Classes and Recognition of Curve Contact Graphs*

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Abstract. Contact graphs are a special kind of intersection graphs of geometrical objects in which the objects are not allowed to cross but only to touch each other. Contact graphs of simple curves, and line segments as a special case, in the plane are considered. Various classes of contact graphs are introduced and the inclusions between them are described, also the recognition of the contact graphs is studied. As one of the main results, it is proved that the recognition of 3–contact graphs is *NP*–complete for planar graphs, while the same question for planar triangulations is polynomial.

1 Introduction

The intersection graphs of geometrical objects have been extensively studied for their many practical applications. Formally the *intersection graph* of a set family $\mathcal{M}$ is defined as a graph $\boldsymbol{G}$ with the vertex set $V(\boldsymbol{G}) = \mathcal{M}$ and the edge set $E(\boldsymbol{G}) = \big\{\{A,B\} \subseteq \mathcal{M} \,|\, A \neq B,\ A \cap B \neq \emptyset\big\}$. Probably the first type studied were interval graphs, see [15],[1]; we may also mention other kinds such as the intersection graphs of chords of a circle [2], or of curves or line segments in the plane [3],[12],[13].

A special type of geometrical intersection graphs—the *contact graphs*, in which the geometrical objects are not allowed to cross but only to touch each other, are considered here. Unlike the general intersection graphs, only a few results are known in this field. There is a nice old result of Koebe [11] about representations of planar graphs as contact graphs of circles in the plane. In [6] a similar result about contact graphs of triangles is proved. The contact graphs of line segments are considered in works of de Fraysseix with de Mendez and of Thomassen [5],[7] and [18]: It is proved that every bipartite planar graph is a contact graph of vertical and horizontal line segments [5] (see also an earlier related work [17]), and for contact graphs of line segments of any direction, with contact of 2 segments in one contact point, a characterization is given in [18].

We follow the ideas of intersection graphs of curves and of contact graphs of segments, and define the contact graphs of simple curves in the plane. We also allow a contact of more than 2 curves in a point—in such situation we consider "one-sided" contacts. We define various classes of curve or line-segment contact graphs, and study mainly the inclusions among them and their recognition.

This paper provides proofs for results announced in [9], concerning the contact graph classes and their recognition. A related paper [10] deals with the chromatic number and the maximal cliques in the curve contact graphs. The complete formal definitions and proofs may be also found in the technical report [8].

2 Curve Contact Representations

2.1 Definitions

Simple curves of finite length (Jordan curves) in the plane are considered. Each curve has two *endpoints* and all of its other points are called interior points; they form the *interior* of the curve. We say that a curve φ *ends in (passes through)* a point X if X is an endpoint (interior point) of φ.

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Definition. A finite set $\mathcal{R}$ of curves in the plane is called a *curve contact representation* of a graph $\boldsymbol{G}$ if interiors of any two curves of $\mathcal{R}$ are disjoint and $\boldsymbol{G}$ is the intersection graph of $\mathcal{R}$. The graph $\boldsymbol{G}$ is called the *contact graph* of $\mathcal{R}$ and denoted by $\boldsymbol{G}(\mathcal{R})$. A curve contact representation $\mathcal{R}$ is said to be a *line segment contact representation* if each curve of $\mathcal{R}$ is a line segment.
A graph $\boldsymbol{H}$ is called a *contact graph of curves* (*contact graph of line segments*) if there exists a curve contact representation (line segment contact representation) $\mathcal{S}$ such that $\boldsymbol{H} \cong \boldsymbol{G}(\mathcal{S})$.

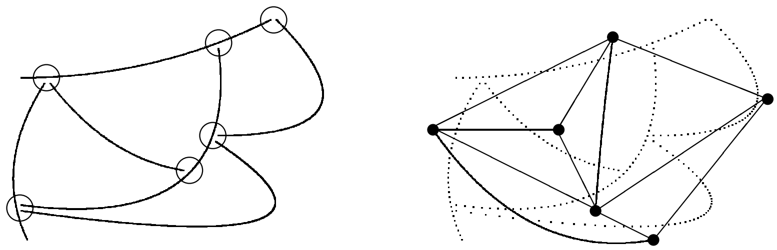

Fig. 1. An example of a curve contact representation of a graph

A curve contact representation is called simply a *representation*, a contact graph of curves simply a *contact graph*. Any subset $\mathcal{S} \subseteq \mathcal{R}$ is called a *subrepresentation* of $\mathcal{R}$. A point C is said to be a *contact point* of a representation $\mathcal{R}$ if it is contained in at least two curves of $\mathcal{R}$. The *degree* of a contact point C in $\mathcal{R}$ is the number of curves of $\mathcal{R}$ containing C, a contact point of degree k is called a *k–contact point.* We say that an endpoint of a curve is *free* if it is not a contact point.

In Figure 1 an example of a curve contact representation and its contact graph are given. For a better view, every contact point is emphasized by a circle around it. Note that for any k–contact point C either all k curves containing C end in C or one curve is passing through C and the other $k-1$ curves end in C.

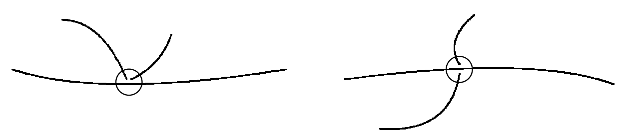

Fig. 2. The difference between one-sided and two-sided contact points

A representation $\mathcal{R}$ is called a *k–contact representation* if each contact point of $\mathcal{R}$ has degree at most k. A representation $\mathcal{R}$ is said to be *simple* if each pair of curves from $\mathcal{R}$ has at most one common contact point. A representation $\mathcal{R}$ is *one-sided* if each of its contact points is one-sided—that means either all its curves are ending in the point, or other curves are going to one side of the passing through curve, see the example in Figure 2. The same definitions of a k–contact or one-sided representations are applied for line segment representations. It is clear that every line segment representation is simple. All these properties of contact representations are transferred to contact graphs, and we refer to contact graphs as k-contact, simple or one-sided contact in the obvious sense. Unless explicitly stated otherwise, *we will consider one-sided contact representations.*

2.2 The structure of contact representations

For a description of a curve contact representation we define the following tool.
The *incidence graph* of a representation $\mathcal{R}$ (denoted by $\boldsymbol{I}(\mathcal{R})$) is a directed bipartite graph,

whose vertices correspond to curves and to contact points of $\mathcal{R}$, and each vertex of a curve is connected with all contact points that lie on it; the edge is oriented from the curve to the contact point iff the point is an endpoint of the curve. We consider here a directed graph as an orientation of undirected graph, i.e. without multiple edges and 2-cycles. An example of the incidence graph of a representation is shown in Figure 3.

