Bounded degree conjecture holds precisely for c-crossing-critical graphs with $c \leq 12$

Drago Bokal
Department of Mathematics and Computer Science, University of Maribor, Maribor, Slovenia
drago.bokal@um.si

Zdeněk Dvořák
Computer Science Institute, Charles University, Prague, Czech Republic
rakdver@iuuk.mff.cuni.cz

Petr Hliněný
Faculty of Informatics, Masaryk University, Brno, Czech Republic
hlineny@fi.muni.cz

Jesús Leaños
Academic Unit of Mathematics, Autonomous University of Zacatecas, Zacatecas, Mexico
jleanos@matematicas.reduaz.mx

Bojan Mohar
Department of Mathematics, Simon Fraser University, Burnaby BC, Canada and
Institute of Mathematics, Physics, and Mechanics, Ljubljana, Slovenia
mohar@sfu.ca

Tilo Wiedera
Theoretical Computer Science, Osnabrück University, Germany
tilo.wiedera@uos.de

Abstract

We study c-crossing-critical graphs, which are the minimal graphs that require at least c edge-crossings when drawn in the plane. For every fixed pair of integers with $c \geq 13$ and $d \geq 1$, we give first explicit constructions of c-crossing-critical graphs containing arbitrarily many vertices of degree greater than d. We also show that such unbounded degree constructions do not exist for $c \leq 12$, precisely, that there exists a constant D such that every c-crossing-critical graph with $c \leq 12$ has maximum degree at most D. Hence, the bounded maximum degree conjecture of c-crossing-critical graphs, which was generally disproved in 2010 by Dvořák and Mohar (without an explicit construction), holds true, surprisingly, exactly for the values $c \leq 12$.

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Related Version This work is based on a SoCG contribution [6] and gives full proofs and a significant strengthening of the announced results.

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1 Introduction

Minimizing the number of edge-crossings in a graph drawing in the plane (the *crossing number* of the graph, see Definition 2.1) is considered one of the most important attributes of a "nice drawing" of a graph. In the case of classes of dense graphs (those having superlinear number of edges in terms of the number vertices), the crossing number is necessarily very high – see the famous Crossing Lemma [1, 16]. However, within sparse graph classes (those having only linear number of edges), we may have planar graphs at one end and graphs with up to quadratic crossing number at the other end. In this situation, it is natural to study the "minimal obstructions" for low crossing number, with the following definition.

Let c be a positive integer. A graph G is called *c-crossing-critical* if the crossing number of G is at least c, but every proper subgraph has crossing number smaller than c. We say that G is *crossing-critical* if it is c-crossing-critical for some positive integer c.

Since any non-planar graph contains at least one crossing-critical subgraph, the understanding of the properties of the crossing-critical graphs is a central part of the theory of crossing numbers.

In 1984, Širáň gave the earliest construction of nonsimple c-critical-graphs for every fixed value of $c \geq 2$ [21]. Three years later, Kochol [14] gave an infinite family of c-crossing-critical, simple, 3-connected graphs, for every $c \geq 2$. Another early result on c-crossing-critical graphs was reported in the influential paper of Richter and Thomassen [20], who proved that c-crossing-critical graphs have bounded crossing number in terms of c. They also initiated research on degrees in c-crossing-critical graphs by showing that, if there exists an infinite family of r-regular, c-crossing-critical graphs for fixed c, then $r \in \{4, 5\}$. Of these, 4-regular 3-critical graphs were constructed by Pinontoan and Richter [19], and 4-regular c-critical graphs are known for every $c \geq 3$, $c \neq 4$ [4]. Salazar observed that the arguments of Richter and Thomassen could be applied to average degree as well, showing that an infinite family of c-crossing-critical graphs of average degree d can exist only for $d \in (3, 6]$, and established their existence for $d \in [4, 6)$. Nonexistence of such families with $d = 6$ was established much later by Hernández, Salazar, and Thomas [11], who proved that, for each fixed c, there are only finitely many c-crossing-critical simple graphs of average degree at least six. The existence of such families with $d \in [\frac{7}{2}, 4]$ was established by Pinontoan and Richter [19], whereas the whole possible interval was covered by Bokal [3], who showed that, for sufficiently large crossing number, both the crossing number c and the average degree $d \in (3, 6)$ could be prescribed for an infinite family of c-crossing critical graphs of average degree d.

In 2003, Richter conjectured that, for every positive integer c, there exists an integer $D(c)$ such that every c-crossing-critical graph has maximum degree less than $D(c)$ [17]. Reflecting upon this conjecture, Bokal in 2007 observed that the known 3-connected crossing-critical graphs of that time only had degrees $3, 4, 6$, and asked for existence of such graphs with arbitrary other degrees, possibly appearing arbitrarily many times. Hliněný augmented his construction of c-crossing-critical graphs with pathwidth linear in c to show the existence of c-crossing-critical graphs with arbitrarily many vertices of every set of even degrees. Only a recent paper by Bokal, Bračič, Derňár, and Hliněný [4] provided the corresponding result for odd degrees, showing in addition that, for sufficiently high c, all the three parameters – crossing number c, rational average degree d, and the set of degrees $D \subseteq \mathbb{N} \setminus \{1, 2\}$ that appear arbitrarily often in the graphs of the infinite family – can be prescribed. They also analysed the interplay of these parameters for 2-crossing-critical graphs that were recently completely characterized by Bokal, Oporowski, Richter, and Salazar [7].

Despite all this research generating considerable understanding of the behavior of degrees in known crossing-critical graphs as well as extending the construction methods of such graphs, the original conjecture of Richter was not directly addressed in the previous works. It was, however, disproved by Dvořák and Mohar [10], who showed that, for each integer $c \geq 171$, there exist c-crossing-critical graphs of arbitrarily large maximum degree. Their counterexamples, however, were not constructive, as they only exhibited, for every such c, a graph containing sufficiently many critical edges incident with a fixed vertex and argued that those edges belong to every c-crossing-critical subgraph of the exhibited graph. On the other hand, as a consequence of [7] it follows that, except for possibly some small examples, the maximum degree in a large 2-crossing-critical graph is at most 6, implying that Richter's conjecture holds for $c = 2$. In view of these results, and the fact that 1-crossing-critical graphs (subdivisions of K_5 and $K_{3,3}$) have maximum degree at most 4, this leaves Richter's conjecture unresolved for each $c \in \{3, 4, \ldots, 170\}$.

The richness of c-crossing-critical graphs is restricted for every c by the result of Hliněný that c-crossing-critical graphs have bounded path-width [12]; this structural result is complemented by a recent classification of all large c-crossing-critical graphs for arbitrary c by Dvořák, Hliněný, and Mohar [9]. We use these results in Section 4 to show that Richter's conjecture holds for $c \leq 12$. The result is stated below. It is both precise and surprising and shows how unpredictable are even the most fundamental questions about crossing numbers.

▶ **Theorem 1.1.** *There exists an integer D such that, for every positive integer $c \leq 12$, every c-crossing-critical graph has maximum degree at most D.*

In fact, one can separately consider in Theorem 1.1 twelve upper bounds D_c for each of the values $c \in \{1, 2, \ldots, 12\}$. For instance, $D_1 = 4$ and the optimal value of D_2 (we know $D_2 \geq 8$) should also be within reach using [7] and continuing research. On the other hand, due to the asymptotic nature of our arguments, we are currently not able to give any "nice" numbers for the remaining upper bounds, and we leave this aspect to future investigations.

We cover the remaining values of $c \geq 13$ in the gap in a very strong sense, by constructing critical graphs with arbitrarily many high-degree vertices:

▶ **Theorem 1.2.** *For every positive integers d and m, there exists a 3-connected 13-crossing-critical graph $G(d, m)$, which contains at least m vertices of degree at least d.*

▶ **Corollary 1.3.** *For every positive integers $c \geq 13$, d and m, there exists a 3-connected c-crossing-critical graph $G(c, d, m)$, which contains at least m vertices of degree at least d.*

The paper is structured as follows. The preliminaries, needed to help understanding the structure of large c-crossing critical graphs are defined in Section 2. We prove Theorem 1.1 in Section 4, and Theorem 1.2 in Section 5. An additional technical treatment and an operation call zip product is needed to establish Corollary 1.3 in Section 6. We conclude with some remarks and open problems in Section 7.

2 Graphs and the crossing number

In this paper, we consider multigraphs by default, even though we could always subdivide parallel edges (while sacrificing 3-connectivity) in order to make our graphs simple. We follow basic terminology of topological graph theory, see e.g. [18].

A *drawing* of a graph G in the plane is such that the vertices of G are distinct points and the edges are simple (polygonal) curves joining their end vertices. It is required that no edge passes through a vertex, and no three edges cross in a common point. A *crossing* is then an intersection point of two edges other than their common end. A *face* of the drawing is a maximal connected subset of the plane minus the drawing. A drawing without crossings in the plane is called a *plane drawing* of a graph, or shortly a *plane graph*. A graph having a plane drawing is *planar*.

The following are the core definitions used in this work.

▶ **Definition 2.1** (crossing number). *The* crossing number $\mathrm{cr}(G)$ *of a graph G is the minimum number of crossings of edges in a drawing of G in the plane. An* optimal drawing *of G is every drawing with exactly* $\mathrm{cr}(G)$ *crossings.*

▶ **Definition 2.2** (crossing-critical). *Let c be a positive integer. A graph G is* c-crossing-critical *if* $\mathrm{cr}(G) \geq c$*, but every proper subgraph G' of G has* $\mathrm{cr}(G') < c$*.*

Let us remark that a c-crossing-critical graph may have no drawing with only c crossings (for $c = 2$, such an example is the Cartesian product of two 3-cycles, $C_3 \square C_3$).

Suppose G is a graph drawn in the plane with crossings. Let G' be the plane graph obtained from this drawing by replacing the crossings with new vertices of degree 4. We say that G' is the plane graph associated with the drawing, shortly the *planarization* of (the drawing of) G, and the new vertices are the *crossing vertices* of G'.

In some of our constructions, we will have to combine crossing-critical graphs as described in the next definition.

▶ **Definition 2.3.** *Let $d = 2$ or 3. For $i \in \{1, 2\}$, let G_i be a graph and let $v_i \in V(G_i)$ be a vertex of degree d that is only incident with simple edges, such that $G_i - v_i$ is connected. Let u_i^j, $j \in \{1, \ldots, d\}$ be the neighbors of v_i. The zip product of G_1 and G_2 at v_1 and v_2 is obtained from the disjoint union of $G_1 - v_1$ and $G_2 - v_2$ by adding the edges $u_1^j u_2^j$, for each $j \in \{1, \ldots, d\}$.*

Note that, for different labellings of the neighbors of v_1 and v_2, different graphs may result from the zip product. However, the following has been shown:

▶ **Theorem 2.4** ([5]). *Let G be a zip product of G_1 and G_2 as in Definition 2.3. Then,* $\mathrm{cr}(G) = \mathrm{cr}(G_1) + \mathrm{cr}(G_2)$*. Furthermore, if for both $i = 1$ and $i = 2$, G_i is c_i-crossing-critical, where $c_i = \mathrm{cr}(G_i)$, then G is $(c_1 + c_2)$-crossing-critical.*

For vertices of degree 2, this theorem was established already by Leaños and Salazar in [15].

