Contact Graphs of Line Segments are *NP*-complete

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Abstract. Contact graphs are a special kind of intersection graphs of geometrical objects in which the objects are not allowed to cross but only to touch each other. Contact graphs of line segments in the plane are considered — it is proved that recognizing line-segment contact graphs, with contact degrees of 3 or more, is an *NP*-complete problem, even for planar graphs. This result contributes to the related research on recognition complexity of curve contact graphs (P. Hliněný: *The classes and recognition of curve contact graphs*, J. Combin. Theory Ser. B 74 (1998), 87–103).

1 Introduction

The intersection graphs of geometrical objects have been extensively studied for their many practical applications. Formally the *intersection graph* of a set family $\mathcal{M}$ is defined as a graph $\boldsymbol{G}$ with the vertex set $V(\boldsymbol{G}) = \mathcal{M}$ and the edge set $E(\boldsymbol{G}) = \big\{\{A,B\} \subseteq \mathcal{M} \mid A \neq B,\ A \cap B \neq \emptyset\big\}$.

A special type of geometrical intersection graph—the *contact graph*, in which the geometrical objects are not allowed to cross but only to touch each other, is considered here. Unlike general intersection graphs, only a few results are known in this field. Probably the first result in this field is the one by Koebe [11] about representation of planar graphs as contact graphs of discs in the plane. Contact graphs of line segments are considered in the works of de Fraysseix, Ossona de Mendez, Thomassen, Pach [3],[18],[4], and, recently, Castro, Cobos, Dana, Márquez, Noy [2], see Section 2.

This paper shows that the recognition of line-segment contact graphs is *NP*-complete. (This was already announced in [8] and sketched in the technical report [7].) Although the paper is primarily concerned with line-segment contact graphs, it is useful to define and to consider more general curve contact graphs here (see [9, 10]). Simple curves of finite length (Jordan curves) in the plane are considered as a generalization of line segments. Each curve has two *endpoints* and all of its other points are called interior points; they form the *interior* of the curve. We say that a curve φ *ends in (passes through)* a point X if X is an endpoint (interior point) of φ.

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Definition. A finite set $\mathcal{R}$ of Jordan curves in the plane is called a *curve contact representation* of a graph $\boldsymbol{G}$ if the interiors of the curves are pairwise disjoint and $\boldsymbol{G}$ is the intersection graph of $\mathcal{R}$. The graph $\boldsymbol{G}$ is called the *contact graph* of $\mathcal{R}$. A curve contact representation $\mathcal{R}$ is said to be a *line-segment contact representation* if each curve of $\mathcal{R}$ is a line segment.
A graph $\boldsymbol{H}$ is called a *contact graph of curves* (*contact graph of line segments*) if there exists a curve contact representation (line-segment contact representation) of a graph $\boldsymbol{G} \cong \boldsymbol{H}$.

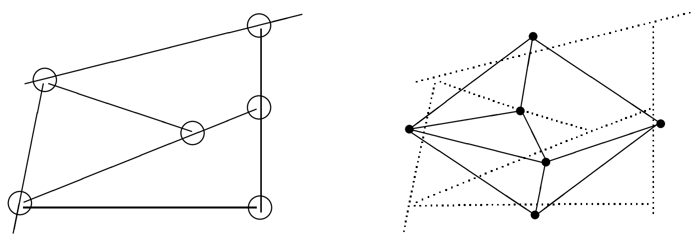

Fig. 1. An example of a line-segment contact representation of a graph.

A point is said to be a *contact point* (*k-contact point*) of a contact representation $\mathcal{R}$ if it is contained in at least two curves (exactly k curves) of $\mathcal{R}$. We say that an endpoint of a curve is *free* if it is not a contact point. In Figure 1, an example of a contact representation and its contact graph are given. For a better view, every contact point is emphasized by a circle around it. Note that for any k–contact point C either all those k curves end in C or one curve passes through C and the other $k-1$ curves end in C.

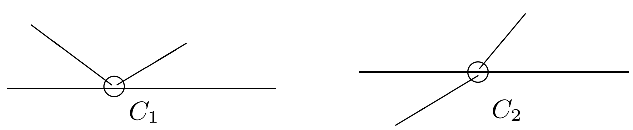

Fig. 2. The difference between one-sided (C_1) and two-sided (C_2) contact points.

The above presented definition allows two types of contact points, as shown in Figure 2. A contact representation is called one-sided if each of its contact points is *one-sided*; that means either all the curves end in the point, or one curve passes through it and the other curves approach it from only one side. We are considering only *one-sided contact representations* in this paper, which seem to be more natural. (However, the *NP* reduction can also be adapted for general two-sided contact representations [7].)

A contact representation is a *k–contact representation* if each of its contact points contains at most k curves. Two contact representations are said to be *similar*, if there is a bijection between their sets of curves and the induced bijection between their contact points, which preserve the cyclic order of curves around the contact points. For the sake of brevity we use the notions "representation" and "contact graph" instead of "line-segment contact representation" and "line-segment contact graph", respectively.