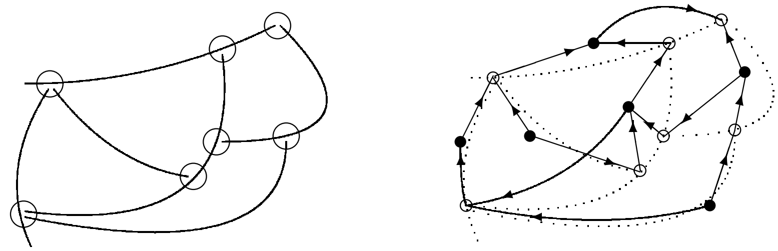

Fig. 3. An example of the incidence graph of a representation

We say that two contact representations are *similar* if their incidence graphs are isomorphic. The next theorem enables us to handle a curve contact representation easier and to describe it using finite (polynomial) space.

Theorem 1. *For each two-sided representation $\mathcal{R}$, $|\mathcal{R}| = n$, there exists a two-sided representation $\mathcal{S}$ similar to $\mathcal{R}$, so that each curve from $\mathcal{S}$ is a piecewise linear curve with its vertices embedded on a grid of size $O(n) \times O(n)$. Additionally, if $\mathcal{R}$ is one-sided, then $\mathcal{S}$ can be also chosen one-sided.*

Proof. We define on $\mathcal{R}$ a plane graph in the following way: The vertices are put into each contact point of $\mathcal{R}$ and into each free endpoint of a curve of $\mathcal{R}$. The edges are exactly the pieces of curves of $\mathcal{R}$ between the vertices. Then, to avoid multiple edges, each edge is subdivided by a new vertex. Clearly, the obtained graph $\boldsymbol{H}$ is a planar graph of size $|V(\boldsymbol{H})| = O(n)$.

By [16], every planar graph on v vertices can be embedded on a $(v-1) \times (v-1)$ grid, which produces the required similar representation $\mathcal{S}$. For one-sided $\mathcal{R}$ we can preserve the cyclic order of edges around the vertices of $\boldsymbol{H}$, hence we get one-sided $\mathcal{S}$. □

Using the previous nice embedding of a contact representation, it is not hard to prove (details can be found in [8]) a useful characterization of incidence graphs of one-sided representations.

Proposition 2.1 *For a graph $\boldsymbol{G}$ there exists a contact representation $\mathcal{R}$ such that $\boldsymbol{G} \cong \boldsymbol{I}(\mathcal{R})$ iff $\boldsymbol{G}$ is a planar directed graph and its vertices can be divided into two independent set $V(\boldsymbol{G}) = A \cup B$ so that the outdegrees in A are at most 2, the outdegrees in B are at most 1 and the total degrees in B are at least 2.*

An immediate consequence of this assertion is:

Proposition 2.2 *Graph $\boldsymbol{G}$ is a 2–contact graph iff $\boldsymbol{G}$ is planar and for each subgraph $\boldsymbol{H} \subseteq \boldsymbol{G}$, $|E(\boldsymbol{H})| \leq 2 \cdot |V(\boldsymbol{H})|$.*

It is more difficult to characterize the line segment 2–contact graphs, the result of [18] solves it.

Theorem (Thomassen).
Graph $\boldsymbol{G}$ is a 2–contact graph of line segments iff $\boldsymbol{G}$ is planar and for each subgraph $\boldsymbol{H} \subseteq \boldsymbol{G}$, $|E(\boldsymbol{H})| \leq 2 \cdot |V(\boldsymbol{H})| - 3$.

Another consequence of Proposition 2.1 is that one-sided 3–contact graphs are also planar. However, there is most likely no such nice characterization (in the meaning of recognition complexity) as the previous ones, due to the results presented in the last section.

Proposition 2.3 *If $\mathcal{R}$ is a 3–contact representation, then there exists a planar drawing of $\boldsymbol{G}(\mathcal{R})$ such that for each 3–contact point X of curves $u, v, w \in \mathcal{R}$ the triangle u, v, w forms a face.*

Sketch of proof. The drawing of the graph $\boldsymbol{G}(\mathcal{R})$ is obtained from the planar incidence graph $\boldsymbol{I}(\mathcal{R})$ by replacing each contact vertex of degree 2 with one edge and each contact vertex of degree 3 with a triangle on its neighbouring vertices. □

For further proofs we need to define an operation of *splitting* a contact point X of curves $\varrho, \sigma_1, \dots, \sigma_k$ along the curve ϱ; it produces a new contact representation in which the contact point X is replaced by k new 2–contact points $X_1, \dots, X_k$ of the pairs of curves $\varrho\sigma_1,\dots,\varrho\sigma_k$. Clearly, using the piecewise-linear embedding from Theorem 1, we may apply this operation on any one-sided contact representation, in both cases of ending or passing-through curve ϱ.

Lemma 2.4. *Let $\mathcal{R}$ be a 2–contact (3–contact) representation. Then there exists a simple 2–contact (simple 3–contact) representation $\mathcal{S}$ of an isomorphic graph $\boldsymbol{G}(\mathcal{S}) \cong \boldsymbol{G}(\mathcal{R})$.*

Sketch of proof. Suppose there exists a pair of curves $\varrho, \sigma \in \mathcal{R}$ such that $X, Y \in \varrho \cap \sigma$, $X \neq Y$. If Y is a 2–contact, we simply shorten the curve ending in Y. If Y is a 3–contact of curves $\varrho, \sigma, \vartheta$, we split Y along the curve ϑ. In both cases we get a representation of an isomorphic graph, and we proceed by induction. □

3 Classes of Contact Graphs

Various classes of (one-sided) contact graphs of curves or line segments, with respect to contact degrees and simplicity, are defined; and inclusions among them are studied. The complete description of the partially ordered set (Figure 4) formed by inclusions among the classes, is presented in Theorem 2, and it is proved by a sequence of following lemmas.

