Testing Graph MSO Properties A Fresh View

Or, can we beat Courcelle's theorem?

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- Yet, more on "optimality": cannot get much *above bd. tree-width*, for MSO_2 by [Kreutzer–Tazari], and col.-MSO_1 by [Ganian et al.]

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- and, can find new wider classes for elementary m.c. of MSO_1

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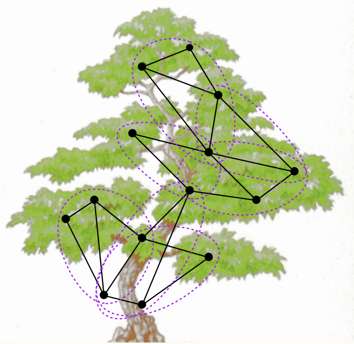
- extensions can enumerate / optimize over solutions...

Courcelle's MSO_2 Theorem, once again

Tree-width $tw(G) \leq k$ if whole G can be covered by bags of size $\leq k+1$, arranged in a "tree-like fashion".



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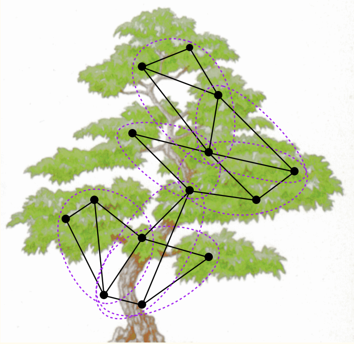

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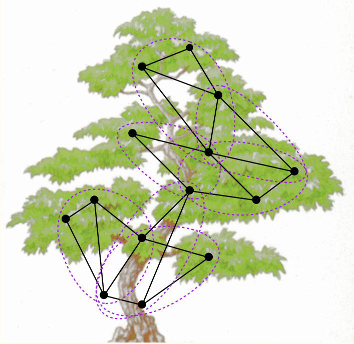

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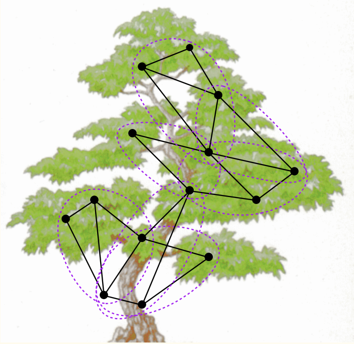

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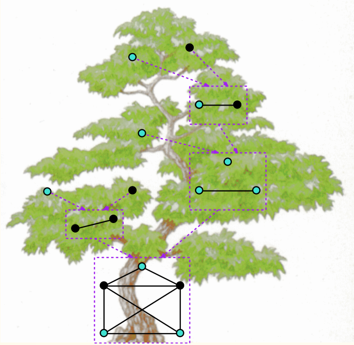
Theorem. (Courcelle)
Assume ϕ is an MSO_2 sentence, and G is of tree-width k, given along with a tree-decomposition. Then $G \models \phi$ can be decided by an FPT algorithm, in time $\mathcal{O}\big(g(k,\phi) \cdot |V(G)|\big)$ for some g.

Courcelle–Makowsky–Rotics MSO_1 Theorem

Clique-width $cwd(G) \le k$ if G given by a *k-expression* (over k-labelled gr.),

k-expression $\sim$ disjoint unions, relabelling, edge-add. between labels.



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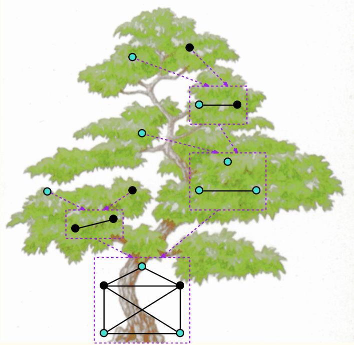

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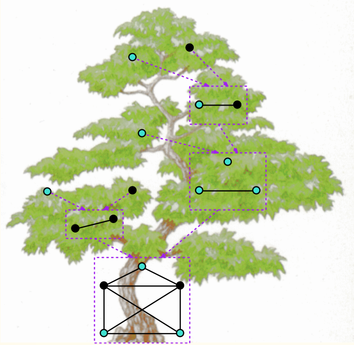

Theorem. (Courcelle–Makowsky–Rotics)

Assume ψ is an MSO_1 sentence, and G is of clique-width k, given along with a k-expression. Then $G \models \psi$ can be decided by an FPT algorithm, in time $\mathcal{O}\big(g(k,\psi) \cdot |V(G)|\big)$ for some g.