3 Structure of c-crossing-critical graphs with large maximum degree

Dvořák, Hliněný, and Mohar [9] recently characterized the structure of large c-crossing-critical graphs. From their result, it can be derived that in a crossing-critical graph with a vertex of large degree, there exist many internally vertex-disjoint paths from this vertex to the boundary of a single face. To keep our contribution self-contained, we give a simple independent proof. We are going to apply this structural result to exclude the existence of large degree vertices in c-crossing-critical graphs for $c \leq 12$.

Structural properties of crossing-critical graphs have been studied for more than two decades, and we now briefly review some of the previous important results which we shall use.

Richter and Thomassen [20] proved the following upper bound:

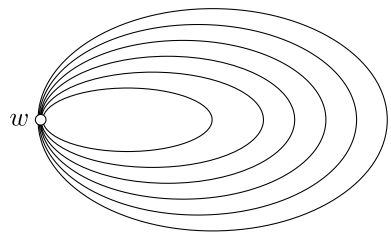

Figure 1 An illustration of a 1-nest.

▶ **Theorem 3.1** ([20]). *Every c-crossing-critical graph has a drawing with at most $\lceil 5c/2+16\rceil$ crossings.*

Hliněný [12] proved that c-crossing-critical graphs have path-width bounded in terms of c.

▶ **Theorem 3.2** ([12]). *There exists a function $f_{3.2} : \mathbb{N} \to \mathbb{N}$ such that, for every integer $c \geq 1$, every c-crossing-critical graph has path-width less than $f_{3.2}(c)$.*

For simplicity, we omit the exact definition of path-width; rather, we only use the following fact [2]. For a rooted tree T, let $b(T)$ denote the maximum depth of a rooted complete binary tree which appears in T as a rooted minor (the *depth* of a rooted tree is the maximum number of edges of a root-leaf path).

▶ **Lemma 3.3.** *For every integer $p \geq 0$, if a graph G either*
- *contains a subtree T which can be rooted so that $b(T) \geq p$, or*
- *contains pairwise vertex-disjoint paths $P_1, \ldots, P_p$ and pairwise vertex-disjoint paths $Q_1, \ldots, Q_p$ such that P_i intersects Q_j for every $i, j \in \{1, \ldots, p\}$,*

then G has path-width at least p.

Hliněný and Salazar [13] also proved that distinct vertices in a crossing-critical graph cannot be joined by too many paths.

▶ **Theorem 3.4** ([13]). *There exists a function $f_{3.4} : \mathbb{N} \to \mathbb{N}$ such that, for every integer $c \geq 1$, no two vertices of a c-crossing-critical graph are joined by more than $f_{3.4}(c)$ internally vertex-disjoint paths.*

As seen in the construction of Dvořák and Mohar [10] and in the construction we give in Section 5, crossing-critical graphs can contain arbitrarily many cycles intersecting in exactly one vertex. However, such cycles cannot be drawn in a nested way. A 1-*nest* of depth m in a plane graph G is a sequence $C_1, \ldots, C_m$ of cycles in G and a vertex $w \in V(G)$ such that, for $1 \leq i < j \leq m$, the cycle C_i is drawn in the closed disk bounded by C_j and $V(C_i) \cap V(C_j) = \{w\}$ (Figure 1). Hernández-Vélez et al. [11] have shown the following.

▶ **Theorem 3.5** ([11]). *There exists a function $f_{3.5} : \mathbb{N} \to \mathbb{N}$ such that, for every integer $c \geq 1$, the planarization of every optimal drawing of a c-crossing-critical graph does not contain a 1-nest of depth $f_{3.5}(c)$.*

The key structure we use in the proof of Corollary 3.13 is a *fan-grid*, which is defined as follows:

▶ **Definition 3.6.** *Let G be a plane graph and let v be a vertex incident with the outer face of G. Let C be a cycle in G, and let the path $C - v$ be the concatenation of vertex-disjoint paths L, $Q_1, \ldots, Q_n$, R in that order. Let H be the subgraph of G drawn inside the closed disk bounded by C. We say that $(v, C, Q_1, \ldots, Q_n)$ is an $(r \times n)$-*fan-grid *with* center v *if*

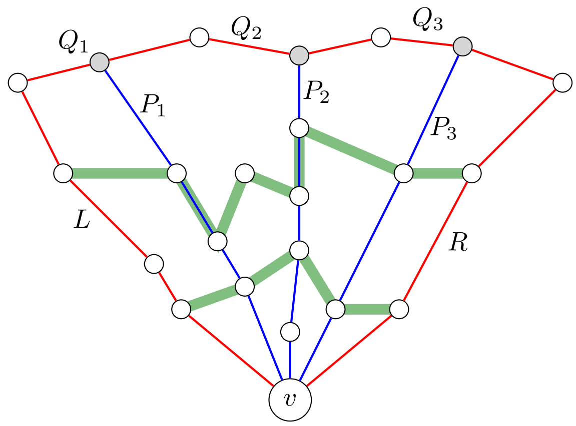

Figure 2 A (2×3)-fan-grid with center v. The rays of this fan-grid (P_1, P_2, and P_3) are colored blue. The underlying cycle C is red, and the two vertex-disjoint paths from $V(L)$ to $V(R)$ are green. These paths are shown in the idealized situation where they cross each of the paths P_i only once.

- *H contains n internally vertex-disjoint paths $P_1, \ldots, P_n$ (the* rays *of the fan-grid), where P_i joins v with a vertex of Q_i for $i = 1, \ldots, n$, and*
- *$H - V(Q_1 \cup \ldots \cup Q_n)$ contains r vertex-disjoint paths from $V(L)$ to $V(R)$. See Figure 2.*

In the argument, we start with a $(0 \times n)$-fan-grid and keep enlarging it (adding new rows while sacrificing some of the rays) as long as possible. The following definition is useful when looking for the new rows. A *comb with teeth* $v_1, \ldots, v_k$ is a tree consisting of a path P (the *spine* of the comb) and vertex-disjoint paths $P_1, \ldots, P_k$ of length at least one, such that P_i joins v_i to a vertex in P. We start with simple observations on combs in trees with many leaves.

▶ **Lemma 3.7.** *There exists a function $f_{3.7} : \mathbb{N}^3 \to \mathbb{N}$ such that the following holds for all integers $D, k \geq 1$ and $b \geq 0$. Let T be a rooted tree of maximum degree at most D satisfying $b(T) \leq b$. If every root-leaf path in T contains less than k vertices with at least two children, then T has at most $f_{3.7}(D, b, k)$ leaves.*

Proof. Let $f_{3.7}(D, b, k) = 1$ if $k = 1$ or $b = 0$, and

$$f_{3.7}(D, b, k) = f_{3.7}(D, b, k-1) + (D-1) \cdot f_{3.7}(D, b-1, k-1)$$

if $k \geq 2$ and $b \geq 1$. We prove the claim by the induction on the number of vertices of T. If $|V(T)| = 1$, then T has only one leaf. Hence, suppose that $|V(T)| \geq 2$. Let v be the root of T and let $T_1, \ldots, T_d$ be the components of $T - v$, where $d = \deg(v) \leq D$. If $d = 1$, then the claim follows by the induction hypothesis applied to T_1; hence, suppose that $d \geq 2$. In particular, $b \geq b(T) \geq 1$ and $k \geq 2$. Then, for all $i \in \{1, \ldots, d\}$, each root-leaf path in T_i contains less than $k-1$ vertices with at least two children. Furthermore, there exists at most one $i \in \{1, \ldots, d\}$ such that $b(T_i) = b$; hence, we can assume that $b(T_i) \leq b - 1$ for $2 \leq i \leq d$. By the induction hypothesis, T_1 has at most $f_{3.7}(D, b, k-1)$ leaves and each of $T_2, \ldots, T_d$ has at most $f_{3.7}(D, b-1, k-1)$ leaves, implying the claim. ◀

▶ **Corollary 3.8.** *For every triple of integers satisfying $D, k \geq 1$ and $b \geq 0$, every rooted tree T of maximum degree at most D, $b(T) \leq b$, and with more than $f_{3.7}(D, b, k)$ leaves contains a comb with k teeth, all of which are leaves in T.*

Proof. By Lemma 3.7, T contains a root-leaf path P with at least k vertices that have at least two children. A subpath of P together with the paths from k of these vertices to leaves forms a comb with k teeth. ◀

Suppose Q is a path and K is a comb in a plane graph G, such that all teeth of K lie on Q and K and Q are otherwise disjoint. We say that the comb is Q-*clean* if both Q and the spine of K are contained in the boundary of the outer face of the subdrawing of G formed by $K \cup Q$.

▷ Observation 3.9. Suppose Q is a path and K is a comb in a plane graph G, such that all teeth of K lie on Q and K and Q are otherwise disjoint. Let $k \geq 2$ be an integer. If Q is contained in the boundary of the face of G and K has at least $3k-1$ teeth, then K contains a Q-clean subcomb with at least k teeth.

Our aim is to keep growing a fan-grid using the following Lemma (increasing r at the expense of sacrificing some of the rays, see the outcome (d)) until we either obtain a structure that cannot appear in a planarization of a c-crossing-critical graph (outcomes (a)–(c)), or are blocked off from further growth by many rays ending in the boundary of the same face (outcome (e)).

▶ **Lemma 3.10.** *There exists a function $f_{3.10} : \mathbb{N}^5 \to \mathbb{N}$ such that the following holds. Let G be a plane graph with a vertex v incident with the outer face. Let D, b, m, r, k, and t be positive integers. Let $n = f_{3.10}(D, b, m, k, t)$. If G contains an $(r \times n)$-fan-grid with center v, then G also contains at least one of the following substructures:*

(a) two vertices joined by more than D internally vertex-disjoint paths, or
(b) a 1-nest of depth greater than m, or
(c) a subtree T which can be rooted so that $b(T) > b$, or
(d) an $((r+1) \times k)$-fan-grid with center v, or
(e) more than t internally vertex-disjoint paths from v to distinct vertices contained in the boundary of a single face of G.

