FO Properties of Interval Graphs

. . . (FO model checking)

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  - or, quantifies vertex and edge sets together $\exists X, Y, E, F$.

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  - nowhere dense classes in general ??

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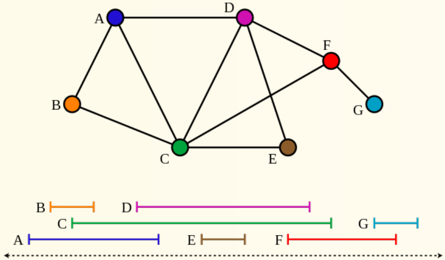

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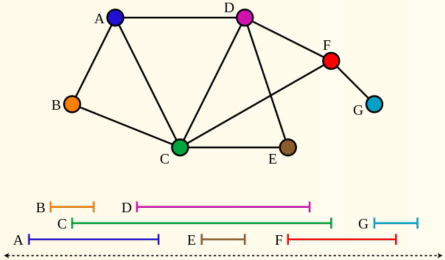

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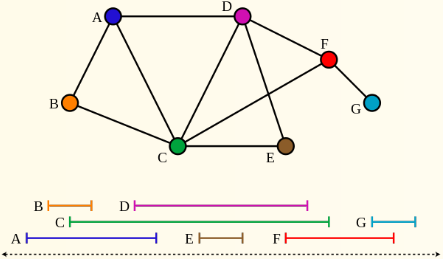

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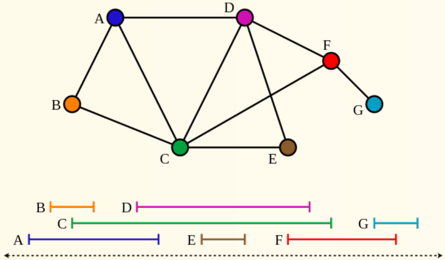

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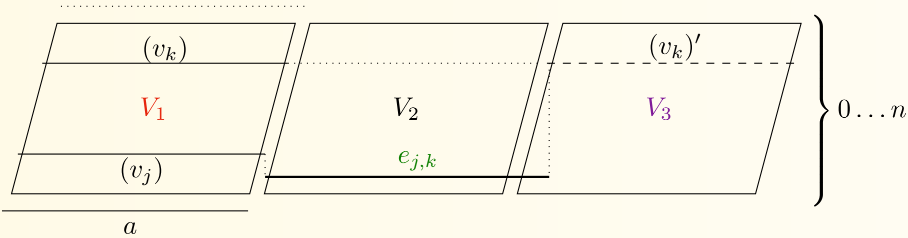
- Note; open/close intervals do not matter.

Interpreting in $[1, 1+\varepsilon]$-interval graphs

- The rough idea in picture (based on three "staircase piles"):



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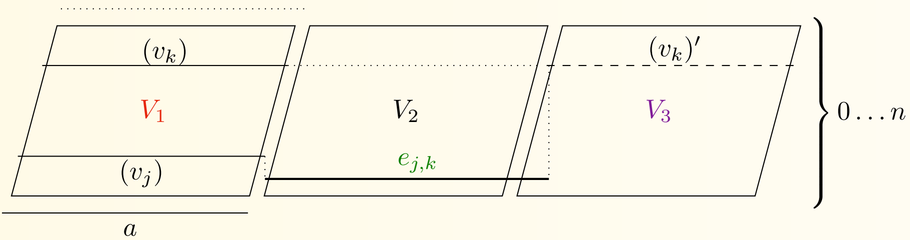

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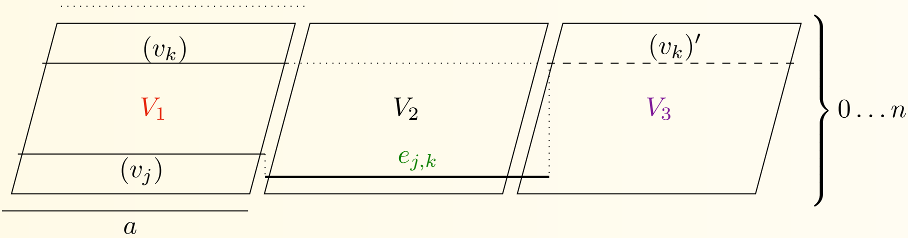

