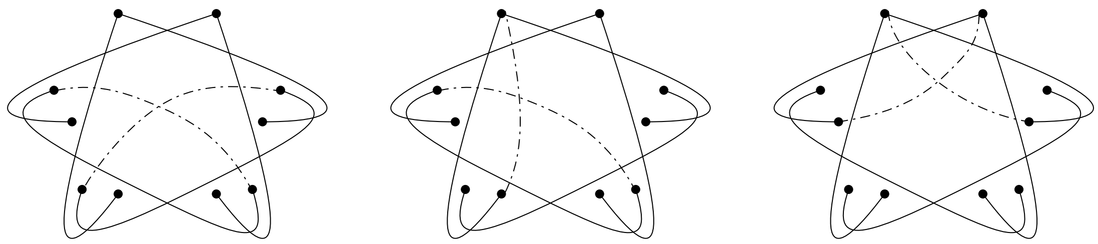
Crossing Numbers Workshop 2019



Telč, Czech Republic

1 Open Problems

1.1 Problem 1: Planarly drawn spanning tree in every/some crossing-minimal drawing

by Tilo Wiedera

Question 1. *Does every crossing minimal drawing contains a planarly drawn spanning tree?*

It seems that long time ago, Petr and Gelasio constructed examples of graphs whose optimal drawings do not contain spanning trees. However, the answer to this question is unknown when only crossing-minimal drawings are considered.

1.2 Problem 2: Planarly drawn spanning tree in every simple drawing of $K_{m,n}$

by Irene Parada

Question 2. *Does every simple drawing of $K_{m,n}$ have a planarly drawn spanning tree?*

Certain conditions that make the answer to this question is positive:

- if the considered drawing is rectilinear;
- if $m = 2$ or 3; and
- when all the vertices of one of the two maximal independent sets are incident with the outer face.

Possibly a stronger statement holds: given a simple drawing of $K_{m,n}$ and the set of edges $E_v \subseteq E(K_{m,n})$ incident with a fixed vertex v, is there there a planarly drawn spanning tree containing E_v? It would be interesting to show whether this holds for pseudolinear/monotone drawings.

1.3 Problem 3: Generalizing 4-to-3 expansion (from high-degree critical construction)

by Drago Bokal

Let G be a graph with a vertex s and three neighbours t_1, t_2 and t_3. Suppose there are two edges of thickness 4 from s to t_1 and t_2, and that there is an edge of thickness one from s to t_3 as indicated in the left of Figure 1. The *4-to-3 expansion* is obtained by removing s and replacing it by three vertices s^1, s^2 and s^3. Edges with certain thickness from and between s^1, s^2 and s^3 are added according to the right of Figure 1.

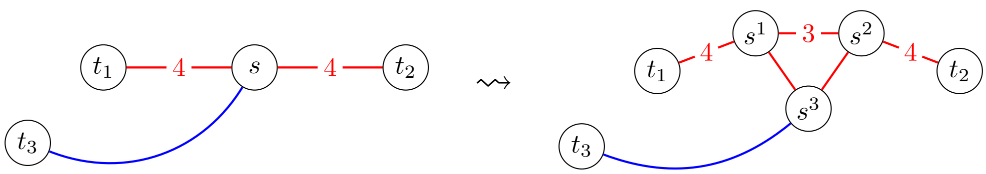


Figure 1: 3-to-4 expansion

In [1], Petr Hliněný introduced 4-to-3 expansions, with an ingenious proof that if G' is a 4-to-3 expansion of a graph G and $cr(G) \geq 13$, then $cr(G') \geq 13$ and G' is 13-crossing critical. The proof uses an interesting fact, that in one of the cases that need to be analyzed, one can find at least 13 crossings on the edges involved in the operation.

Generalizing the 4-to-3 expansion, for an integer $d \geq 4$, define the d-*to*-$(d-1)$ operation the same as before, but replacing the parameters 3 and 4 by d and $d-1$, respectively.