2 Related Results

This section presents other results published on our topic. First let us mention a result describing 2-contact graphs of segments in two directions [3].

Theorem (de Fraysseix, Ossona de Mendez, Pach). *Graph is a contact graph of vertical and horizontal line segments if and only if it is a planar bipartite graph.*

The same result, formulated in terms of visibility representations, was actually discovered earlier in [17]. We also include the following characterization of 2-contact graphs of line segments [18]:

Theorem (Thomassen). *Graph $\boldsymbol{G}$ is a 2–contact graph of line segments if and only if $\boldsymbol{G}$ is planar, and $|E(\boldsymbol{H})| \leq 2 \cdot |V(\boldsymbol{H})| - 3$ for each subgraph $\boldsymbol{H} \subseteq \boldsymbol{G}$.*

Moreover, every triangle-free planar graph can be represented using just three prescribed distinct directions of segments [2].

Theorem (de Castro, Cobos, Dana, Márquez, Noy). *Every triangle-free planar graph is a 2-contact graph of line segments in just three directions.*

Various classes of curve and line-segment contact graphs are defined, and their inclusions are completely described, in [10]. It is worthwhile to notice that 3-contact graphs are always planar. The chromatic number and cliques in curve contact graphs are studied in [9]. The next lemma [10] allows us to describe curve contact representations in polynomial space.

Lemma 2.1. *For every curve contact representation, there exists a similar representation consisting of piecewise linear curves with corners embedded on a grid of quadratic size (linear in both dimensions).*

A description of line-segment contact representations is not so obvious. Namely, it is a question what number precision (i.e. how many bits) is needed to describe an arbitrary line-segment contact representation. Recently, it was proved by de Fraysseix and Ossona de Mendez [4] (see also [5]) that a curve contact representation can be, under certain conditions, "stretched" to a similar line-segment contact representation. A *weak arrangement of pseudolines* is a system of (infinite length) curves in the plane, each two of them having at most one intersection.

Theorem 1. (de Fraysseix, Ossona de Mendez) *For each curve contact representation $\mathcal{R}$ such that the curves of $\mathcal{R}$ are extendable into a weak arrangement of pseudolines, there exists a line-segment contact representation $\mathcal{S}$ similar to $\mathcal{R}$.*

The problem of deciding whether a given graph can be represented as an intersection graph of specified objects, is important when studying intersection or contact graphs. The decision version of the problem is called the *recognition* of those graphs. Many of the intersection graph classes are recognizable in polynomial time (like interval graphs [15], circle graphs [1], etc). On the other hand, many of them are known to be *NP*-hard (like string graphs [12] or 3-ball touching graphs); moreover, it is often not even known whether their recognition belongs to *NP* since a possible representation can be very complex (see [14] for string graphs).

Considering curve contact graphs, we can summarize the main results here. It is easy to recognize curve 2-contact graphs, and the above result of Thomassen gives a polynomial algorithm for recognizing line-segment 2-contact graphs. The situation gets more interesting for 3-contact graphs—the following is proved in [10]:

Theorem 2. (PH) *There is a polynomial algorithm that for a given planar triangulation decides whether it is a curve* 3*–contact graph.*
The recognition of k*–contact graphs of curves is NP–complete for* $k \geq 3$*, even within the class of planar graphs.*

3 Recognition of line-segment contact graphs

The aim of this paper is to show *NP*-completeness of the recognition of line-segment contact graphs, which contributes to the research on recognition complexity of curve contact graphs from [10].

Theorem 3. *The recognition of contact graphs (*k*–contact graphs for* $k \geq 3$*) of line segments is NP–complete, even within the class of planar graphs.*

The original proof of this theorem, sketched in the technical report [7], was complicated and showed only the *NP*-reduction. However, Theorem 1 implies that the problem belongs to *NP*, and recent improvements to the reduction enable us to describe it in a reasonable way.

Lemma 3.1. *The recognition of line-segment contact graphs belongs to NP.*

Proof. Suppose the given graph $\boldsymbol{G}$ has a line-segment contact representation $\mathcal{S}$. By Lemma 2.1, there exists a similar contact representation $\mathcal{T}$ consisting of piecewise linear curves embedded on a grid of quadratic size. It is easy to extend the piecewise linear curves of $\mathcal{T}$ to obtain an arrangement $\mathcal{L}$ of the same crossing type as the straight lines of $\mathcal{S}$ have, which is still embedded on a grid of quadratic size. Naturally, $\mathcal{L}$ is a weak arrangement of pseudolines supporting a contact representation of the graph $\boldsymbol{G}$.

