A Short Proof of Euler–Poincaré Formula

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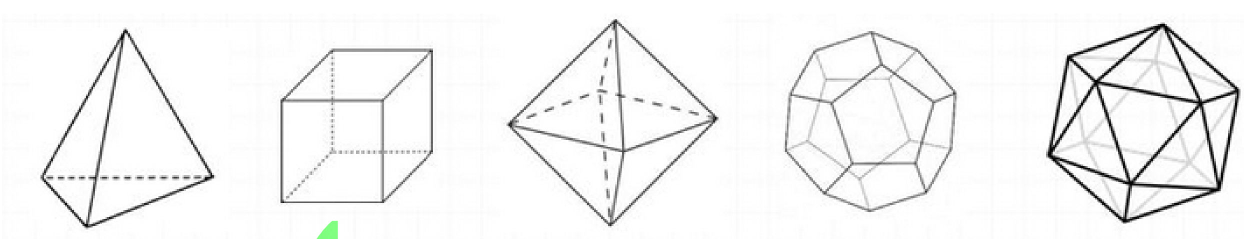
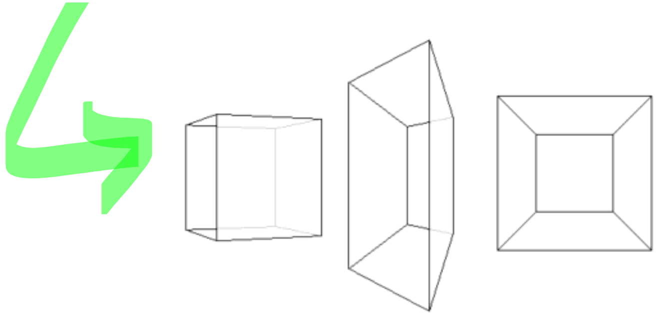
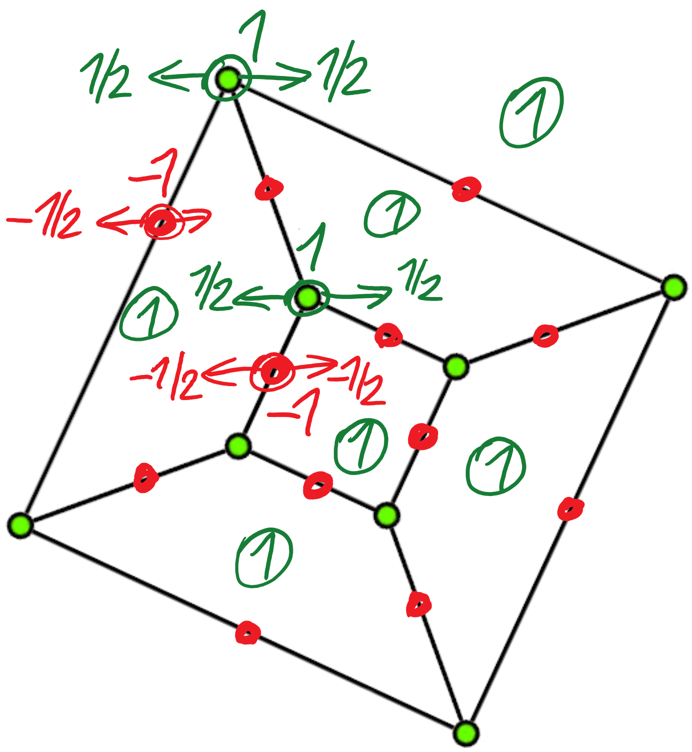
The basic formula (by Euler):

$$\#\text{vertices} - \#\text{edges} + \#\text{faces} = 2$$





A simple discharging proof



Σ of inner face

=

Σ of outer face

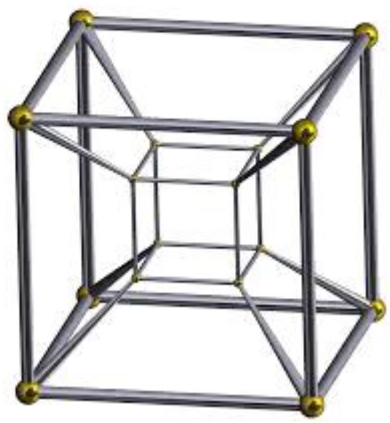
=

The generalized formula in dim. d :

Theorem 1 **("Euler–Poincaré formula"; Schläfli [5] 1852).** *Let P be a convex polytope in $\mathbb{R}^d$, and denote by f^c, $c \in \{0, 1, \ldots, d\}$, the numbers of faces of P of dimension c. Then*