Proof. Let $s_1 = 2f_{3.7}(D, b, 3k+5)$ and $s_2 = 2s_1 m$. For an integer $l \geq 0$, let $d(l) = (t \cdot (s_1 - 1))^l$. Let $f_{3.10}(D, b, m, k, t) = td(s_2) + 1$. Let $(v, C, Q_1, \ldots, Q_n)$ be an $(r \times n)$-fan-grid in G. Let G_1 be the graph obtained from G by removing the vertices and edges drawn in the open disk bounded by C, and let f denote the resulting face bounded by C.

Suppose that, for some $i \in \{1, \ldots, n\}$, there exists a component R of $G_1 - (V(C) \setminus V(Q_i))$ and a set $J \subseteq \{1, \ldots, n\} \setminus \{i\}$ of size more than s_1 such that Q_j has a neighbor in R for every $j \in J$. By symmetry, we can assume that there exists $J' \subseteq J$ of size more than $f_{3.7}(D, b, 3k+5)$ such that $j > i$ for each $j \in J'$. Observe that G contains a tree T with all internal vertices in R and exactly $|J'|$ leaves, one in each of Q_j for $j \in J'$; we root T in a vertex belonging to R. By Corollary 3.8, $\Delta(T) > D$ or $b(T) > b$ or T contains a comb K with $3k+5$ teeth, all of which are leaves of T. In the former two cases, G contains (a) or (c). In the last case, we extract a $(C - v)$-clean subcomb with $k+2$ teeth from K using Observation 3.9 and combine it with a part of the $(r \times n)$-fan-grid in G to form an $((r+1) \times k)$-fan-grid in G, showing that G contains (d).

Therefore, we can assume that the following holds:

($\star$) For every $i \in \{1, \ldots, n\}$, every component R of $G_1 - (V(C) \setminus V(Q_i))$ has neighbors in less than s_1 paths $Q_1, \ldots, Q_n$ other than Q_i.

A C-*bridge* of G_1 is either a graph consisting of a single edge of $E(G_1) \setminus E(C)$ and its ends, or a graph consisting of a component of $G_1 - V(C)$ together with all edges between the component and C and their endpoints. For a C-bridge H, let $J(H)$ denote the set of indices

$j \in \{1, \ldots, n\}$ such that H intersects Q_j. By ($\star$), we have $|J(H)| \leq s_1$. For two C-bridges H_1 and H_2, we write $H_1 \prec H_2$ if $\min(J(H_2)) \leq \min(J(H_1))$, $\max(J(H_1)) \leq \max(J(H_2))$, and either at least one of the inequalities is strict or $J(H_2) \subsetneq J(H_1)$ (note that in the last case, the planarity implies $|J(H_2)| = 2$).

Suppose there exist $s_2 + 1$ C-bridges $H_0, H_1, \ldots, H_{s_2}$ such that $H_0 \prec H_1 \prec \ldots \prec H_{s_2}$. For $1 \leq j \leq m+1$, let $b(j) = 2(j-1)s_1$. If, for $1 \leq j \leq m$, we have $\min(J(H_{b(j)})) > \min(J(H_{b(j+1)}))$ and $\max(J(H_{b(j)})) < \max(J(H_{b(j+1)}))$, then G contains (b). Hence, by symmetry we can assume that there exists $j \in \{1, \ldots, m\}$ such that $\min(J(H_{b(j)})) = \min(J(H_{b(j+1)}))$. Consequently, there exists an index i such that $\min(J(H_p)) = i$, for $b(j) \leq p \leq b(j+1)$. But then $|\bigcup_{p=b(j)}^{b(j+1)} J(H_p) \setminus \{i\}| > s_1$, which contradicts ($\star$).

Consequently, there is no chain of order greater than s_2 in the partial ordering $\prec$. For a C-bridge H, let $l(H)$ denote the order of the longest chain of $\prec$ with the maximum element H. Suppose that $\max(J(H)) - \min(J(H)) > d(l(H))$, and choose a C-bridge H with this property such that $l(H)$ is minimum. Since $|J(H)| \leq s_1$, there exist two consecutive elements j_1 and j_2 of $J(H)$ such that $j_2 - j_1 > t \cdot d(l(H) - 1)$. If $l(H) = 1$, this implies there exists a face f of G such that all paths Q_j with $j_1 \leq j \leq j_2$ contain vertices incident with f, and G contains (e). Hence, suppose that $l(H) > 1$. Let B be the set of bridges $H' \prec H$ such that $j_1 \leq \min(J(H')) \leq \max(J(H')) \leq j_2$ that are maximal in $\prec$ with this property. By the minimality of $l(H)$, every bridge $H' \in B$ satisfies $\max(J(H')) - \min(J(H')) \leq d(l(H) - 1)$. Consequently, there are more than $t + 1$ indices j such that $j_1 \leq j \leq j_2$ and either $j = \max(J(H'))$ for some $H' \in B$ or there does not exists any bridge $H' \prec H$ such that $\min(J(H')) \leq j \leq \max(J(H'))$. Observe there exists a face f of G such that, for each such index j, the path Q_j contains a vertex incident with f. Hence, G again contains (e).

Consequently, we can assume that $\max(J(H)) - \min(J(H)) \leq d(s_2)$, for each C-bridge H. Since $n > td(s_2)$, applying an analogous argument to the C-bridges that are maximal in $\prec$ yields that G contains (e). ◀

To start up the growing process based on Lemma 3.10, we need to show that a fan-grid with many rays exists.

▶ **Lemma 3.11.** *Let G be a 2-connected plane graph with a vertex v incident with the outer face. For every positive integers D, b, and k, if v has degree more than $f_{3.7}(D, b+1, 3k+5)$, then G contains at least one of the following:*

(a) two vertices joined by more than D internally vertex-disjoint paths, or

(c) a subtree T which can be rooted so that $b(T) > b$, or

(d) a $(0 \times k)$-fan-grid with center v.

Proof. Let G' be the graph obtained from G by splitting v into vertices of degree 1, and let S be the set of these vertices. Since G is 2-connected, G' is connected, and thus it contains a subtree T whose leaves coincide with S. Root T arbitrarily in a non-leaf vertex. By Corollary 3.8, $\Delta(T) > D$ or $b(T) > b + 1$ or T has a comb with $3k + 5$ teeth in S. In the first case, (a) holds. In the second case, $b(T - S) > b$ and $T - S$ is a subtree of G, and thus (c) holds. In the last case, we can extract a v-clean subcomb with at least $k + 2$ teeth using Observation 3.9, which gives rise to a $(0 \times k)$-fan-grid with center v in G. ◀

Note that a $((p+1) \times (p+1))$-fan-grid contains two systems of $p+1$ pairwise vertex-disjoint paths such that every two paths from the two systems intersect; hence, by Lemma 3.3 a plane graph of path-width at most p contains neither a $((p+1) \times (p+1))$-fan-grid nor a subtree T which can be rooted so that $b(T) > p$. Hence, starting from Lemma 3.11 and iterating Lemma 3.10 at most $p + 1$ times, we obtain the following.

▶ **Corollary 3.12.** *There exists a function $f_{3.12}: \mathbb{N}^4 \to \mathbb{N}$ such that the following holds. Let G be a 2-connected plane graph. Let D, m, p, and t be positive integers. Let $\Delta = f_{3.12}(D, m, p, t)$. If G has path-width at most p and maximum degree greater than Δ, then G contains at least one of the following:*

(a) two vertices joined by more than D internally vertex-disjoint paths, or

(b) a 1-nest of depth greater than m, or

(e) more than t internally vertex-disjoint paths from a vertex v to distinct vertices contained in the boundary of a single face of G.

We now apply this result to an optimal planarization of a c-crossing-critical graph.

▶ **Corollary 3.13.** *There exists a function $f_{3.13}: \mathbb{N}^2 \to \mathbb{N}$ such that the following holds. Let $c \geq 1$ and $t \geq 3$ be integers and let G be an optimal drawing of a 2-connected c-crossing-critical graph. If G has maximum degree greater than $f_{3.13}(c, t)$, then there exists a path Q contained in the boundary of a face of G and internally vertex-disjoint paths $P_1, \ldots, P_t$ starting in the same vertex not in Q and ending in distinct vertices appearing in order on Q (and otherwise disjoint from Q), such that no crossings of G appear on P_1, P_t, nor in the face of $P_1 \cup P_t \cup Q$ that contains $P_2, \ldots, P_{t-1}$.*

Proof. Let $c' = \lceil 5c/2 + 16 \rceil$, $D = \max(5, f_{3.4}(c) + c')$, $m = f_{3.5}(c)$, $p = f_{3.2}(c) + c'$, and $f_{3.13}(c, t) = f_{3.12}(D, m, p, (c' + 2)t)$.

By Theorem 3.1, G has at most c' crossings. Let G' be the planarization of G. Note that G' is 2-connected, since otherwise a crossing vertex would form a cut in G' and the corresponding crossing in G could be eliminated, contradicting the optimality of the drawing of G. By Theorem 3.2, G has path-width at most $p - c'$, and thus G' has path-width at most p. By Theorem 3.4, G does not contain more than $D - c'$ internally vertex-disjoint paths between any two vertices, and thus G' does not contain more than D internally vertex-disjoint paths between any two vertices. By Theorem 3.5, G' does not contain a 1-nest of depth m. Hence, by Corollary 3.12, G' contains more than $(c' + 1)t$ internally disjoint paths from a vertex v to distinct vertices contained in the boundary of a single face f of G'. Let $Q_1, \ldots, Q_{c'+2}$ be disjoint paths contained in the boundary of f such that, for $i = 1, \ldots, c' + 2$, t of the paths $P_{i,1}, \ldots, P_{i,t}$ from v end in Q_i in order. Let g_i denote the face of $Q_i \cup P_{i,1} \cup P_{i,t}$ containing $P_{i,2}, \ldots, P_{i,t-1}$. Note that the closures of $g_1, \ldots, g_{c'+2}$ intersect only in v and since G' contains at most c' crossing vertices, there exists $i \in \{1, \ldots, c' + 2\}$ such that no crossing vertex is contained in the closure of g_i and v is not in Q_i. Hence, for $j = 1, \ldots, t$, we can set $Q = Q_i$ and $P_j = P_{i,j}$. ◀

4 Crossing-critical graphs with at most 12 crossings

We now use Corollary 3.13 to prove the following “redrawing” lemma.