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- MSO_2 sentence → MSO over the coloured tree.
- (Similarly for clique-width and MSO_1. . .)

The conclusion. Enough to study MSO properties of coloured trees!

4 The Ground: Trees vs. Shrubs

Coloured MSO model checking in time...

$$|T| \cdot \quad 2^{2^{2^{\cdot^{\cdot^{*}}}}} \Big\}\ \text{quant-alt}(\phi) \qquad \text{vs.} \qquad |T| + \quad 2^{2^{2^{\cdot^{\cdot^{*}}}}} \Big\}\ \text{shrub height}$$

About the Shrub Case – FO

Claim. (almost folklore) A given FO sentence ϱ cannot distinguish too many copies of an arb. relational structure R.

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Corollary. For a given tree T, there is (efficiently) a subtree $T' \subseteq T$ such that $T \models \varrho \iff T' \models \varrho$, and T' is of bounded size.

Hence there is a kernelization FPT algorithm with runtime

$$\mathcal{O}\big(|T| + |T'|^q\big) \quad \text{where } |T'| \sim 2^{2^{\cdot^{\cdot^{\cdot}}}}\Big\}^{height}.$$

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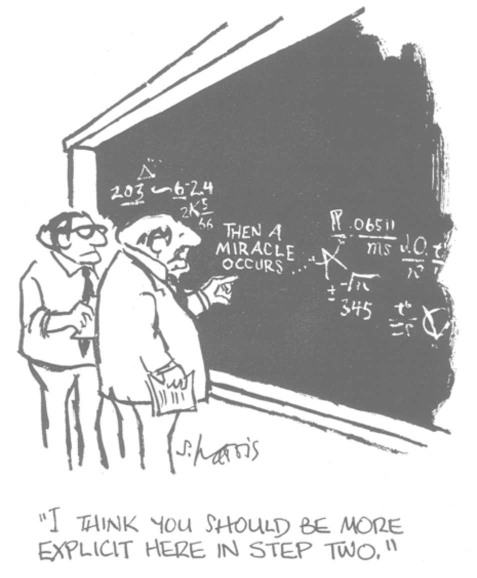

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Cons., the repetition threshold depends on ϕ and on the size of R.

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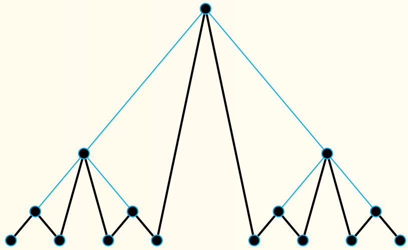

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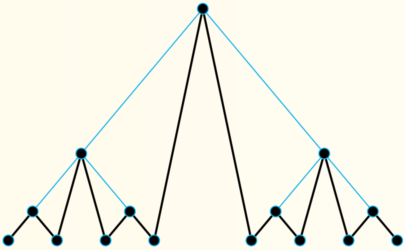

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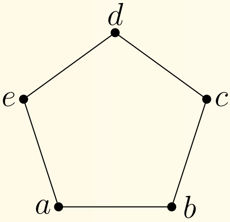

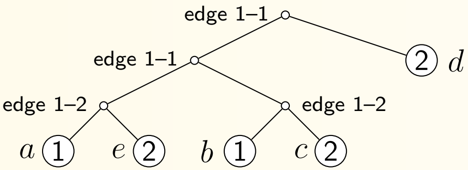

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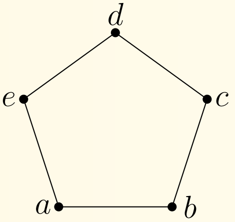

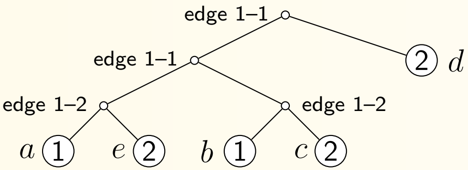

- *Shrub-depth* (of a graph class) = the smallest depth for which all the graphs are m-partite cographs (for some m).

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  That is, to find a reasonably restricted (and still "expressive") fragment of graph MSO giving elementary runtime dependence on the quantifier alternation depth?


HAPPY
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