Question 3. *Let G' be a graph is obtained from G by applying a d-to-$(d-1)$ operation. Is it true that cr$(G') \geq$ cr(G)?*

It seems possible to approach this problem by using general Leighton embedding method that bounds the crossing number.

This might be useful to produce c-crossing-critical graphs with k vertices of arbitrarily large degree for any $c \geq 13$. Current constructions only yield c-crossing-critical graphs with k vertices of arbitrarily large degree for any $c \geq 13k$.

1.4 Problem 4: Bicolouring crossings in a straight-line drawing of K_n

by Birgit Vogtenhuber

Question 4. *Given a straight-line drawing D of K_n, what is the complexity of finding a 2-coloring of the set of edges such that the number of crossings between edges of the same color is minimized?*

Consider the intersection graph X_D (or crossing graph) of a drawing D, whose vertices are the edges of D and there is an edge between two vertices e, $f \in V(X_D)$ if and only if they cross in D.

The 2-coloring required in Question 4 can be viewed as a bipartion (A, B) of $V(X_D)$ that minimizes the number of edges with both endpoints in A or B. Therefore, the considered problem is equivalent to the max cut problem for the complement $\overline{X_D}$.

Observations:

- This question is NP-Hard if general graphs are considered instead of K_n.
- This questions is hard even if points are in convex position

1.5 Problem 5: Characterizing monotone drawings

by Alan Arroyo

A *monotone curve* is the image of a continuous function $\alpha : [0, 1] \to \mathbb{R}^2$ such that, for $x, y \in [0, 1]$, $x < y$ iff $\alpha(x) < \alpha(y)$. A *monotone* (or x-monotone) drawing D is a drawing in which each edge is a monotone curve.

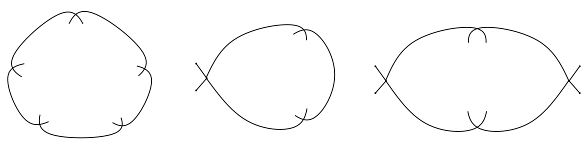


Figure 2: A cloud, a fish and a crab.

Two drawings in $\mathbb{R}^2$ are *homeomorphic* if there is an homeomorphism $\mathbb{R}^2 \to \mathbb{R}^2$ mapping one drawing into the other. A drawing D of a graph G is homomonotone if there is an monotone drawing $\overrightarrow{D}$ of G homeomorphic to D.

There are two known families of not homomonotone drawings: clouds and fish. Two of these creatures are depicted in the left and center drawings in Fig. 2.

Question 5. *Are there more minimal obstructions to homomonotonicity, apart from clouds and fish? Can we find all the obstructions?*

This question is mainly motivated by a recent characterization of pseudolinear drawings that are indeed homomonotone [2]. This characterization is by means of three forbidden obstructions: clouds, fish and crabs (see Figure 2).

To solve this question, one can assume that the considered drawing is not pseudolinear; hence, there is at least one crab.

The hope is that a good understanding of the crabs behaviour maybe sufficient to predict when a drawing is homononotone.

1.6 Problem 6: Well-quasi ordering for surface minors (beyond the plane)

by Gelasio Salazar

Let Σ be a surface. Two graphs embedded in Σ are *combinatorially equivalent* if there is a self-homeomorphism of Σ that takes one to the other. If H and G are graphs embedded in Σ, then H is a Σ-*minor* of G if there is a sequence of vertex deletions, edge deletions and edge contractions performed on Σ that take G to a graph combinatorially equivalent to H.

Question 6. *Is the Σ-minor relation a well-quasi-order for every Σ?*

This question has some motivation coming from Knot Theory. The answer is affirmative when Σ is the plane [3]. For this work, the tools provided by the Graph Minor Theorem were insufficient.
Observations:

- A minor is not necessarily a surface minor.
- Maybe a good use of SPQ-trees can provide a simpler proof for the planar case.
- Seems to be an easy problem when the tree-width is bounded and when the Σ-representativity of a subsequence of Σ-embedded graphs is increasing.