▶ **Lemma 4.1.** *Let G be a 2-connected c-crossing-critical graph. If G has maximum degree greater than $f_{3.13}(c, 6c + 1)$, then there exist integers $r \geq 2$ and $k \geq 0$ such that $kr \leq c - 1$ and G has a drawing with at most $c - 1 - kr + \binom{k}{2}$ crossings.*

Proof. Consider an optimal drawing of G. Let $P_1, \ldots, P_{6c+1}$ be paths obtained using Corollary 3.13 and v their common end vertex. For $2 \leq i \leq 6c - 1$, let T_i denote the 2-connected block of $G - ((V(P_{i-1}) \cup V(P_{i+2})) \setminus \{v\})$ containing P_i and P_{i+1}, and let C_i denote the cycle bounding the face of T_i containing P_{i-1}. Note that if $2 \leq i$ and $i + 3 \leq j \leq 6c - 1$,

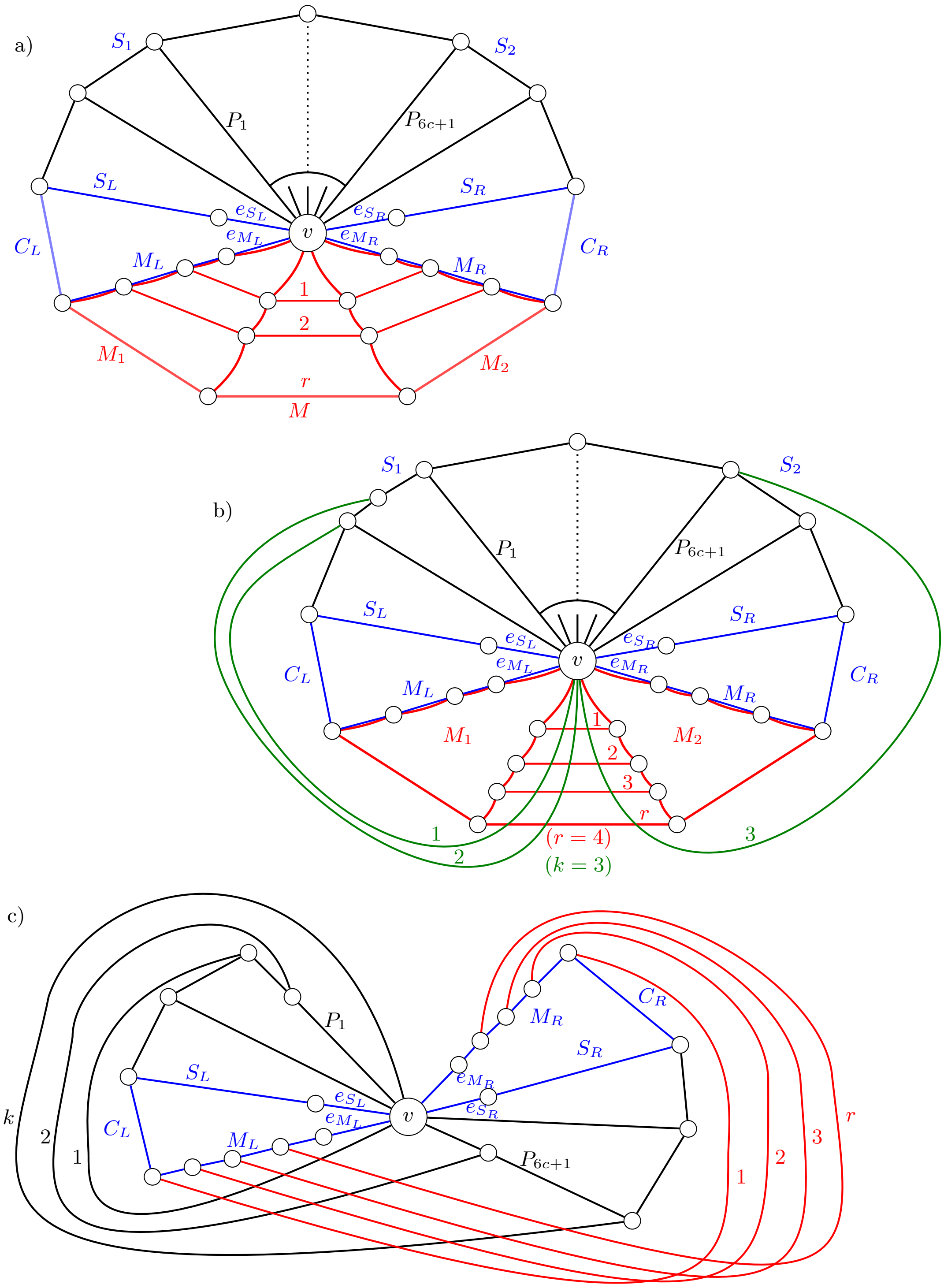

Figure 3 An illustration of the proof of Lemma 4.1. a) The original optimal drawing of G, with subdrawings of M_1 and M_2 (red) that will be glued into the drawing of G_0 from an optimal drawing of $G - e$. b) A drawing of G with at most $c - 1$ crossings, obtained from G_0 (black, blue, green) and M_1, M_2 (red). c) A drawing of G with at most $\binom{c-1-kr+k}{2}$ crossings, obtained from G_0 (black, blue) and M_1, M_2 (red).

then $G - V(T_i \cup T_j)$ has at most three components: one containing $P_{i+2} - v$, one containing $P_1 - v$, and one containing $P_{6c+1} - v$, where the latter two components can be the same.

Let e be the edge of P_{3c+1} incident with v and let G' be an optimal drawing of $G - e$. Since G is c-crossing-critical, G' has at most $c - 1$ crossings. Hence, there exist indices i_1 and i_2 such that $2 \le i_1 \le 3c - 1$, $3c + 2 \le i_2 \le 6c - 1$, and none of the edges of T_{i_1} and T_{i_2} is crossed. Let us set $L = T_{i_1}$, $C_L = C_{i_1}$, $R = T_{i_2}$, and $C_R = C_{i_2}$. Let M, S_1, and S_2 denote the subgraphs of G consisting of the components of $G - V(L \cup R)$ containing $P_{3c+1} - v$, $P_1 - v$, and $P_{6c+1} - v$, respectively, together with the edges from these components to the rest of G and their incident vertices (where possibly $S_1 = S_2$). Let S_L and M_L be subpaths of C_L of length at least one intersecting in v such that $V(S_1 \cap C_L) \subseteq V(S_L)$ and $V(M \cap C_L) \subseteq V(M_L)$. Analogously, let S_R and M_R be subpaths of C_R of length at least one intersecting in v such that $V(S_2 \cap C_R) \subseteq V(S_R)$ and $V(M \cap C_R) \subseteq V(M_R)$. See Figure 3.

We can assume without loss of generality (by circle inversion of the plane if necessary) that neither C_L nor C_R bounds the outer face of $C_L \cup C_R$ in the drawings inherited from G and from G'. Let e_{M_L}, e_{S_L}, e_{S_R}, e_{M_R} be the clockwise cyclic order of the edges of $C_L \cup C_R$ incident with v in the drawing G, where $e_Q \in E(Q)$ for every $Q \in \{M_L, S_L, S_R, M_R\}$. By the same argument, we can assume that the clockwise cyclic order of these edges in the drawing of G' is either the same or e_{M_L}, e_{S_L}, e_{M_R}, e_{S_R}.

In G, L is drawn in the closed disk bounded by C_L, R is drawn in the closed disk bounded by C_R, and M, S_1, and S_2 together with all the edges joining them to v are drawn in the outer face of $C_L \cup C_R$. Since C_L and C_R are not crossed in the drawing G', we can if necessary rearrange the drawing of G' without creating any new crossings[1] so that the same holds for the drawings of L, R, M, S_1, and S_2 in G'. Let $r \ge 1$ denote the maximum number of pairwise edge-disjoint paths in $M - v$ from $V(M \cap C_L - v)$ to $V(M \cap C_R - v)$. By Menger's theorem, $M - v$ has disjoint induced subgraphs M_1' and M_2' such that $V(M - v) = V(M_1') \cup V(M_2')$, $V(M \cap C_L - v) \subseteq V(M_1')$, $V(M \cap C_R - v) \subseteq V(M_2')$, and G contains exactly r edges with one end in M_1' and the other end in M_2'. For $i \in \{1, 2\}$, let M_i be the subgraph of M induced by $V(M_i') \cup \{v\}$. Let F be a path in $M - v$ from $V(M \cap C_L - v)$ to $V(M \cap C_R - v)$ that has in the drawing G' the smallest number of intersections with the edges of $S_1 \cup S_2$, and let k denote the number of such intersections. Let G_0 denote the drawing $G' - (V(M) \setminus V(M_L \cup M_R))$. Since $M - v$ contains r pairwise edge-disjoint paths from $V(M \cap C_L - v)$ to $V(M \cap C_R - v)$ and each of them crosses $S_1 \cup S_2$ at least k times, we conclude that G' has at least kr crossings (and thus $kr \le c - 1$) and G_0 has at most $c - 1 - kr$ crossings.

Suppose first that edges of $C_L \cup C_R$ incident with v are in G' drawn in the same clockwise cyclic order as in G. We construct a new drawing of the graph G in the following way: Start with the drawing of G_0. Take the plane drawings of M_1 and M_2 as in G, "squeeze" them and draw them very close to M_L and M_R, respectively, so that they do not intersect any edges of G_0. Finally, draw the r edges between M_1 and M_2 very close to the curve tracing F (as drawn in G'), so that each of them is crossed at most k times. This gives a drawing of G with at most $(c - 1 - kr) + kr < c$ crossings, contradicting the assumption that G is c-crossing-critical.

Hence, we can assume that the edges of $C_L \cup C_R$ incident with v are in G' drawn in the clockwise order e_{M_L}, e_{S_L}, e_{M_R}, e_{S_R}. If $r = 1$, then proceed analogously to the previous

---

[1] As G is not 3-connected, it is possible that some 2-connected components or some edges of L, R are drawn in the exterior of the disk bounded by C_L, C_R. However, these can be flipped into the interior of C_L, C_R, and after such *rearranging*, C_L, C_R bound the outer face of the drawings of L, R. Similarly, if $S_1 \neq S_2$, either of them could be in the interior of C_L, C_R, and we flip them into the exterior, so that the interior of C_L, C_R contains only drawings of L, R, respecitvely.

paragraph, except that a mirrored version [2] of the drawing of M_2 is inserted close to M_R; as there is only one edge between M_1 and M_2, this does not incur any additional crossings, and we again conclude that the resulting drawing of G has fewer than c crossings, a contradiction. Therefore, $r \geq 2$.

Consider the drawing G', and let q be a closed curve passing through v, following M_L slightly outside C_L till it meets F, then following F almost till it hits M_R, then following M_R slightly outside C_R till it reaches v. Note that q only crosses G_0 in v and in relative interiors of the edges, and it has at most k crossings with the edges. Shrink and mirror the part of the drawing of G_0 drawn in the open disk bounded by q, keeping v at the same spot and the parts of edges crossing q close to q; then reconnect these parts of the edges with their parts outside of q, creating at most $\binom{k}{2}$ new crossings in the process. Observe that in the resulting re-drawing of G_0, the path $M_L \cup M_R$ is contained in the boundary of a face (since q is drawn close to it and nothing crosses this part of q), and thus we can add M planarly (as drawn in G) to the drawing without creating any further crossings. Therefore, the resulting drawing has at most $c - 1 - kr + \binom{k}{2}$ crossings. ◀

It is now easy to prove Theorem 1.1.