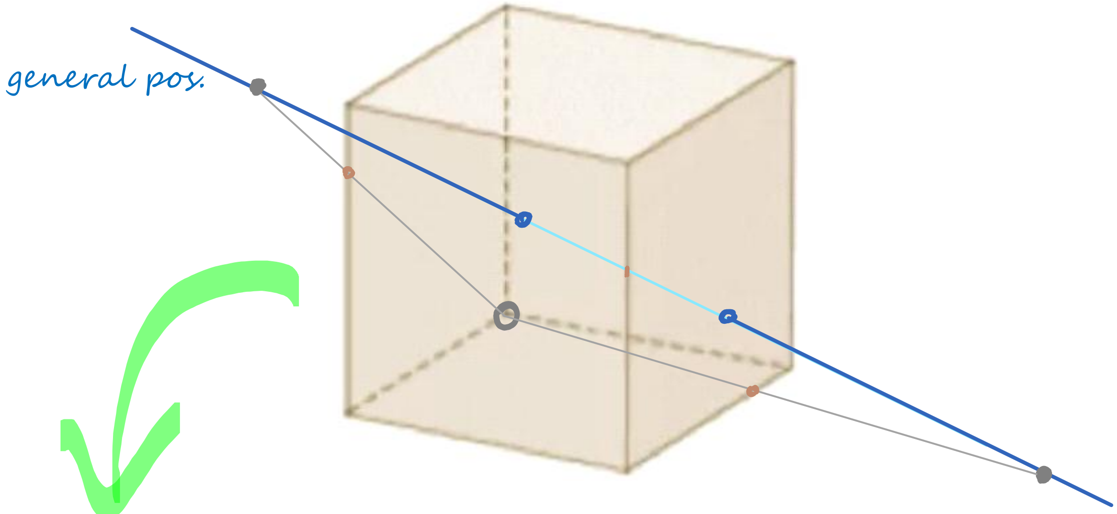
$$f^0 - f^1 + f^2 - \cdots + (-1)^d f^d = 1. \tag{1}$$

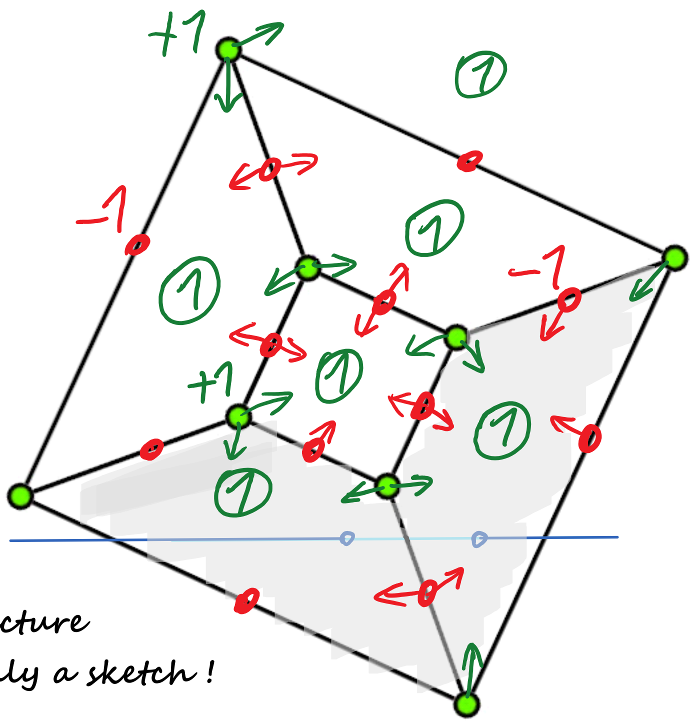
e.g., dim. 4:



A discharging proof again ?

- not quite yet, need a different view in 3D first





picture
only a sketch !

Σ of "clean" face

=

Σ of "pierced" face

=

Generalized discharging proof in dim. d :

- Denote shortly by E.P.[d] our formula

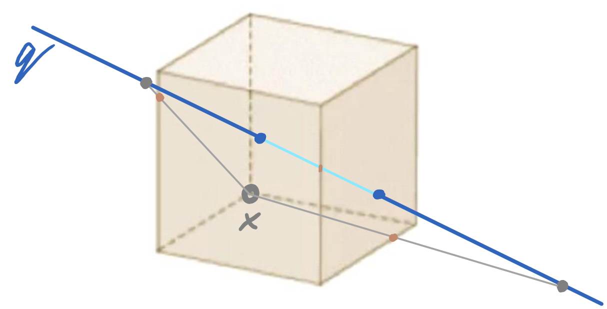
$$f^0 - f^1 + f^2 - \cdots + (-1)^d f^d = 1$$

- Proving by induction on k>1

E.P.[k-1] & E.P.[k] => E.P.[k+1]

as follows...