Definition. For an integer $k \geq 2$, we denote by *CONCUR* (k–*CONCUR*) the class of all contact (k–contact) graphs of curves, by *SCONCUR* (k–*SCONCUR*) the class of all simple contact (simple k–contact) graphs of curves, and by *CONSEG* (k–*CONSEG*) the class of all contact (k–contact) graphs of line segments.

Theorem 2. *All the inclusions among contact graph classes, described by a poset in Figure 4, are strict and no other inclusion holds.*

Lemma 3.1. 2–*CONCUR* $\not\subseteq$ *CONSEG*.

Proof. Let $\boldsymbol{H}$ be the graph constructed from the graph of a regular octahedron by adding 4 new vertices and 6 new edges into each face as is shown in Figure 5. By Proposition 2.2 the graph $\boldsymbol{H}$ is a 2–contact graph.

Suppose there exists a segment contact representation $\mathcal{S}$ of $\boldsymbol{H}$. Since $\boldsymbol{H}$ contains no clique of size 4, $\mathcal{S}$ must be 3–contact. The planar drawing of $\boldsymbol{H}$ is unique and contains no triangular face, so $\mathcal{S}$ is 2–contact by Proposition 2.3. This is a contradiction to the characterization of segment 2–contact graphs because the regular octahedron has 6 vertices and 12 edges. □

Lemma 3.2. 4–*CONCUR* $\not\subseteq$ *SCONCUR*.

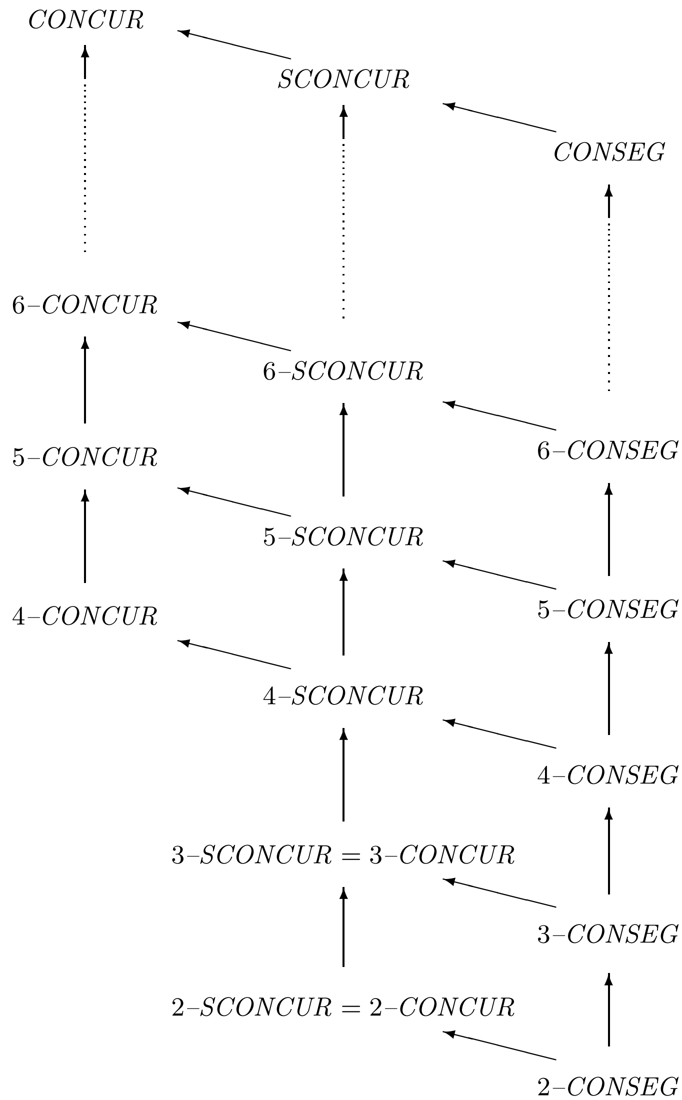

Fig. 4. The inclusions between classes of contact graphs

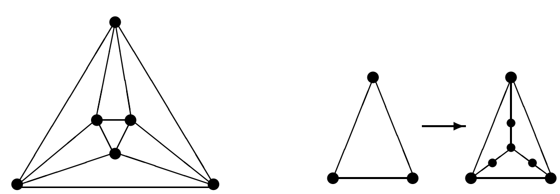

Fig. 5. A 2–contact graph that has no line segment contact representation

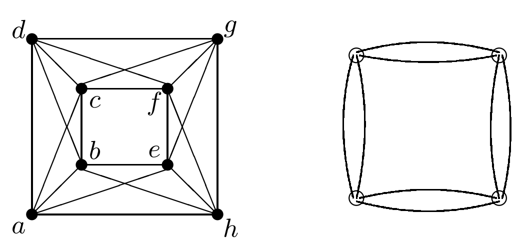

Fig. 6. A 4–contact graph that has no simple contact representation

Proof. In Figure 6 we can see the graph $\boldsymbol{F}$ and its 4–contact representation. Suppose there exists a simple contact representation $\mathcal{R}$ of $\boldsymbol{F}$. Clearly $\mathcal{R}$ is a 4–contact representation. Because $\mathcal{R}$ is simple, both subrepresentations $\mathcal{S}_1 = \{a, b, c, d\}$, $\mathcal{S}_2 = \{c, d, f, g\}$ cannot simultaneously contain a contact point of degree 4. Hence let $\mathcal{S}_1$ be a 3–contact representation, then also $\mathcal{S} = \mathcal{S}_1 \cup \{e, f\} = \{a, b, c, d, e, f\}$ is 3–contact, but the contact graph $\boldsymbol{G}(\mathcal{S})$ is not planar, a contradiction to Proposition 2.3. □