Thus it is enough to guess the contact representation $\mathcal{T}$, and the arrangement $\mathcal{L}$ extending it. It can be checked in polynomial time that $\mathcal{L}$ is a weak arrangement of pseudolines, and that $\mathcal{T}$ is a contact representation of the given graph $\boldsymbol{G}$. (Then $\boldsymbol{G}$ has a line-segment contact representation by Theorem 1.) □

In our proof of the recognition complexity, we will reduce from the PLANAR 3-SAT problem (see [6] for a general overview). It is defined as a special case of the satisfiability problem (a *formula* Φ with a set variables V and a set of clauses C), for which the bipartite *formula graph* $F = \boldsymbol{F}_\Phi$, $V(F) = C \cup V$, $E(F) = \{xc : c \in C,\ x \in c \text{ or } \neg x \in c\}$ is planar with degrees of all vertices bounded by 3. The planar version of the SAT problem is known to be *NP*-complete from the work [16], and has been used in similar geometrical reductions previously (see [12] for an example). Note that we may suppose that each variable has at most 2 positive and at most 2 negated occurrences in Φ; otherwise the formula is reducible.

Before starting the reduction, we need one more technical lemma proved in [10]. Two triangles are neighbouring if they have a common edge.

Lemma 3.2. *If a curve 3–contact representation of a graph* $\boldsymbol{G}$ *contains* f *free endpoints of curves, then it contains at least* $(|E(\boldsymbol{G})| - 2 \cdot |V(\boldsymbol{G})| + f)$ *3–contact points forming non-neighbouring facial triangles in some planar drawing of* $\boldsymbol{G}$.

A natural question arises of how we can force an endpoint of a curve to be free. Generally, if a special subgraph is added that has no free endpoint in its contact subrepresentation, then any other curve adjacent to some curve of this subgraph must use its own endpoint for the adjacency, thus the endpoint is free with respect to the rest of the representation. We say that the endpoint is "eaten". While a simple "end-eating" graph exists for curve contact representations, the situation gets complicated for line-segment contact representations.

Lemma 3.3. *The PLANAR 3-SAT problem reduces to the recognition of line-segment 3-contact graphs.*

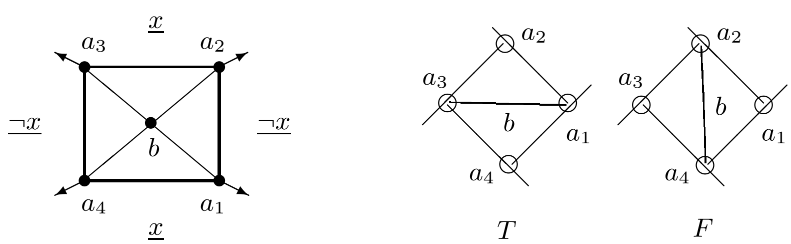

Fig. 3. The variable graph $\boldsymbol{\mathcal{V}}$ of a variable x, with two terminals a_1a_4, a_2a_3 for positive occurences and two terminals a_1a_2, a_3a_4 for negated occurences of x in clauses; and its possible contact representations (encoding logical values T and F).

The proof of this lemma is an extension of the one used for curve contact graphs in [10]. So first, we briefly repeat main steps of that proof, modified specifically for our proof.

Given a planar formula Φ, a graph $\boldsymbol{\mathcal{R}}_\Phi$ that has a line-segment contact representation iff the formula Φ is satisfiable, is constructed as follows: All variable and clause vertices of the formula graph $\boldsymbol{F}_\Phi$ are replaced by copies of special graphs $\boldsymbol{\mathcal{V}}$ and $\boldsymbol{\mathcal{C}}$ from Figures 3 and 4. Everything is arranged within a sufficiently large global frame which is used to "eat" the endpoints of specified line segments. The variable–clause edges are then substituted by paths of connectors crossing the

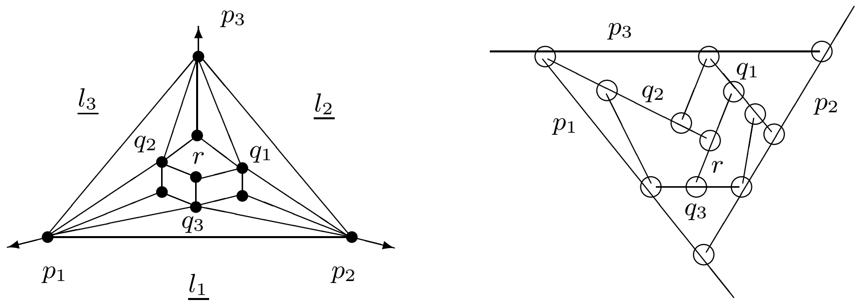

Fig. 4. The clause graph $\boldsymbol{\mathcal{C}}$ of a clause $c = l_1 \vee l_2 \vee l_3$, with terminals p_1p_2, p_2p_3, p_3p_1 for the literals l_1, l_2, l_3 resp.; and one of its possible contact representations (forcing l_3 to be T). Another two representations result by rotation.

frame. The connectors are attached to variable or clause subgraphs or to frame cells in special vertex pairs called terminals.

Formally, the *variable graph* $\boldsymbol{\mathcal{V}}$ is the graph on 5 vertices in Figure 3, and the *clause graph* $\boldsymbol{\mathcal{C}}$ is the graph on 10 vertices presented in Figure 4. (The vertices a_1, a_2, a_3, a_4 of $\boldsymbol{\mathcal{V}}$, and p_1, p_2, p_3 of $\boldsymbol{\mathcal{C}}$, are going to be attached to the global frame—to produce "eaten" endpoints of curves.) Lemma 3.4 shows key properties of these graphs: Two possible ways to represent the graph $\boldsymbol{\mathcal{V}}$ will encode logical values T/F of variables, and three essential representations of the graph $\boldsymbol{\mathcal{C}}$ will determine true literals in clauses.