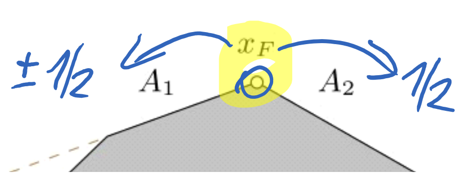
- Consider a polytope in dimension k+1, and
- choose a gen. position **line q** "piercing" two facets:

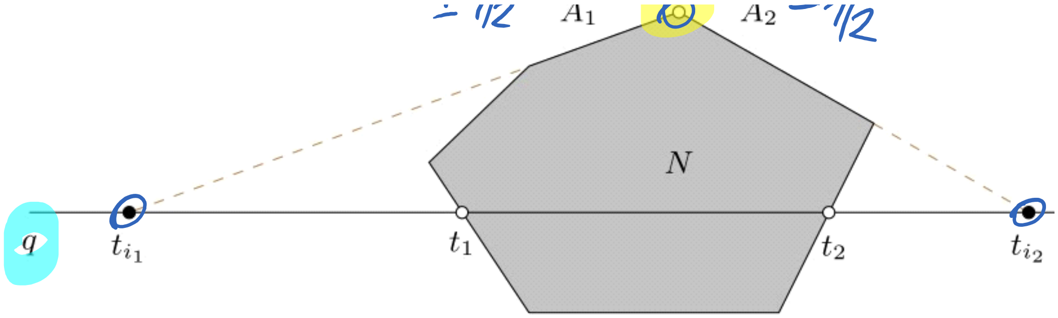


- Charge vertices by +1, edges by -1, polygs. by +1,

... c-dim. faces by $(-1)^c$; c.f. $f^0 - f^1 + f^2 - \cdots + (-1)^d f^d = 1$

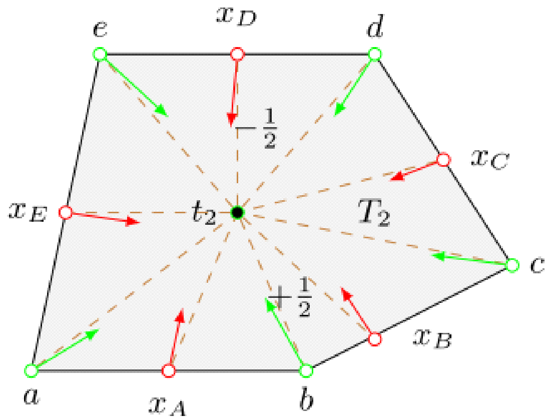
- The **discharging rule** for a face F (of dim. <k):
  - Take (any) point x_F in the rel. interior of F,
  - cut the polytope by the plane through q & x_F,
  - and send the charge from F to the two facets determ. by two edges of x_F in the cut-polyg.





- Now, all faces **discharged to 0** except the facets.

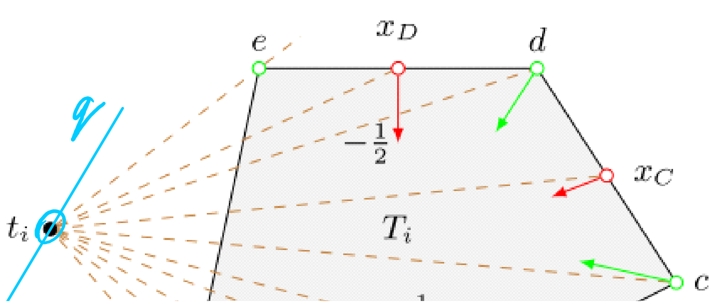
- A **"pierced" facet** : all its faces send into it!



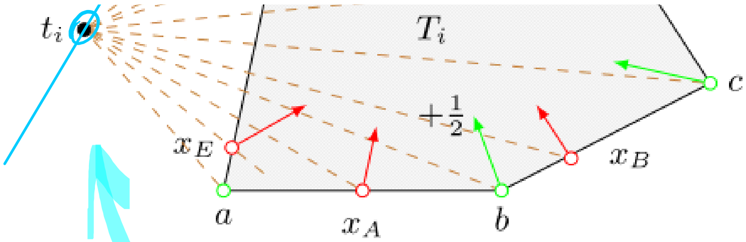
1/2× "E.P.[k]" = 1/2

"times two" = 1

- A **"clean" facet** T gives a more versatile picture...



Σ = ??



Set $t := q \cap$ hyperplane(T), then:

Face F sends half-charge to the facet T

<=>

the straight line $\overline{t\, x_F}$ passes through int(T)

<=>

the face F is destroyed by a projection of T from the point t.

- Last bit - where has the unit charge of the full **polytope** "disappeared" ?

- Nowhere; actually, we have cheated a bit....
    - The "E.P.[k]" formula of each of the two pierced facets used up only 1/2 of the facet charge, and
    - the remaining two halves exactly cancel with unit charge of the whole polytope.

Conclusions

- While there exist other simple inductive combin. proofs of the E.-P. formula, they all assume shellability of polytopes (highly nontrivial).
- We are using only very simple "2D" geometry and basics of linear algebra and convexity.

Thank you for your attention.