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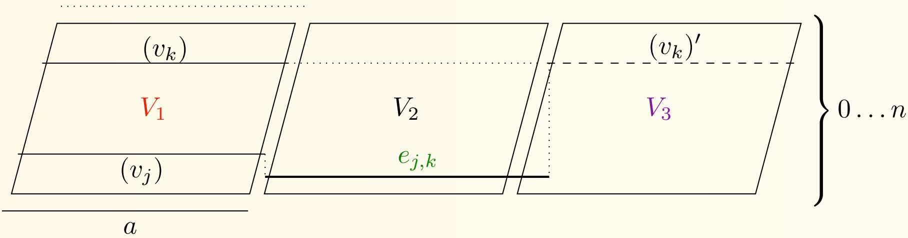

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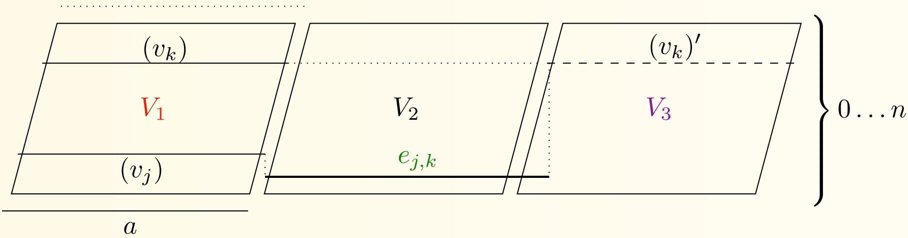

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  - for example, independent and dominating set, subgraph isom., etc.
  - nearly tight result by the previous examples,
  - rather easy to prove for rational L, but difficult otherwise.

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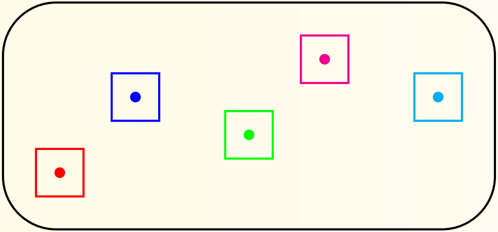

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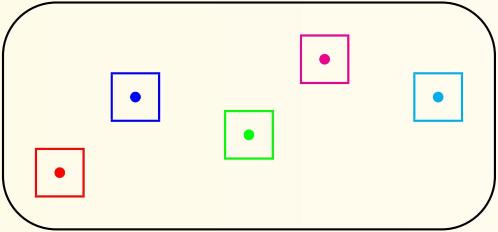

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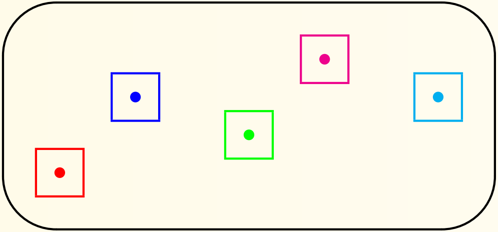

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- Hence, it is enough to solve any given FO property in every local neighbourhood!

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  - e.g., for unit interval $L = \{1\}$ these are $0, 1, 2, \ldots$
- clique-width — simply order the intervals by their distance from the resp. accumulation points $\to$ linear k-expression
  - where $k \sim |L| \cdot$ #accum. points ! (finite in bounded radius)

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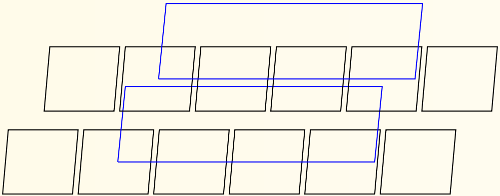

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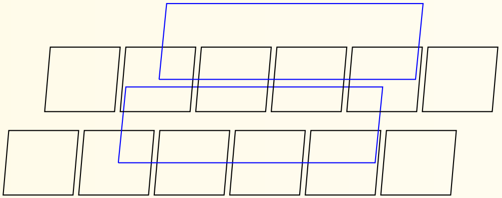

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- Infinitely many accumulation points in a bounded interval $\Rightarrow$ locally unbounded clique-width (not trivial, though).

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- By locality of FO, again, close & scattered can be taken as one accumulation pt. $\rightarrow$ again an irrelevant interval.

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