1.7 Problem 7: Tanglegrams and induced sub-tanglegrams relation, WQO

by László Székely

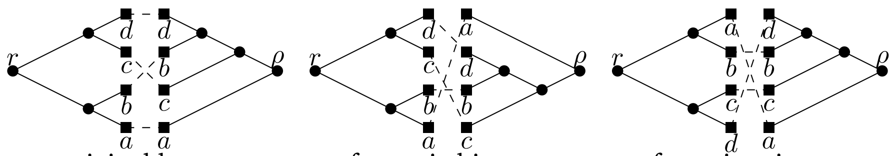


Figure 3: Results of a switch and a mirror operation.

A *tangelgram layout* consists on two rooted binary trees L and R (left and right) and a perfect matching M between their leafs. The left tree L is planarly drawn to the left of the $x = -1$ line where only its leafs are allowed to be on this line; likewise the right tree R is planarly drawn to the right of $x = 1$ with its leafs on this line. The perfect matching M is drawn rectilinearly.

There is an important operation for tangelgrams layouts:

- *Switch:* Given a non-leaf vertex v in any of the two trees L and R, there are two subtrees of descendants of v. These trees are the upper and lower trees at v. The switching operation at v consists on interchanging the upper tree by the lower tree, and the lower tree by the upper (as in Figure 3).

Two tangelgram layout *represent* the same tangelgram, if, after a series of switches, one layout is obtained from the other.

In a rooted binary plane tree T, if one chooses a subset S of the leafs of T, there is a unique subtree T' of T containing S as set of leafs. Moreover, T' is a (subdivided) binary tree. The tree obtained after suppressing the degree 2 vertices of T' is the *binary subtree of T induced by S*.

Given two tangelgram layouts $\mathcal{L}_1 = (R_1, L_1, M_1)$ and $\mathcal{L}_2 = (R_2, L_2, M_2)$, $\mathcal{L}_1$ is a sub-layout of $\mathcal{L}_2$ if $M_1 \subset M_2$ and L_1 and R_1 are the binary subtrees induced by the leafs of L_2 and R_2, respectively, that are incident with M_1. A tangelgram T_1 is a sub-tangelgram of T_2 iff there are layouts $\mathcal{L}_1$ and $\mathcal{L}_2$ of T_1 and T_2, respectively, for which $\mathcal{L}_1$ is a sublayout of $\mathcal{L}_2$.

Question 7. *Is the induced-subtangelgram relation well-quasi ordered?*

This question is partly motivated from the fact that planar tangelgrams (tangelgrams that have a layout without crossings between the matching edges), can be characterized by forbidding two induced-subtangelgrams.

1.8 Problem 8: Big clique big line conjecture (visibility)

by Bodhayan Roy

Given a set S of points in the plane, we say that two points $u, v \in S$ are *visible* if the line segment between u and v does not contain any other point from S.

Question 8. *Is it true that for all integers k, ℓ there exists an integer n such that every finite set of at least n points in the plane contains either ℓ collinear points or k pairwise visible points.*

Kára, Pór, and Wood conjectured that the answer is positive [4]. The assumption that S is finite is crucial, as the conjecture fails for infite point sets.

A possible way to approach this problem would be to consider vertex degrees in point visibility graphs (for a set S of points in the plane, the corresponding visibility graph has vertex set S and for any two $u, v \in S$, uv is an edge iff the points u and v are visible). Onur's conjecture is the following.

Conjecture 1. *For every ϵ there exists n_0 such that every set of $n \geq n_0$ points in the plane contains either 4 collinear points or a point of degree at least $(1 - \epsilon)\, n$.*

So far, we have shown that the statement in Conjecture 1 is true for $\epsilon \geq \sqrt{2} - 1$.