Lemma 3.3. *$(k+1)$–CONSEG $\not\subseteq$ k–CONCUR for $k \geq 2$.*

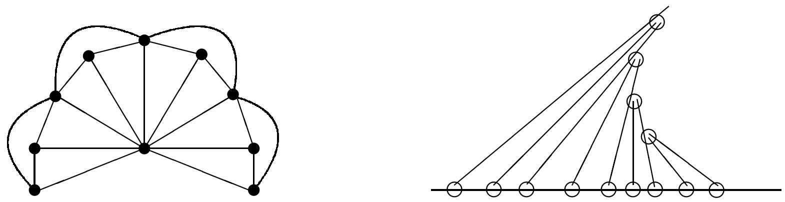

Fig. 7. A 3–contact graph of segments that has no 2–contact representation

Proof. For $k = 2$ we take the graph from Figure 7 that is a 3–contact graph of segments, but it has 10 vertices and 21 edges, so it has no 2–contact representation.

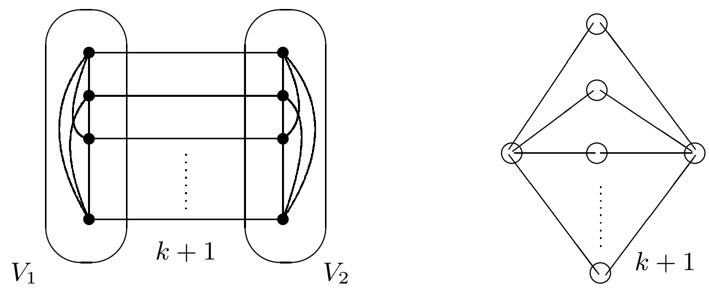

Fig. 8. A $(k+1)$–contact graph of segments that has no k–contact representation

For $k \geq 3$ let $\boldsymbol{G}_k$ be the graph from Figure 8, consisting of two disjoint $(k+1)$–cliques on the vertex sets V_1, V_2 and a perfect matching between them. Clearly there exists a segment $(k+1)$–contact representation of $\boldsymbol{G}_k$.

Firstly we show, using induction on $m \geq 3$, that any m–contact representation $\mathcal{K}$ of the graph $\boldsymbol{K}_{m+1}$ contains four curves forming a 3–contact subrepresentation: It is clear for $m = 3$, otherwise let X be a contact point of degree $m \geq 4$ and ϱ a curve containing X. If the subrepresentation $\mathcal{K} \setminus \{\varrho\}$ is $(m-1)$–contact, we proceed by induction. If not, there is another m–contact point Y in $\mathcal{K}$, and counting endpoints of curves in $\mathcal{K}$ implies that at most one other contact point Z has degree 4 or 5. Then it is enough to take one curve not containing X, one not containing Y, one not containing Z and an arbitrary one.

Let us suppose, for contradiction, that there exists a k–contact representation $\mathcal{R}$ of $\boldsymbol{G}_k$. Denoting by $\mathcal{V}_1, \mathcal{V}_2$ the subrepresentations of complete graphs on the vertex sets V_1, V_2, we apply the previous assertion on $\mathcal{K} = \mathcal{V}_1$, hence there is a 3–contact subrepresentation $\mathcal{S} \subseteq \mathcal{V}_1$, $|\mathcal{S}| = 4$. If we denote by $\mathcal{T} \subseteq \mathcal{V}_2$ its matching subrepresentation in $\mathcal{V}_2$, and split all possible 4–contact points of $\mathcal{T}$ into 2–contacts, producing a new $\mathcal{T}'$, we get a 3–contact representation $\mathcal{S} \cup \mathcal{T}'$ of a graph containing $\boldsymbol{K}_5$ as a minor, which is a contradiction to Proposition 2.3. □

Proof. (of Theorem 2)
All the stated inclusions are clear from definitions, and the equalities for 2– and 3–contact

classes are proved in Lemma 2.4. Finally, one can easily check that the rest of the theorem results from Lemmas 3.1,3.2,3.3. □

4 Recognition of Curve Contact Graphs

The problem to decide, whether a given graph can be represented as an intersection graph of specified objects, is important in studying the intersection graphs. The decision version of the problem is called the *recognition* of intersection graphs (of a special kind). For the interval graphs a simple characterization is given in [15], and a more efficient algorithm for their recognition is in [1]. Circle graphs (intersection graphs of chords of a circle) may be mentioned [2] as other kind of intersection graphs that can be quickly recognized. On the other hand, the recognition of intersection graphs of curves in the plane is proved to be *NP*–hard [12], moreover graphs with at least exponential complex representations [14] are known in that case.

4.1 Contact Representations of Planar Graphs

First we present an important lemma that is used to restrict the existence of certain contact representations of planar graphs.

Lemma 4.1. *Let $\mathcal{R}$ be a two-sided 3–contact representation of a graph $\boldsymbol{G}$ containing f free endpoints of curves. Then the representation $\mathcal{R}$ must contain at least $(|E(\boldsymbol{G})| - 2\cdot|V(\boldsymbol{G})| + f)$ 3–contact points forming non-neighbouring triangles in $\boldsymbol{G}$ (two triangles are said to be neighbouring if they have a common edge).*

Proof. We construct a mapping $t : E(\boldsymbol{G}) \to T(\mathcal{R})$ that assigns to each edge $e \in E(\boldsymbol{G})$ a contact point of $\mathcal{R}$ representing the edge e. If $\mathcal{R}$ is not simple and an edge e is represented by more contact points, we choose an arbitrary one of them. Then for a 2–contact point X, $|\, t^{-1}(X)| \le 1$, and for a 3–contact point X, $|\, t^{-1}(X)| \le 3$.