A (designated) pair of adjacent vertices of a graph $\boldsymbol{G}$ is called a *terminal* of $\boldsymbol{G}$. Suppose that $\mathcal{R}$ is a contact representation of $\boldsymbol{G}$. The terminal uv of $\boldsymbol{G}$ is said to be *available* in $\mathcal{R}$ if there is a 2-contact point of the segments u, v in the representation $\mathcal{R}$. (The availability of a terminal will express its "information state".) The terminals of the variable graph $\boldsymbol{\mathcal{V}}$ are the pairs a_1a_2, a_2a_3, a_3a_4, a_4a_1. The terminals of the clause graph $\boldsymbol{\mathcal{C}}$ are the pairs p_1p_2, p_2p_3, p_3p_1.

Lemma 3.4. *(a) Suppose that $\mathcal{R}$ is a contact representation of the graph $\boldsymbol{\mathcal{V}}$ such that each of the segments a_1, a_2, a_3, a_4 has one free endpoint in $\mathcal{R}$. Then no other endpoint in $\mathcal{R}$ is free. Additionally, either none of the terminals a_1a_4, a_2a_3, or none of a_1a_2, a_3a_4, is available in $\mathcal{R}$.*

(b) Suppose that $\mathcal{S}$ is a representation of the graph $\boldsymbol{\mathcal{C}}$ such that each of the segments p_1, p_2, p_3 has one free endpoint in $\mathcal{S}$. Then no other endpoint in $\mathcal{S}$ is free. Additionally, at least one of the terminals p_1p_2, p_2p_3, p_1p_3 is not available in $\mathcal{S}$.

Proof. Notice that the graphs $\boldsymbol{\mathcal{V}}$, $\boldsymbol{\mathcal{C}}$ have maximal cliques of size 3, and so their representations $\mathcal{R}$, $\mathcal{S}$, respectively, are 3-contact.

(a) The proof follows from Lemma 3.2: Any representation of the graph $\boldsymbol{\mathcal{V}}$ having 4 free endpoints must contain 2 non-neighbouring triangles represented by 3-contact points, and there are just two choices of these triangles—either a_1a_4b and a_2a_3b, or a_1a_2b and a_3a_4b. Hence either the terminals a_1a_4 and a_2a_3, or a_1a_2 and a_3a_4, are not available in $\mathcal{R}$. Moreover, no other endpoint can be free in $\mathcal{R}$.

(b) Similarly, any representation of the graph $\boldsymbol{\mathcal{C}}$ having 3 free endpoints must contain 4 non-neighbouring triangles represented by 3-contact points. Thus one of them must be $p_1p_2q_3$, $p_2p_3q_1$ or $p_1p_3q_2$. Again, there can be no more non-neighbouring triangles, and hence no more free endpoints. □

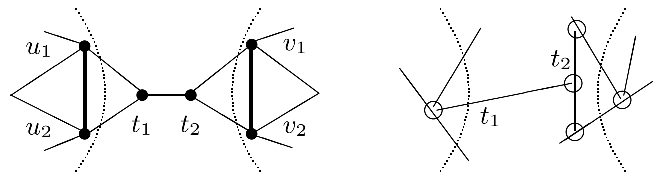

Fig. 5. A connector t_1t_2, connecting terminal u_1u_2 with terminal v_1v_2; and a possible contact representation (transfering the value T from u_1u_2 to v_1v_2).

A *connector* joining two terminals u_1u_2 and v_1v_2 consists of two additional vertices t_1, t_2, and five edges $u_1t_1, u_2t_1, t_1t_2, t_2v_1, t_2v_2$. There are no other edges incident with t_1, t_2. (See Figure 5.) The purpose of a connector is to transfer "information state" between terminals. The next lemma shows a necessary property of a connector joining two terminals.

Lemma 3.5. *Let u_1u_2 and v_1v_2 be terminals in a graph $\boldsymbol{G}$, and t_1t_2 be a connector joining them. Let $\mathcal{R}$ be a contact representation of $\boldsymbol{G}$ such that none of the segments u_1, u_2, v_1, v_2 has a free endpoint within the subrepresentation of $\boldsymbol{G}-\{t_1, t_2\}$. Then at least one of the terminals u_1u_2 or v_1v_2 must be available in $\mathcal{R} - \{t_1, t_2\}$.*

Proof. The edge t_1t_2 of the connector uses one endpoint, say that of t_1, in $\mathcal{R}$. If the edges t_1u_1, t_1u_2 were represented by distinct contact points, one of them would have to use an endpoint of u_1 or u_2, but that endpoint would be free in the subrepresentation of $\boldsymbol{G} - \{t_1, t_2\}$. Therefore the connector triangle $t_1u_1u_2$ is represented by a 3-contact point in $\mathcal{R}$; and consequently, the terminal u_1u_2 is available in $\mathcal{R} - \{t_1\}$. □