A related, but weaker problem regards blocked point sets. A set S of points is *k-blocked* if every point in S can be assigned one of k colors is such a way that every two points $u, v \in S$ are visible if and only if they are assigned different colors. Aloupis et al.[5] conjectured the following.

Conjecture 2. *For every k there exists some n such that every k-blocked set has at most n points.*

1.9 Problem 9: Computing the crossing number parameterized by the vertex cover (treed)

by Petr Hliněný

The goal of this problem is to find reasonably nice and rich classes of graphs where the crossing number is unbounded and efficiently computable. An inspiration for this qustion is the class of simple graphs having bounded vertex cover, for which the crossing number is in FPT when parametrized by the size of a minimum vertex cover.

Question 9. *Is determining the crossing number of edge-weighted graphs FPT, when parametrized by the size of a minimum vertex cover?*

Staying in simple graphs, a natural generalization of a small vertex cover is bounded tree-depth.

Question 10. *Is there an approximation algorithm for the crossing number that is FPT when parametrized by the tree-depth of the graph?*

1.10 Problem 10: Thrackle conjecture for radial drawings

by Radoslav Fulek

A *thrackle* is a drawing of a graph such that each edge is a Jordan arc and every pair of edges meet exactly once. The Thrackle conjecture (posed by Conway in 1960s) states that if a graph $G = (V, E)$ admits a drawing in the plane that is a thrackle, then $|E| \leq |V|$.

Radial drawing is a drawing on $S^1 \times I$ such that the orthogonal projection to I in injective for every edge.

Question 11. *Is the thrackle conjecture true for radial drawings?*

Pach and Sterling showed monotone drawings satisfy the Thrackle conjecture, and hence, if the answer of this question is positive, this would generalize Pach and Sterling result.
Observations:

- Perhaps looking at thrackle radial drawings of small graphs may give some insight.
- An alternative easy problem is to look a drawings such that by removing one edge the drawing becomes monotone.

1.11 Problem 11: Computing clique size in disk graphs of two radii

Onur Çağırıcı

A *biradii disk intersection graph* is a disc intersection graph that can be realized by disks of radius 1 or r for some r.

Question 12. *What is the complexity of MAX CLIQUE for biradii disk intersection graphs?*

1.12 Problem 12: Bipartite midrange crossing constant different than the regular one?

Éva Czabarka

Given a class of graphs $\mathcal{H}$, let $k_{\mathcal{H}}(n, e) = \min\{\text{cr}(G) \ : \ G \in \mathcal{H}, |V(G)| = n, |E(G)| = e\}$. It was shown by Pach, Spencer and Tóth that if $\mathcal{H}$ is the set of all graphs with $n \ll e \ll n^2$,

$$\lim_{n\to\infty} k_{\mathcal{H}}(n, e) \cdot \frac{n^2}{e^3}$$

exists. This limit C is known as the *midrange crossing constant*. Note that this midrange crossing constant is related with the crossing lemma, stating that for any graph with n vertices and $e > 7n$ edges, $\text{cr}(G) \geq k\frac{e^3}{n^2}$ for some constant k (the best known constant is $k = \frac{1}{29}$ due to Ackerman). Indeed, the crossing lemma gives us a positive lower bound for the expression inside the limit. However, it is far from obvious that the midrange crossing constant exists.

The bipartite midrange crossing constant is similarly defined for bipartite graphs.

Question 13. *Does the bipartite midrange crossing constant differ from the regular one?*

A graph class $\mathcal{H}$ is PST if it has a graph with at least on edge and is closed under

1. taking subgraphs;
2. disjoint unions; and
3. vertex cloning.

It can be shown that every PST class has a midrange crossing constant. The set of bipartite graphs is PST, and indeed, every PST class contains the bipartite graphs.

It is unknown whether any two PST classes have different midrange crossing constant. The existence of two different constants would imply that the constant for the class of all graphs is smaller that the one for the bipartite graph, and this naturally motivates Question 13.

References

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