Let $T(\mathcal{R}) = T_1 \cup T_2 \cup T_3$ be a partition of the contact points such that $T_1 = \{X : |\, t^{-1}(X)| \le 1\}$, $T_2 = \{X : |\, t^{-1}(X)| = 2\}$ and $T_3 = \{X : |\, t^{-1}(X)| = 3\}$. In each contact point of T_1 at least one curve must end, each contact point of T_2 or T_3 is a 3–contact point so at least 2 curves must end in it. Therefore, by counting the number of endpoints of curves in $\mathcal{R}$, $|T_1|+2|T_2|+2|T_3| \le 2|V(\boldsymbol{G})|-f$, but from the mapping t we know $|E(\boldsymbol{G})| \le |T_1|+2|T_2|+3|T_3|$, so $|E(\boldsymbol{G})| - 2|V(\boldsymbol{G})| + f \le |T_3|$. That is what we need, because each 3–contact point of T_3 forms a triangle of $\boldsymbol{G}$, and two such triangles cannot share an edge due to the mapping t. □

4.2 A Polynomial Algorithm For Planar Triangulations

A *planar triangulation* is a planar graph, whose all faces, including the outer face, are triangles. We already know by Proposition 2.2 that the recognition of 2–contact graphs is polynomial—the edge number condition may be checked using the polynomial algorithm for network flows, and the planarity is linear. The result [5] is worth mentioning: Every bipartite planar graph is a 2–contact graph of horizontal and vertical line segments. A similar statement for 3–contacts of segments (that also results in a simple algorithm) is included in [7]:

Theorem (de Fraysseix, de Mendez).
A 4–connected planar triangulation is a 3–contact graph of segments iff it is 3–colourable.

We study the curve contact representations of planar triangulations.
Suppose we have a planar triangulation $\boldsymbol{G}$ on n vertices, $n \ge 3$, then $\boldsymbol{G}$ has $3n-6$ edges and $2n-4$ triangular faces. Since a triangulation is 3–connected, the planar drawing of $\boldsymbol{G}$ is unique. By Proposition 2.3, in a 3–contact representation of $\boldsymbol{G}$ each 3–contact point forms a triangular face, i.e. corresponds to a vertex of the dual graph of $\boldsymbol{G}$. In connection with Lemma 4.1 we get the following:

Observation. Let $\boldsymbol{G}$ be a planar triangulation, and $\boldsymbol{H}$ be the dual graph of $\boldsymbol{G}$. If $\boldsymbol{G}$ has a 3–contact representation $\mathcal{R}$, then $\mathcal{R}$ contains, by Lemma 4.1, $(3n-6)-(2n)+0=(n-6)$ 3–contact points corresponding to an independent set of size $n-6=(\frac{1}{2}|V(\boldsymbol{H})|-4)$ in $\boldsymbol{H}$.

From this observation we can already derive a polynomial algorithm for finding a 3–contact representation of a given triangulation.

Theorem 3. *There is a polynomial algorithm that for a given planar triangulation $\boldsymbol{G}$ decides whether $\boldsymbol{G}$ is a 3–contact graph, and finds a representation if one exists.*

Proof. First we show how all independent sets of size at least $(\frac{1}{2}v-4)$ in a connected 3–regular graph $\boldsymbol{H}$ on v vertices may be generated in polynomial time. Let $A \subset V(\boldsymbol{H})$, $|A| \geq (\frac{1}{2}v-4)$ be an independent set, $F \subset E(\boldsymbol{H})$ be the set of all edges of $\boldsymbol{H}$ that are disjoint with A, so the graph $\boldsymbol{H} \setminus F$ is bipartite. Then $|E(\boldsymbol{H})| = \frac{3}{2}v$, $|E(\boldsymbol{H}) \setminus F| \geq 3(\frac{1}{2}v-4) = (\frac{3}{2}v-12)$ and $|F| \leq 12$. In fact, $|F| \in \{0,3,6,9,12\}$, and we simply try all possible choices of such F. If we find out that $\boldsymbol{H}' = \boldsymbol{H} \setminus F$ is bipartite, we must also check whether a 2–colouring of $\boldsymbol{H}'$ may be chosen so that all vertices incident with F have the same colour, then A is formed by the vertices in the other colour. The whole process of generating the independent sets takes about $O(v^{13})$.

Now we return to the problem whether $\boldsymbol{G}$ is a 3–contact graph. If $\boldsymbol{G}$ has a 3–contact representation $\mathcal{R}$, the 3–contact points of $\mathcal{R}$ correspond to an independent set of size at least $\frac{1}{2}v-4$ in the dual graph $\boldsymbol{H}, |V(\boldsymbol{H})| = v$; and this set is surely generated by our algorithm. Conversely, if we get an independent set $A \subset E(\boldsymbol{H})$, $|A| = (\frac{1}{2}v-4)$, we construct the graph $\boldsymbol{I}$ from $\boldsymbol{G}$ by replacing each triangle uvw corresponding to a vertex of A in the dual graph with a new vertex adjacent to each u,v,w, and by subdividing all remaining edges (corresponding to the set F in the dual) with a vertex.

In that way we get a symmetrization of every possible incidence graph of a representation of $\boldsymbol{G}$. Then it is enough to check whether $\boldsymbol{I}$ can be oriented according to the assumptions of Proposition 2.1. Generally, an orientation of a given graph satisfying prescribed maximal outdegrees, can be found in polynomial time using Ford–Fulkerson's algorithm for the maximal flow in a network. Thus the whole process takes only polynomial time. □

Corollary 4.2. *The problem to decide whether a given 4–connected planar triangulation is a two-sided contact graph (and to find the representation) can be solved in polynomial time.*

Proof. The condition of 4–connectivity of the triangulation $\boldsymbol{G}$ means that $\boldsymbol{G}$ contains no non-face triangle, especially, it contains no $\boldsymbol{K}_4$. Thus any contact representation of $\boldsymbol{G}$ is 3–contact, and the 3–contact points must correspond to face triangles. So we may use the above algorithm, and produce a one-sided representation (if any). □

5 3–contact Graphs are *NP*–complete

Unlike the planar triangulations, the problem of deciding whether a given general planar graph is a 3–contact graph, is *NP*–complete (*NP*–hard for line segments). The recognition of one-sided or two-sided contact graphs of curves clearly belongs to *NP* from Theorem 1. For the *NP*–hardness reduction of our problem we use the PLANAR 3–SAT problem, see [4]; that is defined as a special case of the satisfiability problem (a formula Φ with a set variables V and a set of clauses C) for which the bipartite graph $\boldsymbol{F}$, $V(\boldsymbol{F}) = C \cup V$, $E(\boldsymbol{F}) = \{xc : x \in c \text{ or } \neg x \in c\}$, is planar with degrees of all vertices bounded by 3.