Further, we define the *"end-eating" frame* that supports the whole construction. The left-hand side of Figure 6 shows one *cell* of the frame. These cells are arranged into a chain by identifying the vertices f', g', k' of one cell with the vertices f, g, k of the next one (i.e. $f'_1 = f_2, g'_1 = g_2, k'_1 = k_2$, $f'_2 = f_3, g'_2 = g_3, k'_2 = k_3$, $\ldots$, for cells indexed $1, 2, 3, \ldots$ in the chain), as it is shown on the right-hand side of the figure. The frame graph $\boldsymbol{\mathcal{F}}$ is formed by a collection of the cell chains that are stretched between two additional vertices A, B, as sketched in Figure 7. If the particular representation of one cell (Figure 6) is suitably adjusted by a projective transformation, a contact representation of the whole chain can be formed, looking like a thin long belt with concave sides. These chains are extended between the segments A, B, resulting in a contact representation of $\boldsymbol{\mathcal{F}}$ (Figure 7).

Moreover, the edges ff' and gg' of each cell form a matching pair of terminals, which can be used to transfer information across the frame in our reduction. All the necessary properties of the frame graph are shown in the following lemma.

Lemma 3.6. *If the segments A and B have both endpoints free in a contact representation $\mathcal{R}$ of the graph $\boldsymbol{\mathcal{F}}$, then no other segment of this representation has*

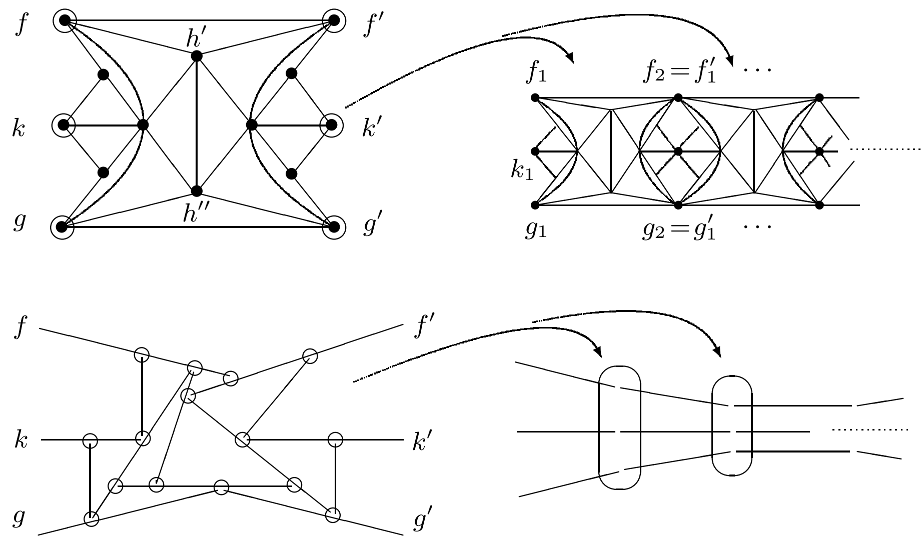

Fig. 6. One cell of the frame graph (with two terminals ff', gg'), forming a chain of cells as illustrated on the right. Below, there is a line-segment contact representation of the cell, extended to a representation of the whole chain. (Each cell representation in the chain can be used either straight or upside-down.)

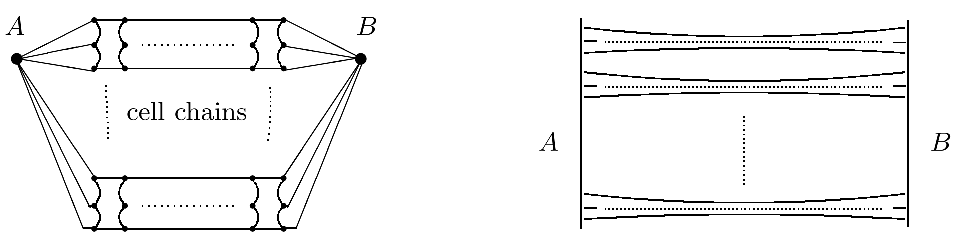

Fig. 7. A sketch of the frame graph $\boldsymbol{\mathcal{F}}$, formed by chains of cells from Figure 6 stretched between two additional vertices A, B, is on the left. On the right, there is a scheme of a possible contact representation of $\boldsymbol{\mathcal{F}}$, having both endpoints of A and both of B free.

a free endpoint. Additionally, for each cell of the frame $\boldsymbol{\mathcal{F}}$, at least one of the terminals ff', gg' is not available in $\mathcal{R}$.

