On “good” and “bad” digraph width measures

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1 How to Measure Graph “Width”

**Tree-width** (Robertson and Seymour) — a real success story:

- FPT algorithms for many problems, incl. all MSO₂
- structurally nice, FPT computable, *just great!*
- related to (even nicer) branch-width
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Recent additions
- an explosion of new directed measures in the past decade...
giving finer resolution for *better algorithmic applications*?
Directed measures: briefly (and chronologically)...

Cycle rank, --- directed path-width, dir. tree-width, $D$-width, entanglement, DAG-width, Kelly-width, DFVS-number, bi-rank-width, K-width, DAG-depth
Directed measures: briefly (and chronologically)...

Cycle rank, —— directed path-width, dir. tree-width, *D-width, entanglement, DAG-width, Kelly-width, DFVS-number, bi-rank-width, K-width, DAG-depth

... as driven by algorithmic use:

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References: ^[JRST01]^[b][LKM08]^[c][GHO10]^[d][FGLS09]^[e][EIS76]^[f][GW06]^[g][DGK09]^[h][GRK09]^[i][FGLS10]^[k][CD06]^[l][KO08]^[m][CLL+08]^[n][vL76]^[p][CMR00]^[q][BDHK06]^[r][OdB07].

\[\text{FPT} \simeq \text{runtime } O\left(f(k) \cdot n^c\right)\]  \[\text{XP} \simeq \text{runtime } O\left(n^{f(k)}\right)\]
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DAG  – directed *acyclic* graph (the *simplest* class ???)
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**DAG-width** – how many cops catch a *visible robber*  
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**Cycle rank** (60’s!) – how “deep” to remove vertices to become acyclic
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Minimum number of *labels* to build the graph using

- create a (labeled) vertex,
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**Bi-rank-width** (Kanté) – related to clique-width / rank-width; i.e. the branch-width of the *bi-cutrank* function on the vertex set.
### How these measures compare

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**Very good:** DAG-width, Kelly-width, DAG-depth

- having nice cops-and-robber *game characterizations*
- monotone under taking subgraphs and some restricted form of *arc contractions*
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and Bad: clique-width, bi-rank-width

- subgraphs can have \textit{much higher} width, e.g. the complete graph (bidirected) has small width while its subgraphs are complex
- still, not so bad since related to so called \textit{vertex minors}
4 and Algorithmic Usefulness

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\(\text{FPT} \simeq \text{runtime } O(f(k) \cdot n^c)\)
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Conclusions from the Table... 

**Very good:** clique-width, bi-rank-width 

- all MSO\(_1\) properties have \textbf{FPT} algorithms 
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- classical digraph problems like dominating set, Steiner tree, max-/min-LOB (outbranching), oriented colouring, etc. are still NP-hard for the measures
- positive algorithmic results seem rather incidental, e.g. Hamiltonian path and related, or some particular algorithms parametrized by the DFVS number
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The contrast: So far we have got no directed measure that is structurally nice and algorithmically useful at the same time!
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OK, but we want a directed measure that is NOT tree-width bounding!
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This “crazy subdivision” measure works well:

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NO, we really do not want a measure like this one, right?
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What about add. monotonicity under *butterfly contractions* (minors)?
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NO, this does not help to dismiss the “crazy” measure either...
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So, what definition of a \textit{directed minor} shall we consider when describing the property of being \textit{“structurally nice”}?
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  - \(\implies\) undirected MSO\(_1\) is the least common denominator!
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I.e., for every undirected $G$, one can efficiently orient (in XP time) the edges of $G$ such that the width is (approximately) optimal over all orientations of $G$. 
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- excessive info. — even knowing a graph is 3-colourable, there is no efficient way to find a colouring (this measure is cheating!)
7 The Conclusion, again

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- **Bi-rank-width is a really good dir. measure** – the best we (can) have?
THANK YOU FOR YOUR ATTENTION