Proof of Theorem 1.1. We prove by induction on c that, for every positive integer $c \leq 12$, there exists an integer Δ_c such that every c-crossing-critical graph has maximum degree at most c. The only 1-crossing-critical graphs are subdivisions of K_5 and $K_{3,3}$, and thus we can set $\Delta_1 = 4$. Suppose now that $c \geq 2$ and the claim holds for every smaller value. We define $\Delta_c = \max(2\Delta_{c-1}, f_{3.13}(c, 6c+1))$. Let G be a c-crossing-critical graph and suppose for a contradiction that $\Delta(G) > \Delta_c$.

If G is not 2-connected, then it contains induced subgraphs G_1 and G_2 such that $G_1 \neq G \neq G_2$, $G = G_1 \cup G_2$, and G_1 intersects G_2 in at most one vertex. Then $c \leq \mathrm{cr}(G) = \mathrm{cr}(G_1) + \mathrm{cr}(G_2)$, and for every edge $e \in E(G_1)$ we have $c > \mathrm{cr}(G - e) = \mathrm{cr}(G_1 - e) + \mathrm{cr}(G_2)$. Hence, $\mathrm{cr}(G_1) \geq c - \mathrm{cr}(G_2)$ and $\mathrm{cr}(G_1 - e) < c - \mathrm{cr}(G_2)$ for every edge $e \in E(G_1)$, and thus G_1 is $(c - \mathrm{cr}(G_2))$-crossing-critical. Similarly, G_2 is $(c - \mathrm{cr}(G_1))$-crossing-critical. Since $\mathrm{cr}(G_1) \geq 1$ and $\mathrm{cr}(G_2) \geq 1$, it follows by the induction hypothesis that $\Delta(G_i) \leq \Delta_{c-1}$ for $i \in \{1, 2\}$, and thus $\Delta(G) \leq \Delta_c$, which is a contradiction.

Hence, G is 2-connected. By Lemma 4.1, there exist integers $r \geq 2$ and $k \geq 0$ such that $kr \leq c - 1$ and $c - 1 - kr + \binom{k}{2} \geq c$, and thus $\binom{k}{2} \geq kr + 1 \geq 2k + 1$. This inequality is only satisfied for $k \geq 6$, and thus the first inequality implies $c \geq kr + 1 \geq 13$. This is a contradiction. Hence, the maximum degree of G is at most Δ_c. ◀

5 Explicit 13-crossing-critical graphs with large degree

We define the following family of graphs, which is illustrated in Figure 4. To simplify the terminology and the pictures, we introduce "thick edges": for a positive integer t, we say that uv is a *t-thick edge*, or an edge of *thickness* t, if there is a bunch of t parallel edges between u and v. Naturally, if a t_1-thick edge crosses a t_2-thick edge, then this counts as $t_1 t_2$ ordinary crossings. By routing every parallel bunch of edges along the "cheapest" edge of the bunch, we get the following important folklore claim:

[2] *Mirrored version of a drawing* is the drawing obtained by reversing the vertex rotations of edges around every vertex and every crossing, and embedding the edges and the vertices accordingly. The name explains that this is homeomorphic to the original drawing seen in a mirror.

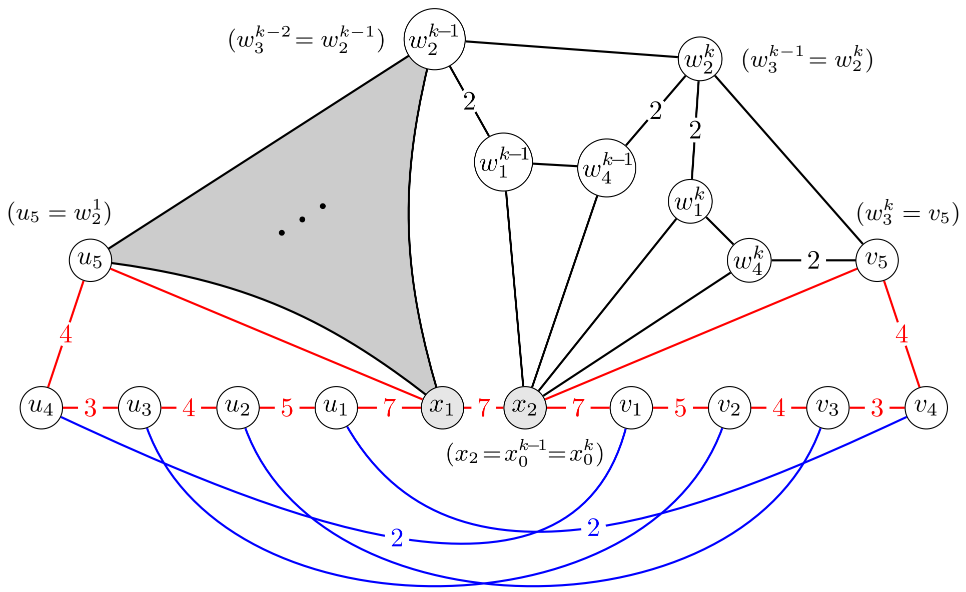

Figure 4 The graph $G_{13}^{(k-2,2)}$ of Definition 5.2, drawn with 13 crossings. The thick edges of this graph have their thickness written as numeric labels, and all the unlabeled edges are of thickness 1. The bowtie part of this graph is drawn in red and blue (where blue edges are those between u_i and v_j vertices), and the wedges are drawn in black. Only the $(k-1)$-th and k-th wedges (incident to x_2) are detailed, while the remaining $k-2$ wedges to the left (which are in this example all incident to x_1) analogously span the grey shaded area.

▷ Claim 5.1. For every graph G, there exists an optimal drawing $\mathcal{D}$ of G, such that every bunch of parallel edges is drawn as one thick edge in $\mathcal{D}$.

▶ **Definition 5.2** (Critical family $\{G_{13}^{(k_1,\dots,k_m)}\}$). *Let $m \geq 1$ and $k, k_1, k_2, \dots, k_m \geq 1$ be integers such that $k = k_1 + \ldots + k_m$. Let C_u be a 6-cycle on the vertex set $\{x_1, u_1, u_2, u_3, u_4, u_5\}$ with (thick) edges x_1u_1, u_1u_2, u_2u_3, u_3u_4, u_4u_5, u_5x_1 which are of thickness $7, 5, 4, 3, 4, 1$ in this order. Analogously, let C_v be a 6-cycle on the vertex set $\{x_m, v_1, v_2, v_3, v_4, v_5\}$ isomorphic to C_u in this order of vertices. Let P_x be a path of length $m-1$ on the vertices $(x_1, x_2, \dots, x_m)$ in this order and with all edges of thickness 7. We denote by B the graph obtained from the union $C_u \cup P_x \cup C_v$ (identifying the vertex x_1 of C_u and P_x and the vertex x_m of C_v and P_x) by adding edges u_2v_3 and u_3v_2, and 2-thick edges u_1v_4 and u_4v_1.*

Let D_i, for $i \in \{1, \dots, k\}$, denote the graph on the vertex set $\{x_0^i, w_1^i, w_2^i, w_3^i, w_4^i\}$ with the edges $x_0^iw_1^i$, $x_0^iw_4^i$, $w_1^iw_4^i$, $w_2^iw_3^i$ and the 2-thick edges $w_1^iw_2^i$ and $w_3^iw_4^i$. From the union $B \cup D_1 \cup \ldots \cup D_k$ we obtain the graph $G_{13}^{(k_1,\dots,k_m)}$ via

- *identifying u_5 with w_2^1 and w_3^k with v_5,*
- *for $i = 2, 3, \dots, k$, identifying w_3^{i-1} with w_2^i, and*
- *for $j = 1, 2, \dots, m$ and all i such that $k_1 + \ldots + k_{j-1} + 1 \leq i \leq k_1 + \ldots + k_j$, identifying x_0^i with x_j of the path P_x.*

This definition is illustrated in Figure 4 for $m = 2$. For reference, we will call the graph B the bowtie *of $G_{13}^{(k_1,\dots,k_m)}$, and the graph D_i the i-th* wedge *of $G_{13}^{(k_1,\dots,k_m)}$.*

▷ Observation 5.3. a) For every $m \geq 1$ and $k_1, k_2, \dots, k_m \geq 1$, the graph $G_{13}^{(k_1,\dots,k_m)}$ is 3-connected and non-planar.
b) For $1 < j < m$, the degree of the vertex x_j equals $2k_j + 14$, and the degree of the vertices x_1 and x_m equals $2k_1 + 15$ and $2k_m + 15$, respectively.

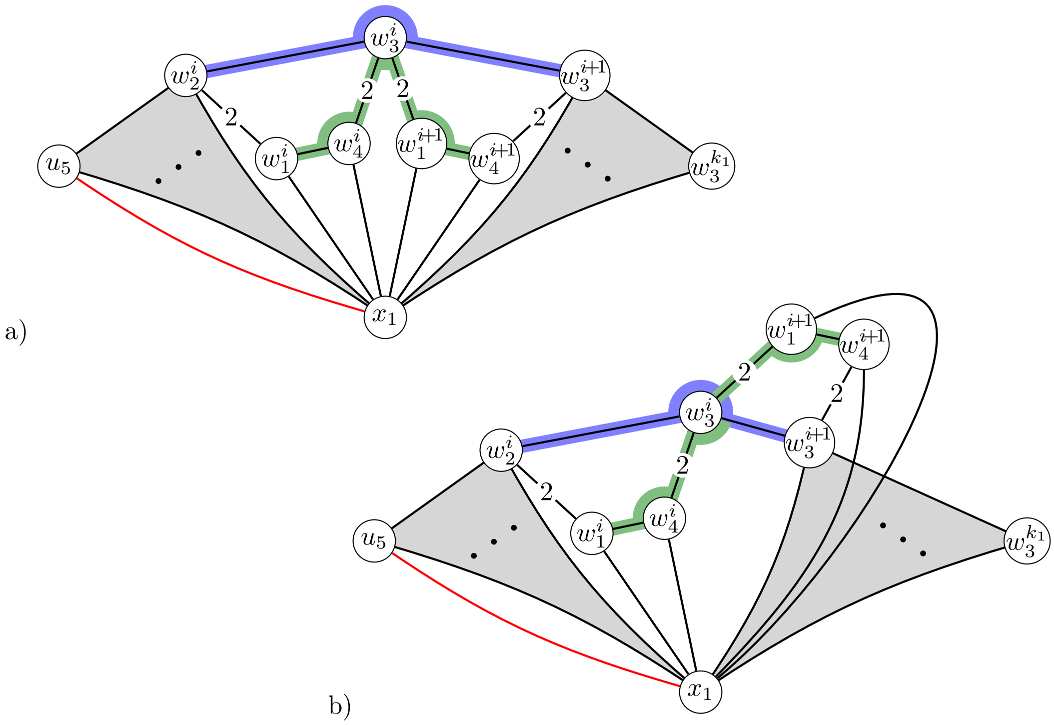

Figure 5 Two cases of vertex w_3^i of the induction step in the proof of Lemma 5.5. In each of them we "shrink" two wedges into one by drawing new edges $w_1^i w_4^{i+1}$ (green) and $w_2^i w_3^{i+1}$ (blue) along the depicted paths. In case (a), this introduces no new crossing, while in case (b) the new crossing between the green and the blue is "paid by" a crossing which must have been on the 4-cycle $(x_1, w_4^i, w_3^i, w_1^{i+1})$ before.