Proof. A closer look shows that one cell (Figure 6) has 14 vertices and 29 edges, and it can contain at most 7 non-neighbouring triangles. If k is the number of chains and c_i is the number of cells in the i-th chain of the graph $\boldsymbol{\mathcal{F}}$, then $\boldsymbol{\mathcal{F}}$ has $\sum_{i=1}^{k}(c_i(14-3)+3)+2 = 11\sum_{i=1}^{k} c_i + 3k + 2$ vertices and $\sum_{i=1}^{k} 29c_i + 2\cdot 3k = 29\sum_{i=1}^{k} c_i + 6k$ edges altogether. Taking cells one by one, the maximal number of non-neighbouring triangles in $\boldsymbol{\mathcal{F}}$ is at most $t(\boldsymbol{\mathcal{F}}) = 7\sum_{i=1}^{k} c_i$. Since there is no 4-clique in $\boldsymbol{\mathcal{F}}$, by Lemma 3.2, any of its contact representations contains at most

$$t(\boldsymbol{\mathcal{F}})+2|V(\boldsymbol{\mathcal{F}})|-|E(\boldsymbol{\mathcal{F}})| = 7\sum_{i=1}^{k} c_i + 2\left(11\sum_{i=1}^{k} c_i + 3k + 2\right) - \left(29\sum_{i=1}^{k} c_i + 6k\right) = 4$$

free endpoints of segments.

To prove the second part of the lemma, notice that if there are 4 free endpoints (of the both segments A, B) in the representation, then the subrepresentation of each cell of $\boldsymbol{\mathcal{F}}$ in $\mathcal{R}$ must contain exactly seven 3-contact points forming non-neighbouring triangles. By simple checking, one of them must be $ff'h'$ or $gg'h''$, and hence the terminal ff' or gg', respectively, is not available in $\mathcal{R}$. □

4 Completing the reduction

Let v be a vertex disjoint with $\boldsymbol{\mathcal{F}}$. We say that a vertex v is *attached* to the frame graph $\boldsymbol{\mathcal{F}}$ if there exist one edge between v and some of the copies of vertices f or g of $\boldsymbol{\mathcal{F}}$. In analogue to a connector, we say that a *false terminator* is joined with a terminal uv if a new vertex is added to the graph, adjacent to both u, v, and attached to the frame $\boldsymbol{\mathcal{F}}$. (False terminators are used to "force availability" of connectors in the reduction.)

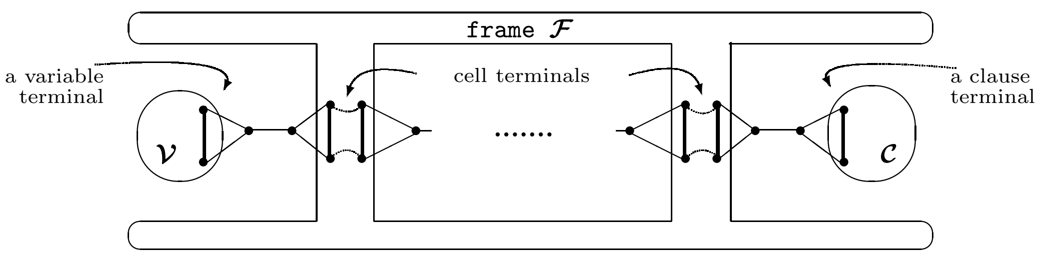

Fig. 8. A scheme of a connector path stretched across the frame from a variable subgraph to a clause subgraph.

Suppose that $\boldsymbol{G} \supset \boldsymbol{\mathcal{F}}$ is a graph containing copies of the variable and clause graphs. We call the *connector path* a sequence of connectors (Figure 8) in $\boldsymbol{G}$, the first one of them joining a terminal of the variable subgraph $\boldsymbol{\mathcal{V}}$ with a terminal of some frame cell in $\boldsymbol{\mathcal{F}}$, the second one joining the opposite terminal of the same cell with a terminal of the next cell, and so on..., up to the last connector joining a terminal of the last cell in the path with a terminal of the clause subgraph $\boldsymbol{\mathcal{C}}$.

Let a PLANAR 3-SAT formula Φ be given as $\Phi = c_1 \wedge c_2 \wedge \ldots \wedge c_l$, where $c_i = \lambda_{i1} \vee \lambda_{i2} \vee \lambda_{i3}$ for $i = 1, \ldots, k$, and $c_i = \lambda_{i1} \vee \lambda_{i2}$ for $i = k+1, \ldots, l$; and let $x_1, x_2, \ldots, x_n$ be the variables of Φ. We say that a graph $\boldsymbol{\mathcal{R}}_{\Phi}$ is a *framed emulator* of a PLANAR 3-SAT formula Φ if it is constructed as follows: The construction starts with the union of the graph $\boldsymbol{\mathcal{F}}$ (the size of which is determined later), of disjoint copies $\boldsymbol{\mathcal{V}}(x_1), \ldots, \boldsymbol{\mathcal{V}}(x_n)$ of the variable graph $\boldsymbol{\mathcal{V}}$, and of disjoint copies $\boldsymbol{\mathcal{C}}(c_1), \ldots, \boldsymbol{\mathcal{C}}(c_l)$ of the clause graph $\boldsymbol{\mathcal{C}}$. All copies of the vertices $a_1, a_2, a_3, a_4 \in V(\boldsymbol{\mathcal{V}})$ and $p_1, p_2, p_3 \in V(\boldsymbol{\mathcal{C}})$ are attached to the frame $\boldsymbol{\mathcal{F}}$. For each literal $\lambda_{ij} = x_m$ ($\lambda_{ij} = \neg x_m$), $i = 1, \ldots, l$, a unique connector path is joining the terminal a_4a_1 or a_2a_3 (a_1a_2 or a_3a_4) of $\boldsymbol{\mathcal{V}}(x_m)$, with a terminal p_jp_{j+1} of $\boldsymbol{\mathcal{C}}(c_i)$. For each clause c_i, $k < i \leq l$, a copy of the false terminator is added to the terminal p_3p_1 of $\boldsymbol{\mathcal{C}}(c_i)$.

