size (weight) c and with m neighbours in X has at least

$$\binom{c}{2} \cdot \left\lfloor \frac{m}{2} \right\rfloor \cdot \left\lfloor \frac{m-1}{2} \right\rfloor \text{ (cluster) crossings.}$$

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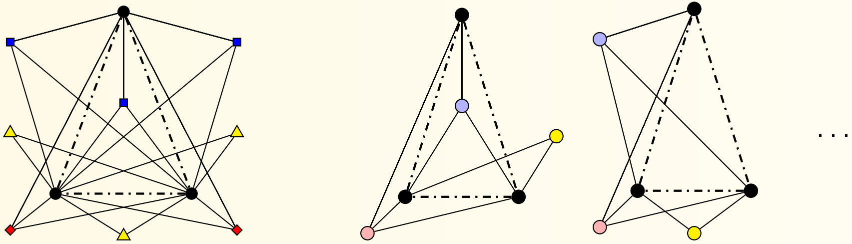

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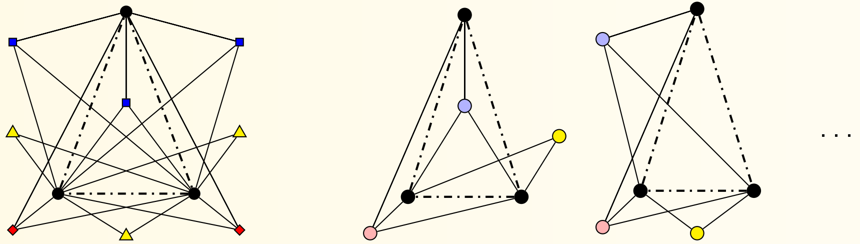

$\rightarrow$ But, what about the cluster weights?

Step II: Integer Quadratic Programming

IQP: to find an optimal solution $\boldsymbol{z}^{\circ}$ to the following optimization problem

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Theorem. [**Lokshtanov**, 2015]
This IQP can be solved in time $f(k,\lambda)\cdot L^{\mathcal{O}(1)}$ where

- $L =$ the length of the combined bit representation of the IQP,
- $\lambda =$ max entry in the matrices $\boldsymbol{A}$, $\boldsymbol{C}$, $\boldsymbol{Q}$ and $\boldsymbol{p}$,
- $k =$ the number of integer variables.

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- Altogether...

$$
\begin{array}{lcll}
\text{Minimize} & f(\boldsymbol{z}) & = \frac{1}{2}\,\boldsymbol{z}^T \boldsymbol{Q} \boldsymbol{z} + \boldsymbol{p}^T \boldsymbol{z} + c_0 & \\
\text{over all} & \boldsymbol{z} & = \left(z_{(1,1)}, \dots, z_{(1,g(1))}, \dots, z_{(l,1)}, \dots, z_{(l,g(l))}\right) & \\
\text{subject to} & \sum_{j=1}^{g(i)} z_{(i,j)} & = g(i) & \text{for } i \in \{1, \dots, l\} \\
 & z_{(i,j)} & \geq 0 & \text{for } (i,j) \in I = \{(1,1), \dots, (l, g(l))\} \\
 & \boldsymbol{z} & \in \mathbb{Z}^{|I|} &
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Thank you for your attention.