5.1 The End-eating graph

In further constructions we need a special graph that has a simple 3–contact representation, but no contact representation in which some curve has a free endpoint. This graph is presented in Figure 9, we denote it by $\boldsymbol{\mathcal{E}}$. The property of having no free endpoint in any contact

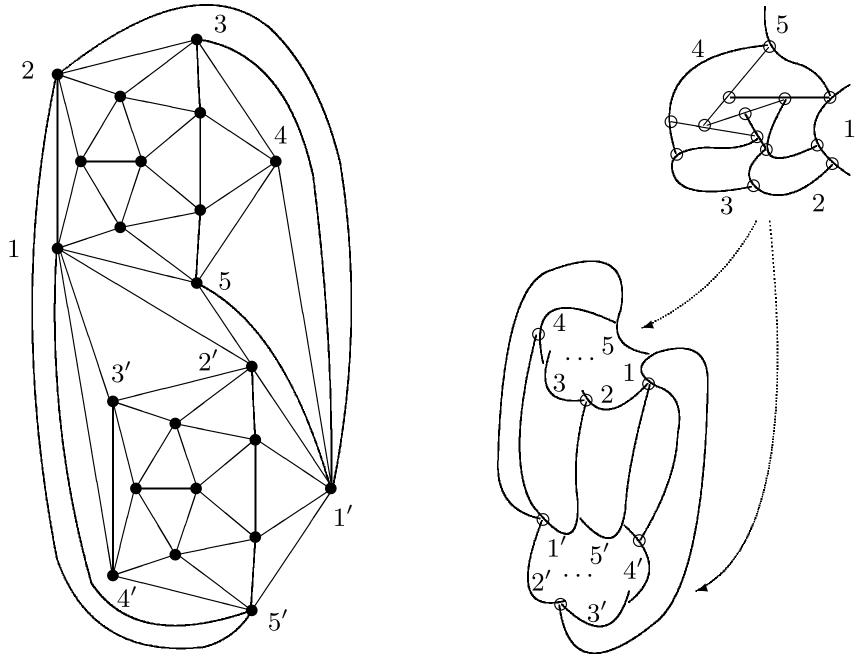

Fig. 9. The end-eating graph $\mathcal{E}$ and a scheme of its contact representation

representation follows from Lemma 4.1; reader may check that $\mathcal{E}$ has 22 vertices, 60 edges, but only 16 non-neighbouring triangles.

The graph $\mathcal{E}$ is used to "eat" an endpoint of a curve in a contact representation—if one of its vertices v is adjacent to some vertex w of another graph $\boldsymbol{G}$, the only way to represent the edge $\{v, w\}$ is to use one end of the curve w in the contact point of v, w. The graph produced by joining vertices $w_1, \ldots, w_k$ of a graph $\boldsymbol{G}$ with disjoint copies of $\mathcal{E}$ in the above sense will be denoted by $ENDEAT(\boldsymbol{G}, w_1, \ldots, w_k)$.

5.2 A Reduction to Satisfiability Problem

Given a formula Φ (of the PLANAR 3–SAT problem), we construct a graph $\boldsymbol{R}(\Phi)$ that has a contact representation iff the formula Φ is satisfiable. We may clearly suppose that each variable has at most 2 positive and at most 2 negated occurrences, otherwise, if some variable has only positive (negated) occurrences, we may set it true (false) and reduce the formula. In the construction all variable and clause vertices of the formula graph $\boldsymbol{F}(\Phi)$ are replaced by special graphs, see Figures 10, 11. Then clauses are connected with their variables by connectors, see Figure 12; connectors are adjacent to variable and clause graphs in special pairs of vertices called terminals, each in a different one.

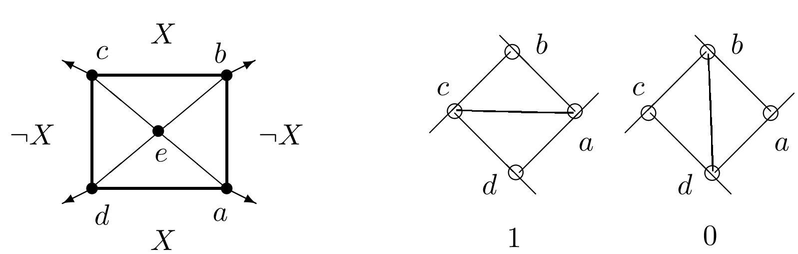

Fig. 10. The variable graph $\mathcal{V}$

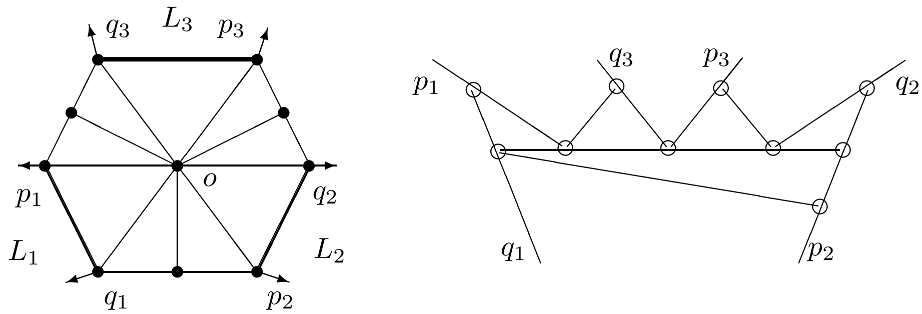

Fig. 11. The clause graph $\boldsymbol{\mathcal{C}}$

The graph $\boldsymbol{\mathcal{V}} = \mathit{ENDEAT}(\boldsymbol{G}, a, b, c, d)$, formed from the graph $\boldsymbol{G}$ in Figure 10 by adding the end-eating graphs to the vertices a, b, c, d, is called the *variable graph*. Similarly, the graph $\boldsymbol{\mathcal{C}} = \mathit{ENDEAT}(\boldsymbol{H}, p_1, q_1, p_2, q_2, p_3, q_3)$, obtained from the graph $\boldsymbol{H}$ in Figure 11, is called the *clause graph*. The pairs of vertices ab, bc, cd, da of the variable graph and p_1q_1, p_2q_2, p_3q_3 of the clause graph are *terminals*.