In order to prove Theorem 1.2, it is enough to consider the graph $G = G_{13}^{(k_1,\dots,k_m)}$ for $m \geq 2$ and $k_1 = \dots = k_m = \lfloor d/2 \rfloor$, and prove that $\mathrm{cr}(G) \geq 13$ and that, for every edge e of G, we get $\mathrm{cr}(G - e) \leq 12$. Before stepping into the proof, we remark that this does not hold for $m = 1$ since $\mathrm{cr}(G_{13}^{(k)}) \leq 12$ for all k (readers aware of the earlier conference paper [6] should note that the similarly looking construction in [6] had the edges u_3u_4 and v_3v_4 of thickness 4 instead of 3).

▶ **Lemma 5.4.** $\mathrm{cr}(G_{13}^{(1,1)}) = 13$.

Proof. Figure 4 outlines a drawing of $G_{13}^{(k_1,k_2)}$ with 13 crossings for all $k_1, k_2 \geq 1$. For the lower bound on $\mathrm{cr}(G_{13}^{(1,1)})$, we use the computer tool *Crossing Number Web Compute* [8] which uses an ILP formulation of the crossing number problem (based on Kuratowski subgraphs), and solves it via a branch-and-cut-and-price routine. Moreover, this computer tool generates machine-readable proofs[3] of the lower bound, which (roughly) consist of a branching tree in which every leaf holds an LP formulation of selected Kuratowski subgraphs certifying that, in this case, the crossing number must be greater than 12. ◀

▶ **Lemma 5.5.** *For every $k_1 \geq 1$ and $k_2 \geq 1$, $\mathrm{cr}(G_{13}^{(k_1,k_2)}) \geq 13$.*

[3] See `http://crossings.uos.de/job/zS43gWV2yd-ZKmit_DNwSg`. Vertices x_1 and x_2 are labeled 0 and 6, respectively. Cycles C_u and C_v traverse vertices $0, 1, 2, 3, 4, 5$ and $6, 7, 8, 9, 10, 11$ in that order, respectively.

Proof. We proceed by induction on $k_1 + k_2$, where the base case $k_1 = k_2 = 1$ is proved in Lemma 5.4. Hence, we may assume that $k_1 \geq 2$, up to symmetry.

Consider a drawing of $G_{13}^{(k_1,k_2)}$ with $c = \mathrm{cr}(G_{13}^{(k_1,k_2)})$ crossings. Let $1 \leq i \leq k_1 - 1$. By Claim 5.1, we may assume that all thick edges are drawn together in a bunch. We now distinguish three cases based on the cyclic order of edges leaving the vertex $w_3^i = w_2^{i+1}$ (the orientation is not important):

- The edges incident to $w_3^i = w_2^{i+1}$, in a small neighbourhood of w_3^i, have the cyclic order $w_3^i w_4^i$, $w_3^i w_1^{i+1}$, $w_3^i w_3^{i+1}$, $w_3^i w_2^i$. See in Figure 5 a), where this cyclic order is anti-clockwise. In this case, we draw a new edge $w_1^i w_4^{i+1}$ along the path $(w_1^i, w_4^i, w_3^i, w_1^{i+1}, w_4^{i+1})$, and another new edge $w_2^i w_3^{i+1}$ along the path $(w_2^i, w_3^i, w_3^{i+1})$ (both new edges are of thickness 1). Then we delete the vertices w_4^i, w_3^i, w_1^{i+1} together with incident edges. The resulting drawing represents a graph which is clearly isomorphic to $G_{13}^{(k_1-1,k_2)}$ – the wedges i and $i + 1$ incident to x_1 have been replaced with one wedge.
  Moreover, thanks to the assumption, we can avoid crossing between $w_1^i w_4^{i+1}$ and $w_2^i w_3^{i+1}$ in the considered neighbourhood of former w_3^i. Therefore, every crossing of the new drawing (including possible crossings of each of the new edges $w_1^i w_4^{i+1}$ and $w_2^i w_3^{i+1}$ among themselves or with other edges) existed already in the original drawing of $G_{13}^{(k_1,k_2)}$, and so $\mathrm{cr}(G_{13}^{(k_1-1,k_2)}) \leq c$. However, $\mathrm{cr}(G_{13}^{(k_1-1,k_2)}) \geq 13$ by the induction assumption, and so $c \geq 13$ holds true in this case.
- The same proof as above works if the cyclic order around w_3^i is $w_3^i w_4^i$, $w_3^i w_1^{i+1}$, $w_3^i w_2^i$, $w_3^i w_3^{i+1}$.
- In a small neighbourhood of $w_3^i = w_2^{i+1}$, the incident edges have (up to orientation reversal) the cyclic order $w_3^i w_4^i$, $w_3^i w_3^{i+1}$, $w_3^i w_1^{i+1}$, $w_3^i w_2^i$. See Figure 5 b). Consider the 4-cycle $C := (x_1, w_4^i, w_3^i, w_3^{i+1})$ which, importantly, uses only single edges of the 2-thick edges incident to w_3^i. In this case of the cyclic order around w_3^i, both sides of C contain a part of the drawing of $G_{13}^{(k_1,k_2)}$. Since $G_{13}^{(k_1,k_2)} - V(C)$ is connected, some edge of C must be crossed. Consequently, the subdrawing of $G_{13}^{(k_1,k_2)} - E(C)$ has $\leq c - 1$ crossings.
  We finish similarly as in the first case, but within $G_{13}^{(k_1,k_2)} - E(C)$: we draw a new edge $w_1^i w_4^{i+1}$ along the path $(w_1^i, w_4^i, w_3^i, w_1^{i+1}, w_4^{i+1})$, and another new edge $w_2^i w_3^{i+1}$ along the path $(w_2^i, w_3^i, w_3^{i+1})$ (both new edges are of thickness 1, and we have so far removed only one of the two edges of each of $w_4^i w_3^i$ and $w_3^i w_1^{i+1}$). These two new edges mutually cross once (at most – in case that the named paths cross also somewhere else than at w_3^i, we may eliminate multiple crossings by standard means). After deleting the original vertices w_4^i, w_3^i, w_1^{i+1}, we hence get a drawing which is again clearly isomorphic to $G_{13}^{(k_1-1,k_2)}$ and has at most $c - 1 + 1 = c$ crossings. Since $\mathrm{cr}(G_{13}^{(k_1-1,k_2)}) \geq 13$ by the induction assumption, $c \geq 13$ holds true also in this case. ◀

▶ **Theorem 5.6.** *For every integers $m \geq 2$ and $k_1, k_2, \ldots, k_m \geq 1$, the graph $G_{13}^{(k_1,\ldots,k_m)}$ is 13-crossing-critical.*

Proof. Let $G = G_{13}^{(k_1,\ldots,k_m)}$ and P_x be the 7-thick path on the vertices $(x_1, x_2, \ldots, x_m)$ from Definition 5.2. We first prove that $\mathrm{cr}(G) \geq 13$. Using Claim 5.1, at most one edge of P_x is crossed, or we already have 14 crossings. So assume that all edges of P_x except possibly $x_j x_{j+1}$ have no crossing. Contracting the edges $E(P_x) \setminus \{x_j x_{j+1}\}$ thus creates no new crossing and results in a valid drawing isomorphic to $G_{13}^{(l_1,l_2)}$ where $l_1 = k_1 + \ldots + k_j$ and $l_2 = k_{j+1} + \ldots + k_m$. We conclude with $\mathrm{cr}(G) \geq \mathrm{cr}(G_{13}^{(l_1,l_2)}) \geq 13$ by Lemma 5.5.

Regarding criticality, our proof strategy can be described as follows. We provide a collection of drawings of our graph G, such that each edge e of G in some of the drawings,