The *skeleton* of a framed emulator $\boldsymbol{\mathcal{R}}_\Phi$ is defined as the subgraph consisting of the frame $\boldsymbol{\mathcal{F}}$, of all copies of the vertices a_1, a_2, a_3, a_4 and p_1, p_2, p_3, and of all copies of connectors and false terminators.

Proof of Lemma 3.3. Suppose, for a while, that any contact subrepresentations of the frame $\boldsymbol{\mathcal{F}}$ is guaranteed to have both endpoints of A and both of B free. Let Φ be the PLANAR 3-SAT formula, given as above. Our proof proceeds in the following steps: First, we prove that if there exists a 3-contact representation of any framed emulator $\boldsymbol{\mathcal{R}}_\Phi$, then the formula Φ is satisfiable. Second, we construct a framed emulator $\boldsymbol{\mathcal{R}}_\Phi$ that has a 3-contact representation if Φ is satisfiable. Third, we show how to force the endpoints of A and of B to be free in a contact representation of $\boldsymbol{\mathcal{R}}_\Phi$.

Let $\mathcal{R}$ be a 3-contact representation of a framed emulator $\boldsymbol{\mathcal{R}}_\Phi$ such that all endpoints of A, B are free in $\mathcal{R}$. Lemmas 3.5 and 3.6 together imply: If two terminals are joined by a connector path in $\boldsymbol{\mathcal{R}}_\Phi$, then at least one of the two terminals must be available in the rest of the emulator. For each variable x_i, $i = 1, \ldots, n$ of Φ we set $x_i = T$ if some of the terminals a_1a_4 or a_2a_3 is available in the subrepresentation of $\boldsymbol{\mathcal{V}}(x_i)$, and we set $x_i = F$ if some of the terminals a_1a_2 or a_3a_4 is available there. This is well defined because of Lemma 3.4(a). We claim that this is a satisfying assignment for Φ.

Indeed, for each $1 \le j \le l$, one of the terminals of the clause subgraph $\boldsymbol{\mathcal{C}}(c_j)$ is not available in its subrepresentation by Lemma 3.4(b). It is easy to see that the non-available terminal is not the one with a false terminator added (if $j > k$). Hence the variable terminal of $\boldsymbol{\mathcal{V}}(x_i)$, that is joined with the non-available clause terminal of $\boldsymbol{\mathcal{C}}(c_j)$ by a connector path, is available is the subrepresentation of $\boldsymbol{\mathcal{V}}(x_i)$. Therefore, by our setting of the variables of Φ, the variable x_i makes the clause c_j to be true; so Φ is satisfiable.

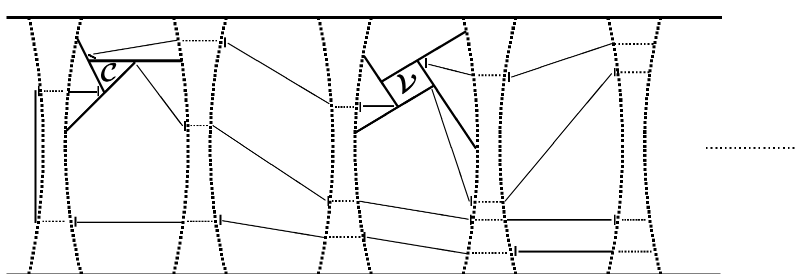

Fig. 9. A scheme of a contact representation of the skeleton of $\boldsymbol{\mathcal{R}}_\Phi$, with connector paths joining variables and clauses (following edges of the planar formula graph).

Conversely, we precise the construction of the graph $\boldsymbol{\mathcal{R}}_\Phi$ so that it has a line-segment contact representation if Φ is satisfiable.

Let m be the number of vertices of the formula graph $\boldsymbol{F}_\Phi$. The frame graph $\boldsymbol{\mathcal{F}}$ used in our construction consists of $2m$ cell chains, each chain formed by $3m$ cells. Recall the scheme of a contact representation of $\boldsymbol{\mathcal{F}}$ from Figures 6,7. Based on that representation, we actually construct a flexible scheme of contact representations for the skeleton of $\mathcal{R}_\Phi$. This scheme is independent the property whether Φ is satisfiable, and the scheme can be completed to a representation of whole $\mathcal{R}_\Phi$ if a satisfying assignment of Φ is given.