Each terminal xy encodes, in a contact representation, a logical value 1 if the edge $\{x, y\}$ is represented by a 2–contact, and 0 otherwise. The terminals bc, da of the variable graph are used to connect it with its positive occurrences in clauses, the terminals ab, cd for its negated occurrences. Thus the left representation from Figure 10 encodes a value TRUE of the variable X, and the right representation a value FALSE. In the clause graph, a terminal of value 0 means that the variable occurrence joined to this terminal must be true to satisfy the clause. The representation from Figure 11 has the terminal p_3q_3 of value 0, so the literal L_3 is forced to be true.

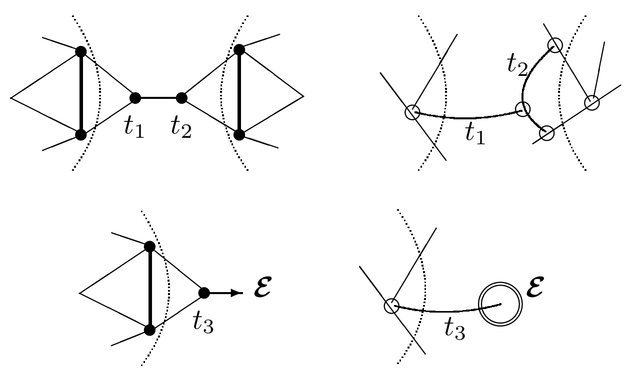

Fig. 12. Schemes of the connector and the false terminator

A subgraph consisting of two vertices t_1, t_2 and five edges joining them together and with two terminals, see the top part of Figure 12, is called a *connector*. A subgraph consisting of one vertex t_3 adjacent by two edges to a terminal and by another edge to a copy of the end-eating graph, see the bottom part of Figure 12, is called a *false terminator*.

Connectors are chosen so that two terminals can be connected iff at least one of them has value 1—either the literal is true or it is not chosen to satisfy the clause. To represent a connector, we use a 3–contact point in the terminal of value 1 and two 2–contact points in the terminal of value 0. The false terminator can be adjacent only to a terminal of value 1 by a 3–contact point, it is used to "fill" the third terminal of a clause with only two literals.

The graph $\boldsymbol{R}(\Phi)$ representing a formula Φ may be formally defined:
Let $\Phi = C_1 \wedge C_2 \wedge \ldots \wedge C_l$, where $C_i = L_{i1} \vee L_{i2} \vee L_{i3}$ for $i = 1, \ldots, k$, and $C_i = L_{i1} \vee L_{i2}$ for $i = k+1, \ldots, l$, and Φ is over a set of variables $x_1, x_2, \ldots, x_n$. The graph $\boldsymbol{R}(\Phi)$ is constructed

as a disjoint union of graphs $\boldsymbol{\mathcal{V}}(x_1), \ldots, \boldsymbol{\mathcal{V}}(x_n)$ and $\boldsymbol{\mathcal{C}}(C_1), \ldots, \boldsymbol{\mathcal{C}}(C_l)$, with connectors added as follows: For each literal $L_{ij} = x_m$ ($L_{ij} = \neg x_m$), $i = 1, \ldots, l$, a disjoint copy of the connector is joining the terminal da or bc (ab or cd) of $\boldsymbol{\mathcal{V}}(x_m)$, so that no terminal is used twice, with a terminal $p_j q_j$ of $\boldsymbol{\mathcal{C}}(C_i)$; and for each clause C_i, $k < i \leq l$, the false terminator is added to the terminal $p_3 q_3$ of $\boldsymbol{\mathcal{C}}(C_i)$.

An example of the graph $\boldsymbol{R}(\Phi)$ for a simple formula Φ is presented in Figure 13, the end-eating graphs added to vertices are schematically drawn by double circles.

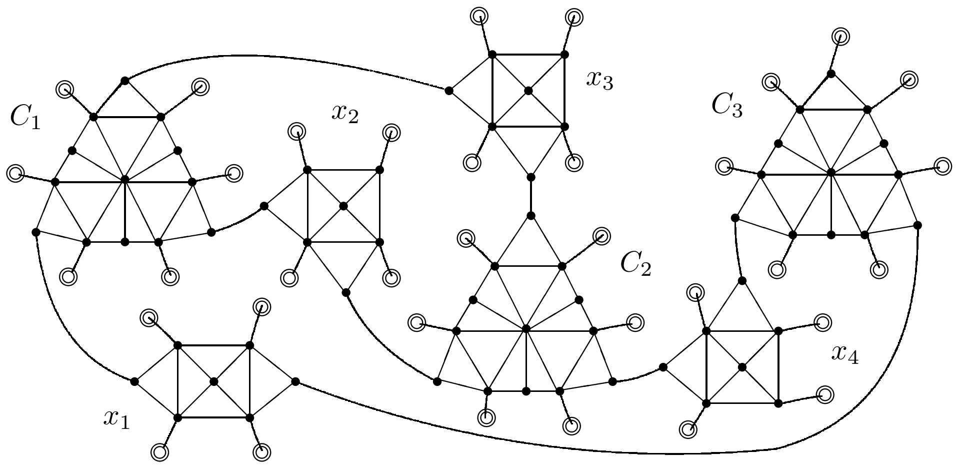

Fig. 13. An example of the graph $\boldsymbol{R}(\Phi)$ for $\Phi = C_1 \wedge C_2 \wedge C_3$, $C_1 = (\neg x_1 \vee \neg x_2 \vee \neg x_3)$, $C_2 = (x_2 \vee x_3 \vee \neg x_4)$, $C_3 = (\neg x_1 \vee x_4)$