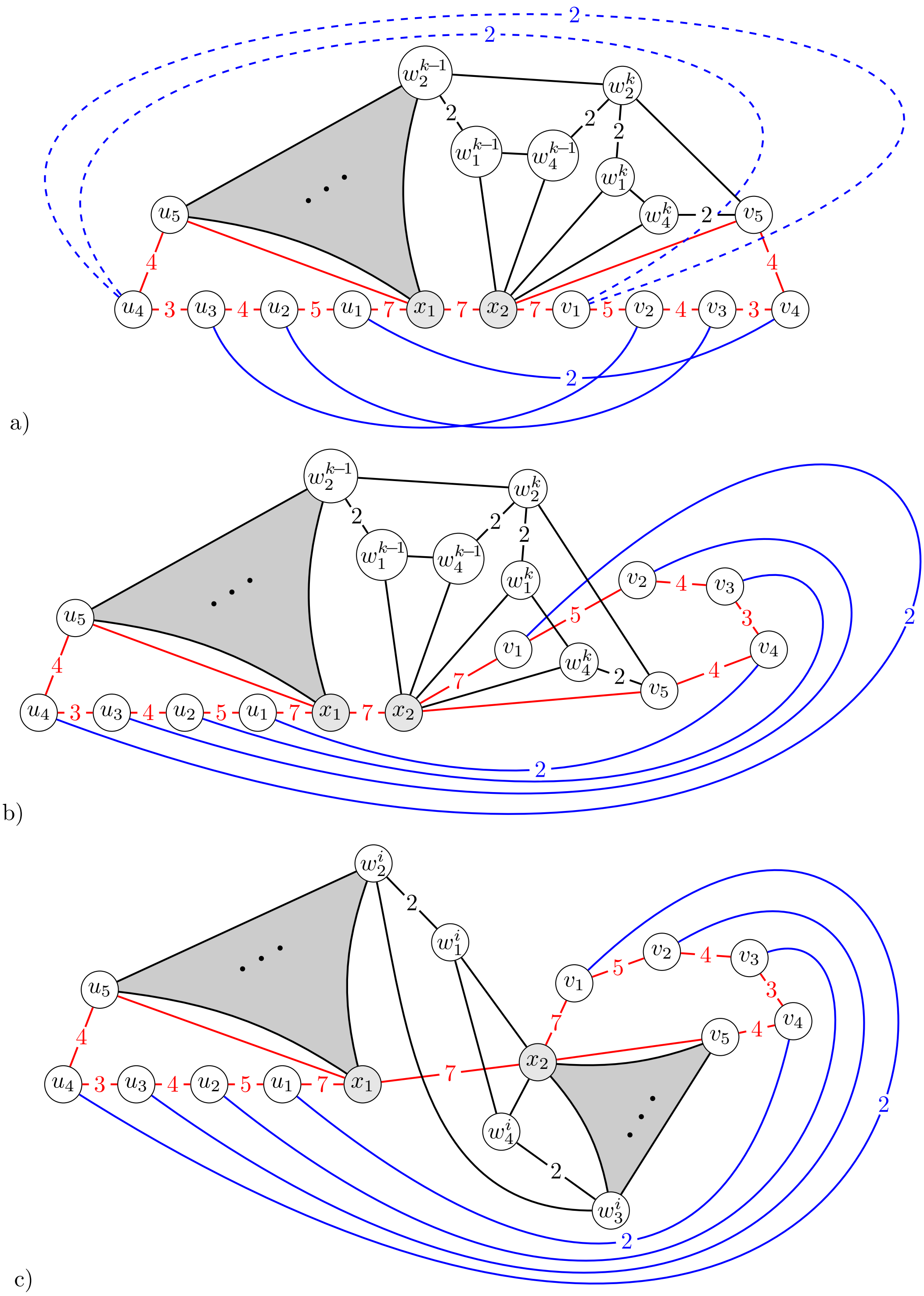

Figure 6 Three drawings of the graph $G_{13}^{(k_1,k_2)}$ with 13 or 14 crossings (where $k_2 = 2$ in cases (a) and (b), while $k_1 = i - 1$ in (c)). These drawings and their straightforward adjustments are used to argue criticality of the bowtie (red) edges of $G_{13}^{(k_1,k_2)}$. The grey areas span the crossing-free wedges of $G_{13}^{(k_1,k_2)}$ which are not detailed in the pictures, similarly as in Figure 4.

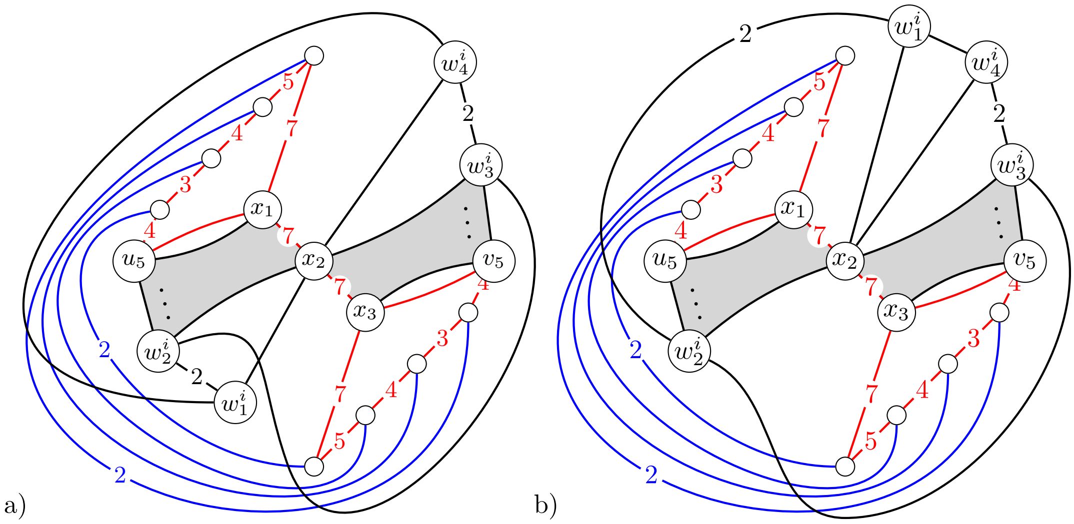

Figure 7 Two drawings of the graph $G_{13}^{(k_1,k_2,k_3)}$, having (a) 13 and (b) 18 crossings. These drawings are used to argue criticality of edges of the i-th wedge. The grey areas span the crossing-free wedges of $G_{13}^{(k_1,k_2,k_3)}$ which are not detailed in the pictures, similarly as in Figure 6.

when deleted, exhibits a "drop" of the crossing number below 13; that is $\mathrm{cr}(G - e) \leq 12$. Note that, for thick edges, we are deleting only one edge of the multiple bunch.

We start with the edges of the bowtie of G. For the blue edges (i.e., $u_2v_3, u_3v_2, u_1v_4, u_4v_1$), this follows immediately from the drawing in Figure 4 in which deleting any blue edge saves crossings. Furthermore, one can easily split the vertices x_1 and x_2 in the picture to produce the full path P_x as needed. For the remaining, red bowtie edges, criticality is witnessed by the three drawings in Figure 6. In the first one (a), which is almost the same as Figure 4, two alternate routings of the edge u_4v_1 show criticality of the edges x_2v_5 and v_4v_5, respectively. We symmetricaly argue about the edges x_1u_5 and u_4u_5. The second one (b) shows criticality of the edge v_1v_2. However, by pulling v_1 in this picture away from x_2 we also certify criticality of x_2v_1, and by pulling v_2 or also v_3 towards x_2 we get criticality of v_2v_3 and v_3v_4. Again, we can easily split the vertices x_1 and x_2 in the drawings to produce the path P_x as needed and without further crossings. The edges x_1u_1, u_1u_2, u_2u_3 and u_3u_4 are symmetric, too.

Consider now a red edge x_jx_{j+1} of P_x. Let the first wedge incident to x_{j+1} be the i-th wedge D_i. We twist the picture from Figure 4 at the edge x_jx_{j+1}, such that the wedges preceding D_i stay above the path P_x, and the wedges succeeding D_i are now below P_x. This is illustrated for $j = 1$ in Figure 6(c). The wedge D_i now crosses the 7-thick edge x_jx_{j+1}, giving a drawing of G with 14 crossings, and so certifying criticality of the edge x_jx_{j+1}, since deleting it drops the number of crossings in this drawing down to 12.

We are left with the last, and perhaps most interesting, cases in which e is an edge in the i-th wedge D_i. We consider a twist of the drawing of G similar to that in Figure 6(c), but this time with the wedge D_i crossing the blue bowtie edges (and itself). This gives a drawing with 13 crossings involving the edges $x_2w_1^i$, $w_1^iw_4^i$ and $w_2^iw_3^i$, which is illustrated in Figure 7(a). Hence the listed edges, and the edge $x_2w_4^i$ by symmetry, are also critical in G, as desired. Finally, we deal, up to symmetry, with the 2-thick edge $w_1^iw_2^i$. A slight adjustement of the last drawing gives a drawing illustrated in Figure 7(b) with exactly 18 crossings which are between the blue edges and $w_2^iw_3^i$, $w_1^iw_2^i$. Since deleting one edge from the 2-thick edge $w_1^iw_2^i$ drops the number of crossings again down to 12, we have shown also criticality of $w_1^i w_2^i$ and the proof is finished. ◀

Theorem 1.2 is now established for $k_1 = \ldots = k_m = \lfloor d/2 \rfloor$.

6 Extended crossing-critical construction

In the previous section, we have constructed an infinite family of 13-crossing-critical graphs with unbounded maximum degree. The construction leaves a natural question about analogous c-crossing-critical families for $c > 13$.

Clearly, the disjoint union of the graph from Theorem 5.6 with $c-13$ disjoint copies of $K_{3,3}$ yields a (disconnected) c-crossing-critical graph with maximum degree greater than d, for every $c \geq 14$. Though, our aim is to preserve also the 3-connectivity property of the resulting graphs.

First, to motivate the coming construction, we recall that the zip product of Definition 2.3 requires a vertex of degree 3 in the considered graphs. However, the graphs of Definition 5.2 have no such vertex, and so we come with the following modification.

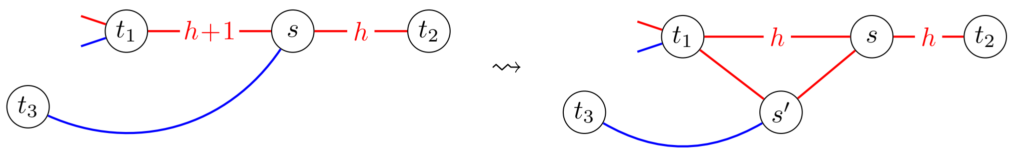

Figure 8 An illustration of the operation of locally introducing a vertex (s') of degree 3 from Lemma 6.1. This operation can be applied, e.g., to vertices $t_1 = v_1$, $s = v_2$, $t_2 = v_3$, and $t_3 = u_3$ of Figure 4.

▶ **Lemma 6.1.** *Assume a graph H with vertices t_1, t_2, t_3 and s such that*

a) vertex s has no more neighbours than t_1, t_2, t_3 in H, the edge t_1s is $(h+1)$-thick, t_2s is h-thick, t_3s is 1-thick, and

b) vertex t_1 is of degree at most $h+5$ in H, or there is a neighbour $w \neq s$ of t_1 such that t_1 is of degree at most $h+3$ in $H - t_1w$.

Other edges of H are not important.

Let H' be created by making the edge t_1s only h-thick, deleting the edge t_3s, and adding a new vertex s' adjacent via three 1-thick edges to the vertices t_1, t_3 and s. See Figure 8. Then $\mathrm{cr}(H') \geq \mathrm{cr}(H)$*. Furthermore, if H is a c-crossing-critical graph and* $\mathrm{cr}(H' - ss') < c$*, then H' is also c-crossing-critical.*

Proof. Assume a drawing $\mathcal{D}$ of H' with $\mathrm{cr}(H')$ crossings. By Claim 5.1, we have st_1 and st_2 drawn each as one thick edge. We consider two cases based on the crossings on ss' in $\mathcal{D}$.

First, there are at least 2 crossings on ss' in $\mathcal{D}$. We modify $\mathcal{D}$ to $\mathcal{D}'$ as follows: delete the current edge ss', and pull the vertices s and s' along their edges to t_1 so that no crossing remains on st_1 and $s't_1$ in $\mathcal{D}'$. This modification does not change the number of crossings on the paths (t_2, s, t_1) and (t_3, s', t_1). Then draw a new (1-thick) edge ss' in $\mathcal{D}'$ closely along the path (s', t_1, s), crossing only some of the edges incident with t_1 (and choosing "the better side" of t_1). Thanks to the assumption (b), this makes only at most 2 crossings on ss' in $\mathcal{D}'$: if t_1 is of degree $h+5$ then we cross at most $[(h+5)-(h+1)]/2 = 2$, and if t_1 is of degree $h+3$ in $H - t_1w$, then we can avoid crossing t_1w and again cross at most $(h+3)-(h+1) = 2$.