We start with a planar embedding of the formula graph $\boldsymbol{F}_\Phi$, and we distribute the vertices of it into each second region of $\mathcal{F}$ (as separated by the cell chains), so that the edges of the formula graph can still be drawn across the frame graph without crossing one another. Then we replace each clause vertex by copies of the three segments p_1, p_2, p_3 of $\mathcal{C}$, and each variable vertex by copies of the four segments a_1, a_2, a_3, a_4 of $\mathcal{V}$, as sketched in Figure 9. One endpoint of each of these segments is attached to a copy of the f or g vertex of the frame $\mathcal{F}$ (the "eaten" endpoints).

The noncrossing edges of the formula graph are replaced by disjoint noncrossing connector paths in the following way: The variable and clause subgraphs of $\boldsymbol{R}$ use only the middle m cells of each cell chain in $\mathcal{F}$, and the connector paths use the top m and the bottom m cells. Those regions of the frame that do not contain formula vertices are used to "switch back" a connector path, and to change between the top and the bottom layer of connector paths, as depicted in Figure 9. Recall that each cell of the frame can be independently represented in two ways that are mirror images of each other (Figure 6). That property allows us to "orient" each connector path in any of the two directions, as needed when completing the representation for satisfiable formula Φ.

The graph $\mathcal{R}_\Phi$ results from the contact graph of the above described representation by completing all vertices of the variable and clause subgraphs. One can easily check that the construction of $\mathcal{R}_\Phi$ is finished in polynomial time. If Φ is satisfiable, then the missing line-segments of variable and clause subgraphs clearly can be completed (according to the satisfying evaluation of variables of Φ) in the representation of the skeleton, and the connector paths can be "oriented" as needed for representing the variable and clause subgraphs.

Finally, a little trick forces both endpoints of the segments A and B to be free in a contact representation of the graph $\mathcal{R}_\Phi$. We make 5 copies of the graph constructed above, and identify the vertices A and B of these copies. (That is, all 5 copies of A make one vertex, and all 5 copies of B make another one vertex.) Then at most 4 of these copies may be "damaged" by using an endpoint of A or B; but the fifth copy satisfies the assumption about free endpoints. □

This completes the proof of Theorem 3, because the graph $\mathcal{R}_\Phi$ is planar, and all of its contact representations are 3-contact since $\mathcal{R}_\Phi$ contains no 4-clique.

References

1. A. Bouchet, *Reducing prime graphs and recognizing circle graphs*, Combinatorica 7 (1987), 243–254.
2. N. de Castro, F.J. Cobos, J.C. Dana, A. Márquez, M. Noy, *Triangle-free planar graphs as segment intersection graphs*, Proceedings Graph Drawing 99, to appear.
3. H. de Fraysseix, P. Ossona de Mendez, J. Pach, *Representation of planar graphs by segments*, 63. Intuitive Geometry (1991), 110–117.
4. H. de Fraysseix, P. Ossona de Mendez, *Stretchability of Jordan arc contact systems*, KAM Preprint Series 98–387 (1998), to appear.

5. M. Las Vergnas, P. Ossona de Mendez, *On barycentric representation of graphs*, in preparation.
6. M.R. Garey, D.S. Johnson, Computers and Intractability, W.H. Freeman and Company, New York 1979.
7. P. Hliněný, *Contact graphs of curves*, KAM Preprint Series 95–285, Dept. of Applied Mathematics, Charles University, Czech republic, 1995.
8. P. Hliněný, *Contact graphs of curves (extended abstract)*, in Graph Drawing (F.J.Brandenburg ed.), Proceedings Graph Drawing '95, Passau September 1995; Lecture Notes in Computer Science 1027, Springer Verlag, Berlin Heidelberg 1996, 312-323.
9. P. Hliněný, *The maximal clique and colourability of curve contact graphs*, Discrete Applied Math. 81 (1998), 59-68.
10. P. Hliněný, *The classes and recognition of curve contact graphs*, J. Combin. Theory Ser. B 74 (1998), 87–103.
11. P. Koebe, *Kontaktprobleme der konformen Abbildung,* Berichte über die Verhandlungen der Sächsischen, Akad. d. Wiss., Math.–Physische Klasse 88 (1936), 141–164.
12. J. Kratochvíl, *String graphs II: Recognizing string graphs is NP–hard*, J. of Comb. Theory Ser. B 1 (1991), 67–78.
13. J. Kratochvíl, J. Matoušek, *Intersection graphs of segments*, J. of Comb. Theory Ser. B 2 (1994), 289–315.
14. J. Kratochvíl, J. Matoušek, *String graphs requiring exponential representations*, J. of Comb. Theory Ser. B 2 (1991), 1–4.
15. C.B. Lekkerkerker, J.C. Boland, *Representation of finite graphs by a set of intervals on the real line*, Fund. Math. 51 (1962), 45–64.
16. D. Lichtenstein, *Planar formulae and their uses*, SIAM J. Comput. 11 (1982), 329–343.
17. R. Tamassia, I.G. Tollis, *A unified approach to visibility representations of planar graphs*, Discrete and Computational Geometry 1 (1986), 321–341.
18. C. Thomassen, presentation at Graph Drawing '93, Paris, 1993.

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