Lemma 1. *The graph $\boldsymbol{R}(\Phi)$ of a given instance Φ of the PLANAR 3–SAT problem is planar; and is a one-sided 3–contact graph if Φ is satisfiable, but has no two-sided contact representation if Φ is not satisfiable.*

Proof. Suppose Φ is satisfiable. We represent the variable and clause graphs by schemes from Figures 10,11, depending on the satisfying values of the variables and on true literals in the clauses. Then we may clearly add all connectors and false terminators by schemes in Figure 12, and the resulting representation of $\boldsymbol{R}(\Phi)$ is one-sided 3–contact.

Conversely, if there exists a two-sided contact representation $\mathcal{R}$ of $\boldsymbol{R}(\Phi)$, it must be 3–contact (there is no 4–clique). By Lemma 4.1, the subrepresentation of the graph $\boldsymbol{\mathcal{V}}(x_m)$ for any variable x_m must contain at least $(8 - 2 \cdot 5 + 4) = 2$ 3–contact points forming non-neighbouring triangles in $\boldsymbol{\mathcal{V}}(x_m)$. Thus there are only two possibilities to represent $\boldsymbol{\mathcal{V}}(x_m)$, as shown in Figure 10, and they determine the logical value of x_m. Also each subrepresentation of the clause graph $\boldsymbol{\mathcal{C}}(C_i)$ must contain at least $(18 - 2 \cdot 10 + 6) = 4$ 3–contact points forming non-neighbouring triangles. This is possible only if at least one of them is the triangle $o\,p_j q_j$, $j \in \{1, 2, 3\}$, so if at least one terminal $p_j q_j$ of $\boldsymbol{\mathcal{C}}(C_i)$ has value 0.

Now it is enough to observe that a connector cannot be represented with both terminals of value 0. Therefore each clause contains a literal that is true in the above constructed valuation of $x_1, \ldots, x_n$, so Φ is satisfiable. □

Theorem 4. *The recognition of contact graphs (simple contact graphs, k–contact graphs for $k \geq 3$) is NP–complete, so is the recognition of two-sided contact graphs (two-sided k–contact graphs for $k \geq 3$).*
The recognition of contact graphs is NP–complete even within the class of planar graphs.

□

5.3 3–contact Graphs of Line Segments

Using more involved methods, a similar reduction can be constructed for contact graphs of line segments (the main problem is with an implementation of the end-eating graph).

Theorem 5. *The recognition of contact graphs (k–contact graphs, two-sided k–contact graphs, for $k \geq 3$) of line segments is NP–hard, even within the class of planar graphs.*

However, we do not know whether the recognition of contact graphs of segments belongs to *NP*.

Problem. *Is there, for every contact graph of line segments, a line segment contact representation with endpoints of the segments embedded on a grid of polynomial size?*

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References

1. K. S. Booth, G. S. Lucker, *Testing for the consecutive ones property, interval graphs, and graph planarity using PQ–tree algorithms*, J. Comp. Systems Sci. 13 (1976), 255–265.
2. A. Bouchet, *Reducing prime graphs and recognizing circle graphs*, Combinatorica 7 (1987), 243–254.
3. G. Ehrlich, S. Even, R.E. Tarjan, *Intersection graphs of curves in the plane*, J. of Comb. Theory Ser. B 21 (1976), 8–20.
4. M.R. Fellows, J. Kratochvíl, M. Middendorf, F. Pfeiffer, *The Complexity of induced minors and related problems*, Algorithmica 13 (1995), 266–282.
5. H. de Fraysseix, P.O. de Mendez, J. Pach, *Representation of planar graphs by segments*, 63. Intuitive Geometry (1991), 110–117.
6. H. de Fraysseix, P.O. de Mendez, P. Rosenstiehl, *On triangle contact graphs*, Combinatorics, Probability and Computing 3 (1994), 233–246.
7. H. de Fraysseix, P.O. de Mendez, to appear (probably not written yet).
8. P. Hliněný, *Contact graphs of curves*, KAM Preprint Series 95–285, Dept. of Applied Math., Charles University, Czech rep., 1995.
9. P. Hliněný, *Contact graphs of curves (extended abstract)*, in Graph Drawing (F.J.Brandenburg ed.), Proceedings Graph Drawing '95, Passau September 1995; Lecture Notes in Computer Science 1027, Springer Verlag, Berlin Heidelberg 1996, 312-323.
10. P. Hliněný, *The maximal clique and colourability of curve contact graphs*, to be published in Discrete Applied Math. 1997.
11. P. Koebe, *Kontaktprobleme der konformen Abbildung,* Berichte über die Verhandlungen der Sächsischen, Akad. d. Wiss., Math.–Physische Klasse 88 (1936), 141–164.
12. J. Kratochvíl, *String graphs II: Recognizing string graphs is NP–hard*, J. of Comb. Theory Ser. B 1 (1991), 67–78.
13. J. Kratochvíl, J. Matoušek, *Intersection graphs of segments*, J. of Comb. Theory Ser. B 2 (1994), 289–315.
14. J. Kratochvíl, J. Matoušek, *String graphs requiring exponential representations*, J. of Comb. Theory Ser. B 2 (1991), 1–4.
15. C.B. Lekkerkerker, J.C. Boland, *Representation of finite graphs by a set of intervals on the real line*, Fund. Math. 51 (1962), 45–64.
16. W. Schnyder, *Embedding planar graphs on the grid*, Proc. First ACM-SIAM Symposium on Discrete Algorithms, ACM Press 1990, 138–147.
17. Tamassia, Tollis, *A unified approach to visibility representations of planar graphs*, Discrete and Computational Geometry 1 (1986), 321–341.
18. C. Thomassen, presentation at Graph Drawing '93, Paris, 1993.

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