Altogether, there are no more crossings in $\mathcal{D}'$ than there were in $\mathcal{D}$. Since st_1 is crossing-free, we can turn st_1 into an $(h+1)$-thick edge and still have at most $\mathrm{cr}(H')$ crossings. Then we delete the edge $s't_1$ and obtain a subdivision of the graph H with at most $\mathrm{cr}(H')$ crossings, which certifies $\mathrm{cr}(H') \geq \mathrm{cr}(H)$.

Second, we assume that there is at most 1 crossing on ss' in $\mathcal{D}$. Let the number of crossings on each edge of the parallel bunch st_1 be a and on the edge $s't_1$ let it be b. If $b \geq a$, then we do the same as previously: delete the edge $s't_1$ and turn st_1 into an $(h+1)$-thick edge. The resulting drawing is a subdivision of H and the new number of crossings is $\mathrm{cr}(H') - b + a \leq \mathrm{cr}(H')$, again certifying $\mathrm{cr}(H') \geq \mathrm{cr}(H)$.

Otherwise, if $b \leq a-1$, there are altogether at most $b+1 \leq a$ crossings along the (1-thick) path (t_1, s', s). We hence make no more crossings than $\mathrm{cr}(H')$ if we redraw the h-thick edge $t_1 s$ closely along the path (t_1, s', s) and "through" the vertex s', creating a subdivision of a graph isomorphic to H (now with s subdividing h-thick edge $s't_2$). Again, the conclusion is that $\mathrm{cr}(H') \geq \mathrm{cr}(H)$.

The last bit is to argue c-crossing-criticality of H' under the additional assumption of the lemma. Consider any edge $e \in E(H') \cap E(H)$, and a drawing $\mathcal{D}$ of $H - e$ with less than c crossings. Since s has only three neighbours in H, the vertex s' can be chosen in $\mathcal{D}$ as subdividing a suitable one of the edges of the $(h+1)$-thick bunch $t_1 s$, the one consecutive to $t_3 s$ in the rotation around s in $\mathcal{D}$. This results in a drawing of $H' - e$ with same number of crossings (less than c). It remains to consider the edges of $E(H') \setminus E(H) = \{t_1 s', t_3 s', ss'\}$. We have got the assumption $\mathrm{cr}(H' - ss') < c$, and drawings of $H' - t_1 s'$ and of $H' - t_3 s'$ with less than c crossings are subdivisions of the corresponding drawings of $H - t_1 s$ and $H - t_3 s$. ◂

Proof of Corollary 1.3. Similarly as in the previous section, we take the 13-crossing-critical graph $G = G_{13}^{(k_1,\ldots,k_m)}$ of Theorem 5.6 for $k_1 = \ldots = k_m = \lfloor d/2 \rfloor$. Then we apply Lemma 6.1 to the vertices $t_1 = v_1$, $s = v_2$, $t_2 = v_3$ and $t_3 = u_3$ of G. This results in a graph G' having a vertex s' of degree 3. Moreover, since $\mathrm{cr}(G' - ss') \leq 11$ which can be easily seen from Figure 4 (we avoid crossings with $u_4 v_1$), we get that G' is 13-crossing-critical.

Hence let $G(13, d, m) = G'$. For $c > 13$, we proceed by induction, assuming that we have already constructed the graph $G(c-1, d, m)$ and it contains a vertex of degree 3. Theorem 2.4 establishes that $G(c, d, m)$, as a zip product of $G(c-1, d, m)$ with 1-crossing-critical $K_{3,3}$, is c-crossing-critical. Furthermore, $G(c, d, m)$ again contains a vertex of degree 3 coming from the $K_{3,3}$ part. ◂

7 Concluding remarks and open problems

While our contribution closes the questions related to the validity of the bounded maximum degree conjecture, the following natural problems remain open:

▷ Problem 7.1. For each $c \leq 12$, determine the least integer $D(c)$ bounding maximum degree of c-crossing-critical graphs.

▷ Problem 7.2. Develop a theory of wedges that parallels the theory of tiles (cf. [19]) for constructively establishing c-crossing-criticality of graphs with large maximum degrees.

Note that with our construction we can get arbitrarily repeated even degrees in $G = G_{13}^{(k_1,\ldots,k_m)}$, cf. Observation 5.3 b), but only two large-odd-degree vertices there. Trying to split the vertex x_1 or x_2 "inside one wedge" does not help either since the resulting critical split edge would be only 6-thick, maintaining the parity of the degree. In general, vertices of high odd degrees in c-crossing critical graphs seem to rely on some local property of the graph, unlike even degrees that can rely simply on sufficiently many relevant paths passing through the vertices. Indeed, the only other known examples of large odd degrees in infinite families of c-crossing-critical graphs are related to staircase-strip tiles [4]. Hence we suggest also the following question:

▷ Problem 7.3. Does there exist, for some/any $c \geq 13$, a family of c-crossing-critical graphs, such that for a prescribed set O of odd integers greater than 3 and each integer m, the family would contain a graph with at least m vertices of each degree in O?

Furthermore, Lemma 6.1 can be applied iteratively to selected vertices of each wedge of the graphs $G = G_{13}^{(k_1,\ldots,k_m)}$ to produce new c-crossing-critical graphs which would be 3-connected and have no double edges within the wedges. However, removing the remaining multiple edges in the bowtie subgraph would require a different approach. Hence, our final problem is:

▷ Problem 7.4. For which c does there exist a family of 3-connected simple c-crossing-critical graphs containing vertices of arbitrarily large degree?

References

1 Miklós Ajtai, Vašek Chvátal, Monroe M. Newborn, and Endre Szemerédi. Crossing-free subgraphs. In Peter L. Hammer, Alexander Rosa, Gert Sabidussi, and Jean Turgeon, editors, *Theory and Practice of Combinatorics*, volume 60 of *North-Holland Mathematics Studies*, pages 9–12. North-Holland, 1982. `doi:10.1016/S0304-0208(08)73484-4`.

2 Daniel Bienstock, Neil Robertson, Paul D. Seymour, and Robin Thomas. Quickly excluding a forest. *J. Comb. Theory, Series B*, 52(2):274–283, 1991. `doi:10.1016/0095-8956(91)90068-U`.

3 Drago Bokal. Infinite families of crossing-critical graphs with prescribed average degree and crossing number. *J. Graph Theory*, 65(2):139–162, 2010. `doi:10.1002/jgt.20470`.

4 Drago Bokal, Mojca Bračič, Marek Dernár, and Petr Hliněný. On degree properties of crossing-critical families of graphs. *Electron. J. Comb.*, 26(1):#P1.40, 2019.

5 Drago Bokal, Markus Chimani, and Jesús Leaños. Crossing number additivity over edge cuts. *Eur. J. Comb.*, 34(6):1010–1018, 2013. `doi:10.1016/j.ejc.2013.02.002`.

6 Drago Bokal, Zdeněk Dvořák, Petr Hliněný, Jesús Leaños, Bojan Mohar, and Tilo Wiedera. Bounded degree conjecture holds precisely for c-crossing-critical graphs with $c \leq 12$. In *35th International Symposium on Computational Geometry (SoCG 2019)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2019.

7 Drago Bokal, Bogdan Oporowski, R. Bruce Richter, and Gelasio Salazar. Characterizing 2-crossing-critical graphs. *Advances in Appl. Math.*, 74:23–208, 2016. `doi:10.1016/j.aam.2015.10.003`.

8 Markus Chimani and Tilo Wiedera. An ILP-based proof system for the crossing number problem. In *ESA 2016*, LIPIcs 57, pages 29:1–29:13, 2016. `doi:10.4230/LIPIcs.ESA.2016.29`.

9 Zdenek Dvořák, Petr Hliněný, and Bojan Mohar. Structure and generation of crossing-critical graphs. In *SoCG 2018*, LIPIcs 99, pages 33:1–33:14, 2018. `doi:10.4230/LIPIcs.SoCG.2018.33`.

10 Zdeněk Dvořák and Bojan Mohar. Crossing-critical graphs with large maximum degree. *J. Comb. Theory, Series B*, 100(4):413–417, 2010. `doi:10.1016/j.jctb.2009.11.003`.

11 César Hernández-Vélez, Gelasio Salazar, and Robin Thomas. Nested cycles in large triangulations and crossing-critical graphs. *J. Comb. Theory, Series B*, 102(1):86–92, 2012. `doi:10.1016/j.jctb.2011.04.006`.

12 Petr Hliněný. Crossing-number critical graphs have bounded path-width. *J. Comb. Theory, Series B*, 88(2):347–367, 2003. `doi:10.1016/S0095-8956(03)00037-6`.

13 Petr Hliněný and Gelasio Salazar. Stars and bonds in crossing-critical graphs. *J. Graph Theory*, 65(3):198–215, 2010. doi:10.1002/jgt.20473.

14 Martin Kochol. Construction of crossing-critical graphs. *Discrete Math.*, 66(3):311–313, 1987. doi:10.1016/0012-365X(87)90108-7.

15 Jesús Leaños and Gelasio Salazar. On the additivity of crossing numbers of graphs. *Journal of Knot Theory and Its Ramifications - JKTR*, 17:1043–1050, 2008. doi:10.1142/S0218216508006531.

16 Frank T. Leighton. *Complexity Issues in VLSI: Optimal Layouts for the Shuffle-exchange Graph and Other Networks.* MIT Press, Cambridge, MA, USA, 1983.

17 Bojan Mohar, Richard J. Nowakowski, and Douglas B. West. Research problems from the 5th Slovenian Conference (Bled, 2003). *Discrete Math.*, 307(3-5):650–658, 2007. doi:10.1016/j.disc.2006.07.013.

18 Bojan Mohar and Carsten Thomassen. *Graphs on Surfaces.* Johns Hopkins University Press, 2001. URL: https://jhupbooks.press.jhu.edu/content/graphs-surfaces.

19 Benny Pinontoan and R. Bruce Richter. Crossing numbers of sequences of graphs II: planar tiles. *J. Graph Theory*, 42(4):332–341, 2003. doi:10.1002/jgt.10097.

20 R. Bruce Richter and Carsten Thomassen. Minimal graphs with crossing number at least k. *J. Comb. Theory, Series B*, 58(2):217–224, 1993. doi:10.1006/jctb.1993.1038.

21 Jozef Širáň. Infinite families of crossing-critical graphs with a given crossing number. *Discrete Math.*, 48(1):129–132, 1984. doi:10.1016/0012-365X(84)90140-7.